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Bertrand-Edgeworth Equilibrium: Manipulable Residual Demand

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Abstract: In this paper we seek to provide a resolution of the Edgeworth paradox for the case where firms are free to supply less than the quantity demanded, the residual demand function is *manipulable* (a generalization of the proportional one) and prices vary over a grid. We demonstrate that a unique equilibrium in pure strategies exist whenever the number of firms is sufficiently large. Interestingly, the equilibrium involves excess production. Moreover, depending on the parameter values, the 'folk theorem' of perfect competition may or may not hold. The results go through even if the firms are asymmetric, or produce to order.

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1 Introduction

The Edgeworth paradox [15] is an important foundational problem in the theory of price competition. In a model of Bertrand duopoly where the firms are free to supply less than the quantity demanded, Edgeworth [15] argues that equilibria in pure strategies generally do not exist.[1] In this paper we seek to provide a resolution of the Edgeworth paradox for the case where firms are free to supply less than the quantity demanded, the residual demand function is a generalization of the proportional one and prices vary over a grid.

There are alternative ways of modeling a game of price competition. In this paper we examine the case where the firms make their price and output decisions simultaneously,[2] as well as the case where the firms produce to order, i.e. they first simultaneously decide on their price levels and then on their output levels.[3] Alternatively, one can assume that firms first decide on their output levels and then on their prices.[4] In this paper, however, we do not examine this framework.

We examine a class of demand functions which we call *manipulable*. We say that a residual demand function is manipulable if, by increasing its output level, a firm can increase the residual demand coming to it. An example would be the proportional residual demand function, while a counter-example would be the parallel residual demand function.[5]

[1] Dixon [11], as well as Friedman [16] provide formal statements of the problem.

[2] This framework has been examined, among others, by Dixon [11], Maskin [22] and Shubik [30,31]. Allen and Hellwig [1] and Vives [34] also use a similar framework for capacity constrained firms.

[3] Papers in this framework include Dixon [12].

[4] See, for example, Davidson and Deneckere [9] and Kreps and Scheinkman [19].

[5] In a companion paper we examine the case where the demand function is non-manipulable, i.e. where a firm cannot increase the demand coming to it by increasing its level of output (see P. Roy Chowdhury, Bertrand-Edgeworth equilibria: Non-manipulable residual demand, mimeo, Jawaharlal Nehru University (2001)).

Moreover, we assume that the price level varies over a grid, where the size of the grid can be arbitrarily small. There are two reasons why, under price competition, a pure strategy equilibrium may not exist. The first reason has to do with the well known open-set problem. The second one is more substantial in nature. Consider some price quantity configuration that is a candidate for being an equilibrium. With few firms in the market, the firms may not supply the whole of the demand coming to them, but produce at a level such that price equals marginal cost. However, if price does equal marginal cost, then it is profitable for the firms to increase their price. But this means that the proposed price quantity configuration cannot be an equilibrium. The grid assumption allows us to side-step the open set problem, and focus on the second problem, which in our view is the essential Edgeworth paradox. This assumption can also be motivated by appealing to the fact that there are minimum currency denominations, or to the practice of integer pricing.[6]

We now briefly summarize our results. We demonstrate that if the number of firms is large enough, then a unique Nash equilibrium exists. We also discuss the limit properties of the equilibrium outcome. The 'folk theorem' of perfect competition suggests that the perfectly competitive outcome can be interpreted as the limit of some oligopolistic equilibrium as the number of firms becomes large. The interesting question is whether in our framework this is true or not.[7] We find that in the limit as the grid size becomes

[6]Some other papers that model such discrete pricing include Dixon [13], Harrington [18], Maskin and Tirole [23], Ray Chaudhuri [27] and Roy Chowdhury [28]. Harrington [18] and Maskin and Tirole [23] examine price games with zero costs, while Dixon [13] examines a game with convex costs and a parallel residual demand function. Both Ray Chaudhuri [27] and Roy Chowdhury [28] examine price games with decreasing average cost functions. In models with discretized strategy spaces, Dasgupta and Maskin [6] discuss the sensitivity of equilibrium outcomes to the size of the grid.

[7]This question has been thoroughly investigated in the context of Cournot competition. See, for example, Novshek [24], and Novshek and Sonnenschein [25].

very small, and the number of firms becomes very large, the price level approaches the competitive one and the output level of each firm becomes vanishingly small. However, the limiting value of aggregate production, as the number of firms goes to infinity, depends on the value of the marginal cost function at the origin. If this is strictly positive then the limiting value of aggregate production is finite. Moreover, as the grid size goes to zero, this finite value converges to the competitive demand level. Thus in this case the 'folk theorem' holds. If, however, this value is zero, then in the limit aggregate production diverges to infinity. In this case the 'folk theorem' can be said to fail.

We then go on to argue that similar results hold even if the cost functions are asymmetric, or if the firms play a two stage game, where in stage 1 the firms decide on their price, and in stage 2 they decide on their output.

We then relate our paper to the existing literature on Bertrand price competition.[8]

One solution to the Edgeworth paradox is to look at mixed strategy equilibria. Examples of this approach include Allen and Hellwig [1,2], Dixon [10], Kreps and Scheinkman [19], Levitan and Shubik [20], Maskin [22], Osborne and Pitchik [26], Shubik [29,30] and Vives [34].[9] Price competition with differentiated products has been examined by Benassy [3], Friedman [16] and Simon [32]. Both Dubey [14] and Simon [31], on the other hand, adopt a general equilibrium framework to argue that if both buyers and sellers are strategic, then a pure strategy equilibrium exists. The approach of the two authors differ, among other things, in the way the rationing rule is modeled. There is also a large literature that examine models of price competition where firms use supply schedules, rather than prices as strategies. We can

[8]We refer the readers to Vives [35] for a more detailed and succinct summary of the literature.

