The Dynamic Instability of Dispersed Price Equilibria.*

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Abstract

We examine whether price dispersion is an equilibrium phenomenon or a cyclical phenomenon. We develop a finite strategy model of price dispersion based on the infinite strategy model of Burdett and Judd (1983). Adopting an evolutionary standpoint, we examine the stability of dispersed price equilibrium under perturbed best response dynamics. We conclude that when both sellers and consumers participate actively in the market, all dispersed price equilibria are unstable leading us to interpret price dispersion as a cyclical process. For a particular case of the model, we prove the existence of a limit cycle.

1 Introduction

Price dispersion, under which different sellers charge different prices for the same homogeneous good is a commonly observed phenomenon. For example, Baylis and Perloff (2002) and Baye and Morgan (2004) have documented price dispersion among internet firms. Similarly, Lach (2002) provides evidence of price dispersion in Israeli product markets. Some recent experimental work by Cason and Friedman (2003), Cason, Friedman, and Wagener (2005), and Morgan, Orzen and Sefton (2006) have also verified the existence of price dispersion. Price dispersion is very puzzling because it seemingly contradicts the "law of one price" of elementary microeconomics. Various models explain price dispersion as an equilibrium phenomenon.[1] The common feature of these models is the presence of heterogeneity among consumers, whether in the number of prices consumers sample before purchasing (Burdett and Judd (1983)), or in search cost (Salop and Stiglitz (1977), Stahl (1989)). Such heterogeneity implies that there are always some consumers who are willing to pay

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[1]Some of these models are Salop and Stiglitz (1977), Varian (1980), Burdett and Judd (1983), Rob (1985), Stahl (1989), Wilde (1992), Benabou (1993).

a price that is not necessarily the lowest prevailing. This prevents the emergence of Bertrand like competition: instead of undercutting each other in attempts to attract more consumers, sellers can earn more by sticking to a higher price and selling to the fraction of consumers who would be willing to buy at that price. We call the resulting mixed equilibrium a dispersed price equilibrium.

While equilibrium analysis is the received approach to solving economic models, it is reasonable to ask whether observed price dispersion might not be an equilibrium phenomenon, but is the manifestation of some disequilibrium occurrence like a price cycle. In contrast to a dispersed equilibria where the proportion of firms charging a particular price is constant over time, in a price cycle, there will be regular fluctuations in that proportion; and the resulting fluctuation in the average market price that is called an Edgeworth cycle.[2] Eckert (2003) and Noel (2003) provide evidence of such Edgeworth cycles in the prices set by firms. Similarly, the experimental data in Cason and Friedman (2003), Cason, Friedman, and Wagener (2005) also suggest the presence of cycles. Lach (2002) also detects patterns of cyclical behavior similar to that in Cason, Friedman, and Wagener (2005).

In this paper, we examine price dispersion from the standpoint of evolutionary game theory.[3] We analyze a discrete analogue of the Burdett and Judd (1983) model and conclude that all dispersed price equilibria of the model are dynamically unstable under the class of *perturbed best response dynamics*. Through simulations, we verify that these dynamics lead naturally to the emergence of disequilibrium behavior in the form of cycles. Our model, therefore, not only rules out equilibrium price dispersion as a robust prediction, but also provides insight into the nature of observed price dispersion. For the general case, rigorously establishing the presence of such cycles is a mathematically intractable problem. However, for a simpler case, we are able to show that such cycles do exist.

Evolutionary game theory seems ideal to analyze market situations like price dispersion. Such perpetual disequilibrium behavior is captured naturally by evolutionary game theory. Moreover, the assumption of myopic agents that underpin evolutionary models is not misplaced in this problem. The number of consumers and sellers in a market are large, and each market participant makes many buying or selling decisions. The impact of any single decision on utility obtained by an agent will be very small, particularly if the items involved are items of daily consumption like sugar or coffee as in the study by Lach (2002). Hence, agents are unlikely to expend a substantial amount of reasoning resources in making these decisions.