[9]Some of these papers use the fixed point theorems for discontinuous games developed by Dasgupta and Maskin [6,7].

mention, among others, Grossman [17] and Mandy [21]. Borgers [4] studies the outcome if one applies a process of iterated elimination of dominated strategies in a Bertrand-Edgeworth game.[10]

In a series of interesting papers Dixon [11,12,13], examines various aspects of pure strategy equilibria of Bertrand-Edgeworth games. Dixon [13] is specially interesting as he also adopts a framework where prices vary discretely. Dixon [11,12,13], however, proves existence for the parallel residual demand function (or generalizations thereof) and not for the manipulable one. Both Dasgupta and Maskin [7] and Maskin [22] prove existence in mixed strategies for proportional residual demand functions. Allen and Hellwig [1] studies the nature of such equilibria, as well as their limiting properties. To the best of our knowledge, however, ours is the only paper that solves for the pure strategy Nash equilibrium of a Bertrand-Edgeworth game with manipulable residual demand functions.

The rest of the paper is organized as follows. Section 2 introduces and analyzes the basic model. Section 3 considers some extensions of the basic model. Section 4 concludes. Finally, proofs of some of the results have been collected together in the appendix.

2 The Model

There are n identical firms, all producing the same homogeneous good. The market demand function is $q = d(p)$ and the common cost function of all the firms is $c(q)$.

Throughout we maintain the following assumptions on the demand and the cost functions.

Assumption 1. $d(p)$ is negatively sloped and intersects the price axis

[10] Dastidar [8] uses a Bertrand-Chamberlin framework, where firms have to supply all demand, to demonstrate that equilibria in pure strategies exist.

at some price $p^{\max}$, where $0 < p^{\max} < \infty$.

Assumption 2. The cost function $c(q)$ is twice differentiable, increasing and strictly convex. Moreover, $p^{\max} > c'(0)$.

We then specify the residual (or the contingent) demand function. Let $R_i(P, Q)$ denote the residual demand facing the i-th firm when the price and the quantity vectors are given by $P = \{p_1, \cdots, p_n\}$, and $Q = \{q_1, \cdots, q_n\}$. Define $\underline{p}$ to be the minimum element in P such that at least some of the firms charging this price has a strictly positive output level. Then if the total production of all firms charging $\underline{p}$ is greater than $d(\underline{p})$, then we assume that all firms who charge a price greater than $\underline{p}$ obtain no demand, thus ensuring that $R_i(P, Q)$ is indeed a *residual* demand function. Moreover, let the number of firms charging the price $\underline{p}$ be m, and let the output vector of these m firms be $(q_1, \cdots, q_m)$. Then the residual demand facing the firms charging the price $\underline{p}$ is given by

$$R_i(P, Q) = \begin{cases} q_i, & \text{if } \sum_{j=1}^m q_j \leq d(\underline{p}), \\ \gamma(q_i, \sum_{j\neq i}^m q_j) d(\underline{p}), & \text{if } \sum_{j=1}^m q_j > d(\underline{p}), \end{cases} \tag{1}$$

where $0 \leq \gamma(q_i, \sum_{j\neq i}^m q_j) \leq 1$.

Notice that if the aggregate production of all the firms exceed $d(\underline{p})$, then the residual demand coming to the i-th firm depends on the i-th firm's output, as well as that of the other firms. In other words, the residual demand is *manipulable* via the output level.

We assume that $\gamma(q_i, \sum_{j\neq i}^m q_j)$ satisfies the following assumption.

Assumption 3. (i) The derivatives $\gamma_1(q_i, \sum_{j\neq i}^m q_j)$, $\gamma_{11}(q_i, \sum_{j\neq i}^m q_j)$ and $\gamma_{12}(q_i, \sum_{j\neq i}^m q_j)$ are well defined.

(ii) $\gamma_1(q_i, \sum_{j\neq i}^m q_j) > 0$, $\gamma_{11}(q_i, \sum_{j\neq i}^m q_j) < 0$, $\gamma_{12}(q_i, \sum_{j\neq i}^m q_j) < 0$ and $\gamma_{11}(q_i, \sum_{j\neq i}^m q_j) - \gamma_{12}(q_i, \sum_{j\neq i}^m q_j) < 0$.

(iii) $\gamma_1(x,(n-1)x)$ is decreasing in x. Moreover, $\lim_{x\to 0}\ \gamma_1(x,(n-1)x) = \infty$ and $\lim_{x\to\infty}\ \gamma_1(x,(n-1)x) = 0$.

(iv) If $\lim_{n\to\infty}(n-1)x(n) = L$, where L is finite, then $\lim_{n\to\infty}\gamma_1(x(n),(n-1)x(n)) = \frac{1}{L}$.

(v) If $\lim_{n\to\infty}(n-1)x(n) = \infty$, then $\lim_{n\to\infty}\gamma_1(x(n),(n-1)x(n)) = 0$.

(vi) If $\lim_{n\to\infty} x(n) = D$, where $D > 0$, then $\lim_{n\to\infty}\gamma_1(x(n),(n-1)x(n)) = 0$.

Note that assumption 3 only imposes restrictions on those firms who charge the price $\underline{p}$. No restrictions are imposed on those firms who charge prices higher than $\underline{p}$. In this sense the assumption is quite general.

It is easy to see that the proportional residual demand function satisfies assumption 3. In this case

$$\gamma(q_i, \sum_{j\neq i}^{m} q_j) = \frac{q_i}{\sum_{j=1}^{m} q_j}.$$ [11]

We assume that prices vary over a grid. Our results, however, hold for any grid size, no matter how small. Thus our analysis is true even if we approximate the continuous case arbitrarily closely. Define the set of feasible prices $F = \{p_0, p_1, \cdots\}$, where $p_0 = 0$, and $p_i = p_{i-1} + \alpha$, $\forall i \in \{1, 2, \cdots\}$, where $\alpha > 0$.

The i-th firm's strategy consists of simultaneously choosing *both* a price $p_i \in F$ and an output $q_i \in [0,\infty)$. All firms move simultaneously. We solve for the pure strategy Nash equilibrium of this game.

The supply function of a firm charging a price of p is given by $\min\{c'^{-1}(p), R_i(P,Q)\}$.[12] Thus we follow Edgeworth [15] in assuming that firms are free to supply less than the quantity demanded, rather than Chamberlin [5], who assumes that firms meet the whole of the demand coming to them.

[11] Observe that in this case $\gamma_1(q_i, \sum_{j\neq i} q_j) = \frac{\sum_{j\neq i}^{m} q_j}{(\sum_{j=1}^{m} q_j)^2}$ and $\gamma_1(x,(n-1)x) = \frac{(n-1)}{n^2 x}$.

[12] Since the cost function is strictly convex, $c'^{-1}(p)$ is well defined.

We then introduce a few more notations.

Let p^* be the minimum $p \in F$ such that $p > c'(0)$.[13] In words, p^* is the minimum price on the grid which is strictly greater than $c'(0)$. Since $p^* \in F$, let $p^* = p_j$ for some integer j.

Moreover, let $q^* = c'^{-1}(p^*)$ and let n^* be the smallest possible integer such that $\forall N \geq n^*$,

$$\frac{d(p^*)}{N} < c'^{-1}(p^*) = q^*.$$

Thus if N is greater than n^*, the price is p^* and all the firms produce $\frac{d(p^*)}{N}$, then the market price is strictly greater than marginal costs.

Define $q'(n-1)$ as satisfying the following equation:

$$p^* d(p^*) \gamma_1(q, (n-1)q) = c'(q). \tag{2}$$

Thus if the market price is p^* and all the firms produce $q'(n-1)$, then, for all firms, marginal revenue equals marginal cost. It is easy to see that $q'(n-1)$ is decreasing in n.[14] We are going to argue that for n large, the outcome where all the firms charge p^* and produce $q'(n-1)$, can be sustained as a Nash equilibrium.

We then introduce a series of lemmas that we require for our analysis. The proofs of all the lemmas have been collected together in the appendix.

Lemma 1. $\lim_{n\to\infty} p^* d(p^*) \gamma_1(\frac{d(p^*)}{n-1}, d(p^*)) > \lim_{n\to\infty} c'(\frac{d(p^*)}{n-1})$.

Given lemma 1, we can define N_1 to be the smallest possible integer such that $\forall n \geq N_1$,

$$p^* d(p^*) \gamma_1(\frac{d(p^*)}{n-1}, d(p^*)) > c'(\frac{d(p^*)}{n-1}).$$

Lemma 2. $\forall n \geq N_1, \ (n-1)q'(n-1) > d(p^*)$.

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[13] We assume that α is not too large so that $p^* < p^{\max}$.

[14] Notice that given assumption 3(iii), $q'(n-1)$ is well defined. That $q'(n-1)$ is decreasing in n, follows from Eq. (2) and the fact that $\gamma_{12} < 0$ and $\gamma_1(x, nx)$ is decreasing in x.

Consider an outcome such that all the firms charge p^* and produce $q'(n-1)$. Then lemma 2 suggests that if $n \geq N_1$, then the residual demand facing any firm that charges a price greater than p^* would be zero. This follows since the total production by the other firms will be enough to meet $d(p^*)$. Moreover, lemma 2 also implies that $\forall n \geq N_1$, $q'(n-1) > 0$.

Next define

$$\hat{\pi} = \max_q \begin{cases} p^* d(p^*)\gamma(q, (n^*-1)q^*) - c(q), & \text{if } q > d(p^*) - (n^*-1)q^*, \\ p^* q - c(q), & \text{otherwise.} \end{cases} \tag{3}$$

The interpretation of $\hat{\pi}$ is as follows. Suppose that n^* of the firms charge p^*, and all other firms charge a higher price. Moreover, out of the n^* firms, (n^*-1) of the firms produce q^* and the remaining firm produces q. Then $\hat{\pi}$ denotes the maximum profit that this firm can earn if it chooses its output level optimally.

Next consider some $\overline{p}_i \in F$, such that $\overline{p}_i > p^*$. Let $\overline{q}_i$ satisfy $\overline{p}_i = c'(\overline{q}_i)$. Let $\hat{n}_i$ be the minimum integer such that $\forall k \geq \hat{n}_i$, $\frac{d(\overline{p}_i)}{k} < \overline{q}_i$ and

$$\frac{\overline{p}_i d(\overline{p}_i)}{k} - c(\frac{d(\overline{p}_i)}{k}) < \hat{\pi}.$$

Lemma 3 below provides an interpretation of $\hat{n}_i$.

Lemma 3. *If the number of firms charging $\overline{p}_i$ is greater than or equal to $\hat{n}_i$, then the profit of some of these firms would be less than $\hat{\pi}$.*

We need a further definition.

Definition. $N_2 = \sum_{i=j+1}^{k} \hat{n}_i + n^* - 1$.

We then argue that for n sufficiently large, the unique equilibrium involves all firms charging the price p^*, producing $q'(n-1)$ and selling $\frac{d(p^*)}{n}$.

Proposition 1. *Let $n \geq \max\{N_1, N_2\}$. Then the unique equilibrium involves all the firms charging p^*, producing $q'(n-1)$ and selling $\frac{d(p^*)}{n}$.*

Proof.

Existence. The proof is divided into two steps.

Step 1. Since, from lemma 2, $(n-1)q'(n-1) > d(p^*)$, it is not possible for any firm to increase its price and gain, as the deviating firm will have no residual demand. Of course, from the definition of p^* it follows that undercutting is not profitable either.