From a methodological point of view, this paper also addresses the criticism that evolutionary game theory, despite the potentially rich set of predictions it can offer, has found little application in addressing substantive economic problems.[4] It opens up a broad area of economic interest—

[2]Edgeworth (1925) was the first to theorize on the presence of price cycles. He argued that in the presence of capacity constraints, the Bertrand prediction of prices being driven down to the marginal cost level would not materialize. Instead, sellers would myopically reduce prices by small amounts when there is excess capacity but jump to higher prices when capacity constraints are binding.

[3]Hopkins and Seymour (2002) were the first to introduce evolutionary ideas to the study of price dispersion. We discuss their work at the end of this section.

[4]Some papers related to application of evolutionary game theory are on externality pricing and macroeconomic

situations of persistent disequilibrium—where evolutionary game theory can be fruitfully applied.

We first construct a discrete analogue of the continuous strategy space model of Burdett and Judd (1983).[5] We have a population of sellers with a strategy set consisting of a finite number of prices. Consumers observe a certain number of prices after paying a search cost for every price they choose to sample and buy at the cheapest price observed. This model yields price dispersion as an equilibrium phenomenon, in which sellers charge different prices and consumers are differentiated by whether they observe only one price or two prices.

We then turn to the evolutionary analysis of our static model. The focus of the evolutionary analysis is the stability properties of the dispersed price equilibria. We conduct our evolutionary analysis using the class of *perturbed best response* (PBR) dynamics. These dynamics are so called because they are generated by slightly perturbing the payoffs of agents and then allowing agents to optimize against the prevailing social state.[6] The shocks ensure that even as players play nearly optimally with respect to the unperturbed payoffs, they chose mixed strategies that vary continuously with respect to the social state. These features make perturbed best response more consistent with myopic decision making than pure best response since the choice of strategy does not change abruptly when the social state changes. Technically, these shocks ensure that the resulting dynamic is smooth, and hence, can be analyzed using linearization techniques.[7]

In Section 5, we establish the main technical result of this paper—that dispersed price equilibria are unstable under PBR dynamics (Proposition 5.8). This result leads naturally to the emergence of price cycles. We provide a numerical simulation of such a cycle in Section 4. Since it is mathematically intractable to rigorously establish the presence of these cycles, we focus on a simple case in Section 6 where such a proof is possible. In this case, consumers are exogenously programmed to sample only one or two prices. Under the additional assumption of *monocyclicity* (Hofbauer, 1995), we are able to rigorously establish the presence of price cycles.

This paper is related to the earlier work of Hopkins and Seymour (2002). These authors perform an evolutionary analysis of price dispersion under the class of positive definite adaptive (PDA) dynamics[8] (Hopkins 1999). Their conclusion is that dispersed equilibria are unstable under these dynamics. The conclusions in that paper, however, have certain ambiguities mainly due to the fact that their finite dimensional analysis is employed in the context of a game with infinite strategy sets. In particular, their definitions of payoff functions and their dynamic analysis are not

spillovers (Sandholm (2002, 2005)), natural selection and animal behavior (Maynard Smith (1982), Hofbauer and Sigmund (1988)), congestion in highway and computer networks (Monderer and Shapley (1996), Sandholm (2001)), emergence of residential segregation (Dokumaci and Sandholm (2007)).

[5]We focus on the discrete case to avoid the technical complications involved with the evolutionary analysis of continuous strategy games.

[6]The introduction of shocks implies that the rest points of the perturbed best response dynamics are not Nash equilibria of the game. Instead, rest points coincide with the set of *perturbed equilibria*. Our stability analysis will therefore relate to the stability of perturbed equilibria. For low levels of shocks, however, perturbed equilibria lie very close to Nash equilibria. So, if perturbed equilibria are unstable, we can conclude that society moves away from Nash equilibria as well.

[7]The most well known perturbed best response dynamic is the logit dynamic (Fudenberg and Levine, 1998).

[8]This class of dynamics describes the behavior of agents who imitate successful opponents, and it includes the replicator dynamic as its prototype.

entirely satisfying from a technical point of view.