Step 2. We then argue that none of the firms can change its output level and gain. Suppose firm i produces q_i, while the other firms produce $q'(n-1)$. Then the profit of the i-th firm

$$\pi_i(q_i, q', p^*) = p^* d(p^*)\gamma(q_i, (n-1)q') - c(q_i). \tag{4}$$

Clearly,

$$\frac{\partial \pi_i}{\partial q_i}(q_i, q', p^*) = p^* d(p^*)\gamma_1(q_i, (n-1)q') - c'(q_i). \tag{5}$$

Observe that the profit function is concave in p_i [15] and $\frac{\partial \pi_i(q_i,q',p^*)}{\partial q_i}|_{q_i=0} > 0$.[16] We then notice that

$$\frac{\partial \pi_i(q_i, q', p^*)}{\partial q_i}|_{q_i=q'} = p^* d(p^*)\gamma_1(q', (n-1)q') - c'(q'). \tag{6}$$

Finally note that setting

$$\frac{\partial \pi_i(q_i, q', p^*)}{\partial q_i}|_{q_i=q'} = 0, \tag{7}$$

we obtain Eq. (2).

[15]This follows since $\frac{\partial^2 \pi_i(q_i,q',p^*)}{\partial q_i^2} = p^* d(p^*)\gamma_{11}(q_i, (n-1)q') - c''(q_i) < 0$.

[16]Suppose not, i.e. let $p^* d(p^*)\gamma_1(0, (n-1)q') - c'(0) \leq 0$. Then,

$$\begin{aligned} c'(q'(n-1)) &= p^* d(p^*)\gamma_1(q'(n-1), (n-1)q'(n-1)) \\ &< p^* d(p^*)\gamma_1(0, (n-1)q'(n-1)) \ \text{(since } \gamma_{11} < 0) \\ &\leq c'(0), \end{aligned}$$

which is a contradiction.

Uniqueness. The proof is in several steps.

Step 1. We first argue that all the firms must be producing strictly positive amounts in equilibrium. Suppose to the contrary that firm i has an output level of zero. Consider the aggregate output produced by all the firms charging p^*.[17] Suppose its less than $d(p^*)$.

(i) Let the i-th firm charge p^*. Since $p^* > c'(0)$, the profit of firm i would increase if it produces a sufficiently small amount.

(ii) Next consider the case where the total production by the firms charging p^* is greater than $d(p^*)$. Without loss of generality let these firms be $1, \cdots, m$, where $m < i$, and let $q_1 > 0$. Note that

$$\begin{aligned} \frac{\partial \pi_i}{\partial q_i}|_{q_i=0} &= p^* d(p^*)\gamma_1(0, \sum_{j=1}^{m} q_j) - c'(0) \\ &> p^* d(p^*)\gamma_1(q_1, \sum_{j=2}^{m} q_j) - c'(q_1) \text{ (since } \gamma_{11} - \gamma_{12} < 0), \\ &= \frac{\partial \pi_1}{\partial q_1} = 0. \end{aligned}$$

But this implies that firm i can increase its output slightly and gain.

Step 2. We then argue that there cannot be some p_i $(\in F) > p^*$ such that some firms charge p_i and supply a positive amount.

Suppose to the contrary that such a price exists. This implies that the total number of firms charging p^*, say $\tilde{n}$, can be at most $n^* - 1$. Suppose not, i.e. let the number of firms be n^* or more. Moreover, let the aggregate production by these $\tilde{n}$ firms be less than $d(p^*)$.[18] Clearly, all $\tilde{n}$ firms must be producing q^*. (Since there is excess demand at this price, the residual demand constraint cannot bind, and the output level of all firms must be such that price equals marginal cost.) But this implies that total production

[17] Clearly, all firms charging prices less than p^* would have an output level of zero.

[18] Of course, if the aggregate production by the firms is greater than $d(p^*)$, then the residual demand at any higher price is zero and we are done.

is greater than $d(p^*)$. (This follows from the definition of n^*). But this is a contradiction.

Now consider some $p_i > p^*$. Clearly, the number of firms charging p^* is less than $\hat{n}_i$. Since otherwise some of these firms would have a profit less than $\hat{\pi}$. But they can always ensure a profit of $\hat{\pi}$ by charging p^*. Thus the total number of firms producing a strictly positive amount is less than N_2, thus contradicting step 1. Hence all the firms must be charging p^*.

Step 3. Let $\tilde{q} = (\tilde{q}_1, \cdots, \tilde{q}_n)$, denote the equilibrium output vector. We first establish that the equilibrium output vector must be symmetric. Suppose not, and without loss of generality let $\tilde{q}_2 > \tilde{q}_1 > 0$. Then,

$$\begin{aligned} \frac{\partial \pi_1}{\partial q_1}|_{\tilde{q}} &= p^* d(p^*)\gamma_1(\tilde{q}_1, \sum_{i \neq 1} \tilde{q}_i) - c'(\tilde{q}_1) \\ &> p^* d(p^*)\gamma_1(\tilde{q}_2, \sum_{i \neq 2} \tilde{q}_i) - c'(\tilde{q}_2) \text{ (since } \gamma_{11} - \gamma_{12} < 0) \\ &= \frac{\partial \pi_2}{\partial q_2}|_{\tilde{q}}. \end{aligned} \tag{8}$$

This, however, is a contradiction, since in equilibrium $\frac{\partial \pi_1}{\partial q_1}|_{\tilde{q}} = 0 = \frac{\partial \pi_2}{\partial q_2}|_{\tilde{q}}$.