Our analysis differs from that of Hopkins and Seymour significantly. First, since our dynamic model is explicitly finite dimensional, it is free from the technical ambiguities present in Hopkins and Seymour (2002).[9] Secondly, we go further than Hopkins and Seymour in considering not only instability of equilibrium, but also the presence of cycles. Moreover, the two classes of dynamics–PDA dynamics and perturbed best response dynamics–are distinct. Nevertheless, the general approach of Hopkins and Seymour has influenced us a great deal, particularly in view of results in Hopkins (1999) that show how concepts derived from PDA dynamics can be used to analyze perturbed best response dynamics.

2 Finite Approximation of the Burdett and Judd Model

The Burdett and Judd (1983) price dispersion model is a game with a continuous strategy space. There are a continuum of homogeneous firms, all selling the same good at a price belonging to the set $[0, 1]$. We interpret 0 as the cost for the sellers and 1 as the common reservation price of consumers which is known to the sellers. Each firm chooses a price independently. Consumers observe prices set by a certain number of different firms and then buy one unit of the commodity at the lowest observed price provided that price do not exceed 1. If more than one observed firm is charging the minimum price, the consumer randomizes uniformly between them.

To avoid the technical difficulties associated with the evolutionary analysis of a continuous strategy game, we develop a finite analogue of the original Burdett and Judd model. We construct a sequence of finite approximation $\{S^n\}_{n\in\mathbf{Z}_{++}}$ of the strategy set $S = [0, 1]$. The set S^n consists of $(n + 1)$ prices $\left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\right\}$. We denote $p_i = \frac{i}{n}$. Given the probability distribution x, the notation x_i refers to the probability of strategy p_i. If we need to emphasize the size of the strategy set, we use the notations p_i^n and x_i^n.

For consumers, a strategy is to sample a certain number of prices before deciding to purchase at the cheapest price sampled. Hence, consumer behavior can be summarized by the distribution $(y_1, y_2, \cdots y_r)$, with y_m being the proportion of consumers who are sampling m prices. Here, r is a finite number that represents the maximum number of prices any consumer samples.[10] Our main focus is on the general case where the distribution y emerges endogenously. However, in establishing our general result, we will need to examine the case when the distribution y is specified exogenously as means to establish our conclusions about the more general case.

We denote the population of sellers as population 1 and that of consumers as population 2. We also assume that each population is of mass 1, which allows us to identify a population state with

[9] In particular, since we avoid any admixture of finite and infinite dimensional issues, our payoff functions have clearly identifiable Nash equilibria. So the evolutionary dynamics that we use have well defined rest points which make our stability analysis meaningful.

[10] In the original model, there is no such upper limit. In our case, it is necessary to impose this upper limit in order to define evolutionary dynamics. The imposition of this limit, however, makes no significant difference to equilibrium behavior. In any mixed equilibrium in the model with endogenous consumer behavior, consumers sample either one or two prices.

a point in the appropriate simplex.

The set $\Delta_1^n = \{x \in \mathbf{R}_+^{n+1} : \sum_{i=0}^{n} x_i = 1\}$ is the set of states in population 1. The set of states in population 2 is $\Delta_2 = \{y \in \mathbf{R}_+^r : \sum_{m=0}^{r} y_m = 1\}$. The set of social states is thus $\Delta^n = \Delta_1^n \times \Delta_2$. A social state is $(x, y) \in \Delta^n$. Given the social state (x, y), (x_i, y_m) is to be interpreted respectively as the proportion of sellers who are charging price p_i and the proportion of consumers who are sampling m prices.

We now specify the payoff functions of our model. First, we consider the sellers. Let us fix the strategy set S^n of sellers and the distribution y of consumer types. The payoff that a firm receives by charging a price $p_i \in S^n$ depends upon p_i, the distribution y, and the distribution x of prices chosen by the other firms.