Step 4. Finally, we argue that there cannot be another symmetric equilibrium where the (common) output level of the firms is different from $q'(n-1)$. Clearly, in any symmetric equilibrium, the production level of all the firms must satisfy Eq. (2). Recall, however, that Eq. (2) has a unique solution. Hence the claim follows. ■

Notice that lemma 2 implies that $\forall n \geq N_1,\ nq'(n-1) > d(p^*)$. Thus this equilibrium involves excess production. This is interesting as this suggests that under certain conditions price competition could lead to inefficiency.[19]

The basic idea behind the existence result is quite simple. We demonstrate that if the number of firms is large enough, then competition will drive

[19] In a different framework Vives [33] examines the efficiency of Bertrand and Cournot equilibria.

all the firms to excess production in an attempt to manipulate the residual demand. This excess production ensures that if any of the firms charge a price greater than p^*, then the residual demand facing this firm will be zero. Thus none of the firms have an incentive to charge a price which is greater than p^*. Undercutting p^* is not profitable anyway. Finally we argue that the quantity decisions are optimal as well.

We then turn to the limit properties of this equilibrium. We need another lemma before we can proceed.

Lemma 4. $\lim_{n\to\infty} q'(n-1) = 0$.

It is easy to see that as the size of the grid becomes small and the number of firms becomes large, the equilibrium price approaches the competitive one and the output level of each firm becomes vanishingly small (from lemma 4).

Recall, however, that the equilibrium involves excess production. The next proposition examines whether in the limit aggregate production, $nq'(n-1)$, approaches the demand level or not.

Proposition 2. *(i) If* $c'(0) = 0$*, then* $\lim_{n\to\infty} nq'(n-1) = \infty$.
(ii) If $c'(0) > 0$*, then* $\lim_{n\to\infty} nq' = d(p^*)\frac{p^*}{c'(0)}$.

Proof. Recall that from lemma 4 it follows that $\lim_{n\to\infty} q'(n-1) = 0$. Hence $\lim_{n\to\infty} nq'(n-1) = \lim_{n\to\infty}(n-1)q'(n-1)$. Moreover, from Eq. (2), assumption 3(ii) and the fact that $q'(n-1)$ is decreasing in n it follows that $(n-1)q'(n-1)$ is increasing in n.[20]

[20] Suppose the number of firms increase from n to $n+1$, so that $q'(n) < q'(n-1)$. Now suppose to the contrary that $(n-1)q'(n-1) \geq nq'(n)$. Then

$$\begin{aligned} p^* d(p^*)\gamma_1(q'(n), nq'(n)) &> p^* d(p^*)\gamma_1(q'(n-1), (n-1)q'(n-1)) \text{ (since } \gamma_{11}, \gamma_{12} < 0) \\ &= c'(q'(n-1)) \text{ (from Eq. (2))} \\ &> c'(q'(n)), \end{aligned}$$

(i) Let $c'(0) = 0$, and suppose to the contrary that $\lim_{n\to\infty}(n-1)q'(n-1) = l$, where l is finite. Then

$$\begin{aligned} & \lim_{n\to\infty} p^* d(p^*)\gamma_1(q'(n-1),(n-1)q'(n-1)) \\ = \; & \frac{p^* d(p^*)}{l} \text{ (from assumption 3(iv))} \\ > \; & 0 = c'(0) = \lim_{n\to\infty} c'(q'(n-1)), \end{aligned}$$

where the last equality follows from lemma 4. But this contradicts Eq. (2).

(ii) Let $c'(0) > 0$ and suppose to the contrary that $\lim_{n\to\infty}(n-1)q'(n-1)$ diverges to infinity. In that case

$$\lim_{n\to\infty} p^* d(p^*)\gamma_1(q'(n-1),(n-1)q'(n-1)) = \lim_{n\to\infty} c'(q'(n-1)),$$

which, from assumption 3(v) and lemma 4, implies that $c'(0) = 0$. But this is a contradiction. Hence let $\lim_{n\to\infty}(n-1)q'(n-1) = L$, where L is finite. We can then simply mimic the earlier argument to show that $L = d(p^*)\frac{p^*}{c'(0)}$.
■

Thus the limiting behavior of the aggregate production level, $nq'(n-1)$, depends on the value of $c'(0)$. If $c'(0) = 0$, then aggregate production increases without bounds.[21] Thus in this case the folk theorem fails to hold. If, however, $c'(0) > 0$, then aggregate production converges to $d(p^*)\frac{p^*}{c'(0)}$. Note that as α goes to zero this term goes to $d(c'(0))$. Thus in this case we can claim that the folk theorem continues to hold.

which contradicts Eq. (2).

[21] As an example consider the case where the demand function is $q = a - p$, the cost function is cq^t, where $t \geq 1$ and $\gamma(q_i, \sum_{j\neq i}^m q_j) = \frac{q_i}{\sum_{j=1}^m q_j}$. It is now straightforward to demonstrate that the total output

$$nq'(n-1) = p^* d(p^*)[(n-1)n^{t-2}]^{\frac{1}{t}}.$$

Clearly, $\lim_{n\to\infty} nq'(n-1) = \infty$.

In the context of capacity constrained price competition, Allen and Hellwig [1] study the limit outcome when the residual demand function is proportional in nature. They find that as the market becomes large the equilibrium prices converge (in distribution) to the competitive price. Note that the limiting procedure adopted in Allen and Hellwig [1] is somewhat different from that in the present paper. Under their approach, not only is the number of firms taken to infinity, but moreover, firm size (i.e. capacity level) is taken to zero. In contrast we keep the cost function of the firms unchanged. Thus we find that in order to achieve the competitive outcome firms are not necessarily required to be small. This is interesting because perfect competition is generally motivated in terms of markets with an infinite number of infinitesimally small firms.

3 Extensions

In this section we examine some extensions of the basic model.

We first examine the case where the cost functions are asymmetric. Let there be m types of firms with the cost function of the i-th type being given by $c_i(q)$. Moreover, let $c_1'(0) < c_2'(0) < \cdots < c_m'(0)$.

We then introduce a series of notations. Define p^{**} as the minimum $p \in F$ such that $p > c_1'(0)$. Next define n_1^*, q_1^*, $q_l'(n^l - 1)$, $\hat{n}_i^l$, N_1^l and N_2^l in a manner similar to that of n^*, q^*, $q'(n-1)$, $\hat{n}_i$, N_1 and N_2 respectively, only taking care to use the cost function of the l-th type, $c_l(q)$, instead of $c(q)$.