Given p_i, x, y and r, the payoff received by a seller is a function $\pi_i : \Delta^n \to \mathbf{R}$ defined by[11]

$$\pi_i(x) = p_i \left[y_1 + \sum_{m=2}^{r} m y_m \left\{ \sum_{k=0}^{m-1} \frac{g_{(k,i)}^m(x)}{k+1} \right\} \right] \tag{1}$$

where

$$g_{(k,i)}^m(x) = \binom{m-1}{k}(x_i)^k \left(\sum_{j>i} x_j\right)^{m-1-k}. \tag{2}$$

The expected mass of consumers who sample the firm is $\sum_{m=1}^{r} m q_m$.[12] If a consumer samples m firms including the firm in question, $g_{(k)}^m(x)$ is the probability that the price p_i chosen by the firm is the minimum of the m prices and is also chosen by k other firms. Hence, the probability that the consumer buys from the firm is $\sum_{k=0}^{m-1} \frac{g_{(k)}^m(x)}{k+1}$. Uniform randomization by consumers accounts for division by $k + 1$.

Example 2.1 *If the distribution of consumer types is* $\{y_1, y_2\}$, *then*

$$\pi_i(x) = p_i(y_1 + 2y_2(\sum_{j>i} x_j + \frac{x_i}{2})).$$

If the distribution is $\{y_1, y_2, y_3\}$, *then*

$$\pi_i(x) = p_i(y_1 + 2y_2(\sum_{j>i} x_j + \frac{x_i}{2}) + 3y_3((\sum_{j>i} x_j)^2 + x_i \sum_{j>i} x_j + \frac{x_i^2}{3})).$$

We now consider the payoff function of consumers. A strategy for a consumer is now the number of prices that is to be sampled before purchasing. We assume that consumers have to pay a cost

[11] The payoff function differs from the one in Burdett and Judd (1983) in that we have to account for the possibility of sellers choosing equal prices. Burdett and Judd ignore this possibility since in their setting, all mixed equilibrium are absolutely continuous probability measures.

[12] The expected number of m price samplers who sample a particular firm is $m y_m$. Hence, the expected measure of consumers who will sample a firm is $y_1 + \sum_{m=2}^{r} m y_m$.

$c > 0$ for every price they choose to sample. The parameters c and r are assumed to be common to all consumers. Consumers are therefore a priori homogeneous. If each price quotation is a random draw from the probability distribution p, then the expected cost of purchasing when m prices are observed is given by the function $C_m : \Delta_1^n \to \mathbf{R}$ defined by,

$$C_m(x) = mc + m\sum_{i=0}^{n} p_i x_i \left\{ \sum_{k=0}^{m-1} \frac{g_{(k,i)}^m(x)}{k+1} \right\} \tag{3}$$

with $g_{(k,i)}^m(x)$ defined in (2). The interpretation of the cost function is as follows. Suppose a consumer is randomly sampling m prices. If one of the prices he observes is p_i, then $g_{(k)}^m(x)$ is the probability that p_i is the minimum of the m prices and that the consumer has observed k other equal prices. Uniform randomization leads to division by $(k+1)$. We multiply by m since p_i can be observed in any of the m draws. Hence, $mx_i \sum_{k=0}^{m-1} \frac{g_{(k,i)}^m(x)}{k+1}$ represents the probability of paying p_i.

Consumers' payoff is the negative of (3). It is important to note that the cost function is independent of consumers' aggregate behavior given by the distribution y. This fact will have important consequences for the stability properties of mixed equilibria.

2.1 Equilibria with Fixed Consumer Types

In deriving the Nash equilibria of our model, we follow the general strategy in Burdett and Judd (1983). We first derive equilibria by fixing the distribution $\{y_i\}_{i=1}^r$ of consumer types. The results derived in this case then allows us to solve the model in the more general case when consumer search behavior is endogenous.

With y fixed, the payoff function of the game is given by (1). First, let us consider the case $0 < y_1 < 1$. This case is important for the emergence of dispersed Nash equilibria. It is the presence of some uninformed consumers that prevents prices from falling to the competitive level. On the other hand, if all consumers are uninformed, then 1 is a dominant strategy. Hence, for dispersed price equilibria to emerge, this condition must be satisfied.[13]

We now show that for n sufficiently large, all equilibria are mixed equilibria. This follows from the following lemma, which shows that as n gets large, the probability attached by any Nash equilibrium on any single price must go to zero.