We are now in a position to write down the next proposition.

Proposition 3. *Assume that $\alpha < c_2'(0) - c_1'(0)$ and $n^1 \geq \max\{N_1^1, N_2^1\}$. Then the 'unique' equilibrium involves all firms of type 1 charging p^{**}, producing $q_1'(n^1 - 1)$ and selling $\frac{d(p^{**})}{n^1}$. The output level of all other firms is zero.*

Note that the term unique is within quotes since the price charged by firms of type i, where $i \geq 2$, is indeterminate.

Finally we examine the case where the firms play a two stage game where, in stage 1, all the firms simultaneously announce prices, and in stage 2, they simultaneously decide on their output levels. For this case we revert to the assumption of symmetric costs. We solve for the unique "subgame perfect Nash equilibrium" of this game.[22]

Proposition 4. *Consider a two stage game where the firms first announce prices, and then their output levels. Let* $n \geq \max\{N_1+1, N_2\}$. *Then the following strategies constitute a unique "subgame perfect Nash equilibrium":*

Stage 1. *All firms simultaneously announce the price* p^*.

Stage 2. *Suppose that in stage 1 all firms announce* p^*. *Then in stage 2, all firms produce* $q'(n-1)$, *and sell* $\frac{d(p^*)}{n}$.

Next suppose that in stage 1, $(n-1)$ *of the firms announce* p^*, *while the remaining firm charges a higher price. Then, in stage 2, all the firms charging* p^* *produce* $q'(n-2)$ *and sell* $\frac{d(p^*)}{n-1}$. *The other firm has an output of zero.*

4 Conclusion

In this paper we re-examine the Edgeworth paradox. We demonstrate that a unique Nash equilibrium exists whenever the residual demand function is manipulable, prices vary on a grid and there are a large number of firms. Interestingly this equilibrium involves excess production. We find that as the grid size goes to zero, and the number of firms becomes large, the equilibrium price converges to the competitive one. Depending on the value of $c'(0)$, however, aggregate production may, or may not converge to the demand

[22]The term subgame perfect Nash equilibrium is within quotes because we do not solve for the equilibrium strategies for all possible subgames in stage 2. Whether equilibrium strategies exist in every subgame is an open question.

level. Thus whether the folk theorem holds or not depends critically on $c'(0)$. These results continue to hold even when the firms are asymmetric, or produce to order.

5 Appendix

Proof of Lemma 1. Notice that

$$\begin{aligned}\lim_{n\to\infty} p^* d(p^*)\gamma_1(\frac{d(p^*)}{n-1}, d(p^*)) &= p^* d(p^*)\frac{1}{d(p^*)} \ \ \text{(from assumption 3(iv))}\\ &= p^*\\ &> c'(0) = \lim_{n\to\infty} c'(\frac{d(p^*)}{n-1}). \quad \blacksquare\end{aligned}$$

Proof of Lemma 2. Suppose not, i.e. let $q'(n-1) \leq \frac{d(p^*)}{n-1}$. Observe that

$$\begin{aligned}& p^* d(p^*)\gamma_1(q'(n-1), (n-1)q'(n-1))\\ \geq\ & p^* d(p^*)\gamma_1(\frac{d(p^*)}{n-1}, d(p^*)) \ \ \text{(since } \gamma_1(x, nx) \text{ is decreasing in } x\text{)}\\ >\ & c'(\frac{d(p^*)}{n-1}) \ \ \text{(since } n \geq N_1\text{)}\\ \geq\ & c'(q'(n-1)) \ \ \text{(since } q'(n-1) \leq \frac{d(p^*)}{n-1}\text{)}.\end{aligned}$$

This, however, violates Eq. (2). $\blacksquare$

Proof of Lemma 3. Let the number of firms charging $\overline{p}_i$ be k, where $k \geq \hat{n}_i$. First consider the case where none of the other firms charge prices that are less than $\overline{p}_i$. Clearly, if all the firms charging $\overline{p}_i$ produce identical amounts then the maximum profit of all such firms is $\frac{\overline{p}_i d(\overline{p}_i)}{k} - c(\frac{d(\overline{p}_i)}{k})$. Since $k \geq \hat{n}_i$, this is less than $\hat{\pi}$.

Now consider the case where the output level of the firms charging $\overline{p}_i$ are not the same. Clearly, if the aggregate production by all such firms are less than equal to $d(\overline{p}_i)$, then some of the firms would be producing and selling less than $\frac{d(\overline{p}_i)}{k}$, and consequently would have a profit less than $\frac{\overline{p}_i d(\overline{p}_i)}{k} - c(\frac{d(\overline{p}_i)}{k}) < \hat{\pi}$. Whereas, if the aggregate production of such firms is greater than $d(\overline{p}_i)$, then some firms would sell less than $\frac{d(\overline{p}_i)}{k}$, while their production would be larger. Again their profit would be less than $\frac{\overline{p}_i d(\overline{p}_i)}{k} - c(\frac{d(\overline{p}_i)}{k})$.

Finally, if some of the other firms charge less than $\overline{p}_i$, then the residual demand at $\overline{p}_i$ would be even less than $d(\overline{p}_i)$. We can now mimic the earlier argument to claim that some of the firms charging $\overline{p}_i$ would have a profit less than $\frac{\overline{p}_i d(\overline{p}_i)}{k} - c(\frac{d(\overline{p}_i)}{k})$. ■

Proof of Lemma 4. Suppose to the contrary that $\lim_{n\to\infty} q'(n-1) = D$, where $D > 0$. Then

$$\begin{aligned}\lim_{n\to\infty} p^* d(p^*)\gamma_1(q'(n-1),(n-1)q'(n-1)) &= 0 \text{ (from assumption 3(vi))} \\ &< c'(D) = \lim_{n\to\infty} c'(q'(n-1)).\end{aligned}$$

This, however, violates Eq. (2). ■

Proof of Proposition 3. The idea of the proof is very similar to that in Proposition 1.