Lemma 2.2 *Let the type distribution $\{y_1, y_2, \cdots y_r\}$ satisfy $0 < y_1 < 1$. Let $\overline{x}^n$ be a Nash equilibrium of the game with strategy set S^n. Then, for all strategies p_i^n, $\overline{x}_i^n \to 0$ as $n \to \infty$.*

Proof. In the Appendix. ■

[13] If $y_1 \in (0, 1]$, then when n is large, there are positive prices that are dominated. In fact, any price that is less than $\underline{p} = y_1(\sum_{m=1}^{r} my_m)^{-1}$ is dominated by price 1. The lowest possible payoff obtained from charging 1 is y_1 whereas the highest possible payoff from any price p_i is $p_i \sum_{i=1}^{r} my_m$. This gives us $\underline{p}$. For positive dominated strategies, we require $\underline{p} \geq \frac{1}{n}$ which implies $n \geq (\sum_{m=1}^{r} my_m)y_1^{-1}$.

The intuition behind this result is as follows. As the number of prices increase, the difference between any two successive prices goes to zero. Hence, if the weight on any price remains bounded away from zero, any seller charging that price can deviate to the price immediately below that. While the two prices are nearly the same, the probability of being the minimum price sampled increases significantly. In the appendix, we show that this intuition works for all prices except the first two positive prices. But for n large, these prices are dominated by 1 and so can be ignored.

We therefore conclude that for n sufficiently large, the only Nash equilibria are mixed strategy Nash equilibria if $0 < y_1 < 1$.[14]

We now consider the two special cases $y_1 = 0$ and $y_1 = 1$. The only Nash equilibria in these cases are pure strategy equilibria.

Lemma 2.3 *1. Let $y_1 = 1$. Then, for all n, the only Nash equilibrium is $x_n^n = 1$, i.e. all firms charge the highest price 1.*

2. Let $y_1 = 0$. Then, for any n, there are always two pure strategy Nash equilibria. One Nash equilibrium is $x_0^n = 1$, i.e., all firms charge price 0. Another Nash equilibrium is $x_1^n = 1$. In the particular case where $y_2 = 1$, $x_2^n = 1$ is also a Nash equilibrium. Moreover, for all n, there exist no other Nash equilibria.

Proof. In the Appendix. ■

Part (1) follows because 1 is then the dominant price. The proof of part (2) is somewhat tedious but the intuition is largely that of Bertrand competition. For the special case where $y_2 = 1$, $x_2^n = 1$ is a non strict Nash equilibrium.[15]

2.2 Equilibrium with Endogenous Consumer Behavior

We can now characterize the equilibria of the complete model in which consumer behavior emerges endogenously. First, we present the following lemma that has important implications for the characterization of Nash equilibria. In this lemma, we show that the cost function defined in (3) is convex in the number of prices a consumer chooses to observe, provided that the strategy distribution in the population of sellers is mixed.

Lemma 2.4 *Let the population 1 state x be mixed. Let F be the distribution function of x. Hence, $F_i = \sum_{j \leq i} x_j$. Then, the cost function $C_m(x)$ is strictly convex in m.*

[14] Let $\lceil a \rceil$ the smallest integer strictly larger than a. If $n > \frac{\lceil L \rceil}{\underline{p}}$ where $L = 1 + y_1 (\sum_{m=2}^{r} (m-1) \, y_m)^{-1}$, then there is no pure strategy equilibrium in the game with strategy set S^n

[15] Lemmas 2.2 and 2.3 are very analogous to the corresponding result in the (Lemma 2) in the original Burdett and Judd (1983) model. They find that if $y_1 = 1$, the only equilibrium is the monopoly equilibrium whereas if $y_1 = 0$, the only equilibrium is the competitive equilibrium. If $0 < y_1 < 1$, the unique equilibrium is an absolutely continuous probability measure with compact and connected support. One significant difference is that in the finite case, there may be more than one mixed strategy equilibrium.

Proof. It can be shown through some tedious manipulation that

$$C_m(x) = mc + \frac{1}{n}\sum_{i=0}^{n}(1 - F_i)^m \tag{4}$$

For any number $b \in (0, 1)$, $(1 - b)^m$ is strictly convex in m. Hence, as long as the distribution x is not a pure strategy, $C_m(x)$ will be strictly convex in m. ■

A feature of (4) is that the price term p_i does not appear in it. This is because prices are uniformly spaced due to which their effect is incorporated in the $\frac{1}{n}$ term. The convexity of the cost function implies that it is minimized at either a unique integer m^* or two successive integers m^* and $m^* + 1$.