Existence. Notice that since $\alpha < c_2'(0) - c_1'(0)$, it follows that $p^{**} < c_i'(0)$, for all $i \geq 2$. Thus no firm of type j, where $j \geq 2$ can profitably charge a price of p^{**}. For type 1 firms we can simply mimic the proof in Proposition 1 to claim that they cannot have a profitable deviation.

Uniqueness. The proof is in several steps.

Step 1. We first argue that all the firms of type 1 must be producing strictly positive amounts in equilibrium. Suppose to the contrary that firm i (of type 1) has an output level of zero. Consider the aggregate output produced by all the firms charging p^{**}.

(i) Suppose its less than $d(p^{**})$. Let the i-th firm charge p^{**}. Since $p^{**} > c'(0)$, for a sufficiently small output level, the profit of firm i would increase.

(ii) Next consider the case where the total production by the firms charging p^{**} is greater than $d(p^{**})$. Without loss of generality let these firms be

$1, \cdots, m$, where $m < i$, and let $q_1 > 0$. Note that

$$\begin{aligned}
\frac{\partial \pi_i}{\partial q_i}|_{q_i=0} &= p^{**} d(p^{**}) \gamma_1(0, \sum_{j=1}^{m} q_j) - c'(0) \\
&> p^{**} d(p^{**}) \gamma_1(q_1, \sum_{j=2}^{m} q_j) - c'(q_1) \text{ (since } \gamma_{11} - \gamma_{12} < 0), \\
&= \frac{\partial \pi_1}{\partial q_1} = 0.
\end{aligned}$$

But this implies that firm i can increase its output slightly and gain.

Step 2. We then argue that there cannot be some p_i ($\in F$) $> p^{**}$ such that some firms of type 1 charge p_i and supply a positive amount.

Suppose to the contrary that such a price exists. This implies that the total number of type 1 firms charging p^{**}, say $\tilde{n}$, can be at most $n_1^* - 1$. Suppose not, i.e. let the number of such type 1 firms be n_1^* or more. In that case, if the aggregate production by these $\tilde{n}$ firms is less than $d(p^{**})$, then all $\tilde{n}$ firms must be producing q_1^*. But this implies that total production is greater than $d(p^{**})$. (This follows from the definition of n_1^*). But this is a contradiction.

Now consider some $p_i > p^{**}$. Clearly, the number of type 1 firms charging p^{**} is less than $\hat{n}_i^1$. Thus the total number of type 1 firms producing a strictly positive amount is less than N_2^1, thus contradicting step 1. Hence all firms of type 1 must be charging p^{**}.

Step 3. Let $\tilde{q}$, denote the equilibrium output vector of type 1 firms. We first establish that this vector must be symmetric. Suppose not, and without loss of generality let $\tilde{q}_2 > \tilde{q}_1 > 0$, where both the firms are of type 1. Then,

$$\begin{aligned}
\frac{\partial \pi_1}{\partial q_1}|_{\tilde{q}} &= p^{**} d(p^{**}) \gamma_1(\tilde{q}_1, \sum_{i \neq 1} \tilde{q}_i) - c'(\tilde{q}_1) \\
&> p^{**} d(p^{**}) \gamma_1(\tilde{q}_2, \sum_{i \neq 2} \tilde{q}_i) - c'(\tilde{q}_2) \text{ (since } \gamma_{11} - \gamma_{12} < 0)
\end{aligned}$$

$$= \frac{\partial \pi_2}{\partial q_2}|_{\tilde{q}}. \tag{9}$$

This, however, is a contradiction, since in equilibrium $\frac{\partial \pi_1}{\partial q_1}|_{\tilde{q}} = 0 = \frac{\partial \pi_2}{\partial q_2}|_{\tilde{q}}$.

Step 4. Finally, we argue that there cannot be another symmetric equilibrium where the (common) output level of the firms is different from $q_1'(n^1 - 1)$. Clearly, in any symmetric equilibrium, the production level of all the firms must satisfy

$$p^{**} d(p^{**}) \gamma_1(q, (n^1 - 1)q) = c_1'(q).$$

It is easy to see that this equation has a unique solution. The argument is similar to that for the uniqueness of $q'(n-1)$.

Finally, since type 1 firms exhaust the demand at p^{**}, the output level of all firms of other types must be zero.■

Proof of Proposition 4.

Existence.

Stage 2. Notice that $n > n-1 \geq N_1$. Hence $q'(n-1)$ satisfies Eq. (2), and $q'(n-2)$ satisfies

$$p^* d(p^*) \gamma_1(q, (n-2)q) = c'(q).$$

Hence we can mimic the argument in Proposition 1 to argue that the output decisions are optimal.

Stage 1. It is easy to see that since $n-1 \geq N_1$, $(n-2)q'(n-2) > d(p^*)$. Hence $(n-1)q'(n-2) > d(p^*)$. Thus the total production by other firms in stage 2, i.e. $(n-1)q'(n-2)$, is greater than $d(p^*)$. Hence any firm that charges a price greater than p^* would have a residual demand of zero.

Uniqueness. The proof is in several steps.

Step 1. We first argue that all the firms must be producing strictly positive amounts in equilibrium. Suppose to the contrary that firm i has an output level of zero. We argue that this firm can charge p^*, supply a positive amount and gain. Suppose that firm i charges p^*.

(i) Following such a strategy, let the aggregate output produced by all other firms charging p^* be less than $d(p^*)$. Since $p^* > c'(0)$, for a sufficiently small output level, the profit of firm i would increase.

(ii) Next consider the case where the total production by the other firms charging p^* is greater than $d(p^*)$. Without loss of generality let these firms be $1, \cdots, m$, where $m < i$, and let $q_1 > 0$. Suppose to the contrary it is optimal for firm i to produce nothing. Note that

$$\begin{aligned}
\frac{\partial \pi_i}{\partial q_i}|_{q_i=0} &= p^* d(p^*)\gamma_1(0, \sum_{j=1}^{m} q_j) - c'(0) \\
&> p^* d(p^*)\gamma_1(q_1, \sum_{j=2}^{m} q_j) - c'(q_1) \text{ (since } \gamma_{11} - \gamma_{12} < 0), \\
&= \frac{\partial \pi_1}{\partial q_1} = 0.
\end{aligned}$$

But this implies that firm i can increase its output slightly and gain.

We can now simply mimic steps 2, 3 and 4 in the uniqueness part of Proposition 1 to establish uniqueness. ■

References

1. B. Allen and M. Hellwig, Bertrand-Edgeworth oligopoly in large markets, *Rev. Econ. Stud.* 53 (1986), 175-204.

2. B. Allen and M. Hellwig, Bertrand-Edgeworth duopoly with proportional residual demand, *Int. Econ. Rev.* 34 (1993), 39-60.

3. J.P. Benassy, On the role of market structure in imperfect competition, *Rev. Econ. Stud.* 56 (1989), 217-234.

4. T. Borgers, Iterated elimination of dominated strategies in Bertrand-Edgeworth games, *Rev. Econ. Stud.* 59 (1992), 163-176.

5. E.H. Chamberlin, " The theory of monopolistic competition," Harvard University Press, Cambridge MA, 1933.

6. P. Dasgupta and E. Maskin, The existence of equilibrium in discontinuous economic games, I: Theory, *Rev. Econ. Stud.* 53 (1986), 1-26.

7. P. Dasgupta and E. Maskin, The existence of equilibrium in discontinuous economic games, II: Applications, *Rev. Econ. Stud.* 53 (1986), 27-41.

8. K.G. Dastidar, On the existence of pure strategy Bertrand equilibrium, *Econ. Theory* 5 (1995), 19-32.

9. C. Davidson and R. Deneckere, Long run competition in capacity, short run competition in price, and the Cournot model, *RAND J. Econ.* 17 (1986), 404-415.

10. H. Dixon, The existence of mixed strategy equilibria in a price-setting oligopoly with convex costs, *Econ. Letters* 16 (1984), 205-212.

11. H. Dixon, Approximate Bertrand equilibria in a replicated industry, *Rev. Econ. Stud.* 54 (1987), 47-62.

12. H. Dixon, Bertrand-Edgeworth equilibria when firms avoid turning customers away, *J. Ind. Econ.* 39 (1990), 131-146.

13. H. Dixon, Integer pricing and Bertrand-Edgeworth oligopoly with

strictly convex costs: Is it worth more than a penny? *Bull. Econ. Res.* 45 (1993), 257-268.

14. P. Dubey, Price-quantity strategic market games, *Econometrica* 50 (1982), 111-126.

15. F. Edgeworth, La teoria pura del monopolio, *Giorn. Econ.* 40 (1897), 13-31.

16. J.W. Friedman, On the strategic importance of prices versus quantities, *RAND J. Econ.* 19 (1988), 607-622.

17. S.J. Grossman, Nash equilibrium and the industrial organization of markets with large fixed costs, *Econometrica* 49 (1981), 1149-1172.

18. J. Harrington, A reevaluation of perfect competition as the solution to the Bertrand price game, *Math. Soc. Sci.* 17 (1989), 315-328.

19. D.M. Kreps and J.A. Scheinkman, Quantity precommitment and Bertrand competition yield Cournot outcomes, *Bell J. Econ.* 14 (1983), 326-338.

20. R. Levitan and M. Shubik, Price duopoly and capacity constraints, *Int. Econ. Rev.* 13 (1972), 111-122.

21. D.V. Mandy, Nonuniform Bertrand competition, *Econometrica* 60 (1992), 1293-1330.

22. E. Maskin, The existence of equilibrium with price setting firms, *Amer. Econ. Rev.* 76 (1986), 382-386.

23. E. Maskin and J. Tirole, The theory of dynamic oligopoly II: Price competition, kinked demand curves and Edgeworth cycles, *Econometrica* 56 (1988), 571-600.

24. W. Novshek, Cournot equilibrium with free entry, *Rev. Econ. Stud.* 67 (1980), 473-486.

25. W. Novshek and H. Sonnenschein, Walrasian equilibria as limit-points of non-cooperative equilibria, part II: Pure strategies, *J. Econ. Theory* 30 (1983), 171-187.

26. M.J. Osborne and C. Pitchik, Price competition in a capacity con-

strained duopoly, *J. Econ. Theory* 33 (1986), 238-260.

27. P. Ray Chaudhuri, The contestable outcome as a Bertrand equilibrium, *Econ. Letters* 50 (1996), 237-242.

28. P. Roy Chowdhury, Limit-pricing as Bertrand equilibrium, forthcoming *Econ. Theory* (2001).

29. M. Shubik, A comparison of treatments of the duopoly problem: Part II, *Econometrica* 23 (1955), 417-431.

30. M. Shubik, "Strategy and market structure," Wiley, New York 1959.

31. L.K. Simon, Bertrand, the Cournot paradigm and the theory of perfect competition, *Rev. Econ. Stud.* 51 (1984), 209-230.

32. L.K. Simon, Bertrand price competition with differentiated commodities, *J. Econ. Theory* 41 (1987), 304-332.

33. X. Vives, On the efficiency of Bertrand and Cournot equilibria with product differentiation, *J. Econ. Theory* 31 (1984), 166-175.

34. X. Vives, Rationing rules and Bertrand-Edgeworth equilibria in large markets, *Econ. Letters* 18 (1986), 140-153.

35. X. Vives, "Oligopoly pricing: Old ideas and new tools," MIT Press, Cambridge and London 1999.