Let us fix the strategy size n. We now establish certain facts about Nash equilibrium when consumer behavior is endogenous. These results are analogous to the infinite dimensional case. We first show that monopoly pricing is always an equilibrium irrespective of the number of prices available. Apart from this, there will exist no other pure strategy equilibrium for any n. Next, we argue that at any mixed equilibrium, consumers will sample either only one price or two prices.

Theorem 2.5 *In the game with endogenous consumer behavior.*

1. *$\{x_n^n = 1, y_1 = 1\}$ is always a Nash equilibrium. This is the monopoly equilibrium.*

2. *For all n, the only other Nash equilibria are mixed equilibria in which both producers and consumers randomize.*

3. *In any mixed equilibrium, $0 < y_1 < 1$ and $y_1 + y_2 = 1$. Consumers sample at most two prices.*

Proof. We prove each statement in turn.

1. If all sellers are charging the highest price, then the cost minimizing strategy for consumers is to sample just one price. On the other hand, if all consumers are searching just once, then sellers' profits are maximized by charging the highest price.

2. We first rule out equilibria in which $y_i = 1$ for $i > 1$. If $y_i = 1$ for $i > 1$, then by Lemma 2.3, the only possible equilibria are pure equilibria where firms charge either the zero price, or p_1^n or p_2^n. Since all firms charge the same price, the cost minimizing strategy for consumers is uniquely $y_1 = 1$. But then sellers will charge the highest price and we are back to the monopoly equilibrium. Next, suppose consumers randomize but all producers charge a single price. Then, all consumers will deviate to sampling just one price.

3. At any mixed equilibrium, we must have $0 < y_1 < 1$. If $y_1 = 0$, then by lemma 2.3, all sellers charge the same price. But then, all consumers sample just once. If $y_1 = 1$, then all firms charge the highest price and we have the monopoly equilibrium. Hence $0 < y_1 < 1$ which implies that sampling one price is one of the cost minimizing strategies of the consumers.

Since sellers play a mixed strategy at the Nash equilibrium, the cost function is strictly convex. This implies that the only other cost minimizing strategy is to sample two prices. Thus, $y_1 + y_2 = 1$.

This completes the proof. ■

Example 2.6 *Consider the game with* $n = 5$, $r = 3$, *and* $c = 0.07$. *Hence the strategy set of sellers is* $S = \left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}$. *Consumers observe a maximum of three prices. The monopoly equilibrium is a pure equilibrium. Apart from this, there are nine mixed equilibria, all having the characteristic* $0 < y_1 < 1$ *and* $y_1 + y_2 = 1$. *We list the mixed equilibria in the appendix. One particular mixed equilibrium we will focus on to illustrate our results on instability is* $x^* = (0, 0, 0, 0.4684, 0.4176, 0.1140)$ *and* $y^* = (0.6680, 0.3319, 0)$.

Theorem 2.5 have exact counterparts in the infinite case (see Theorem 2 in Burdett and Judd (1983)). The only pure equilibrium in the infinite game is the monopoly equilibrium. Any mixed equilibria is characterized by $0 < y_1 < 1$ and $y_1 + y_2 = 1$. For the infinite dimensional case, it is possible to go further and show that there may be zero, one or two mixed equilibria, depending on c. In the finite case, we don't have such an exact result on the number of equilibria that can exist.

3 Perturbed Best Response Dynamics

We now consider the dynamic analysis of our price dispersion model. We model dynamic behavior in the two populations by using perturbed best response (PBR) dynamics.[16] Our objective is to analyze the stability properties of dispersed price equilibria under perturbed best response dynamics. If we are able to show that all such equilibria are unstable, then we need to conclude that observed price dispersion is the result of persistent disequilibrium.

In order to motivate these dynamics, we focus on the one population price dispersion game with fixed consumer behavior. Since agents are myopic, their perceived payoffs on which they base their decisions are always a function of the current social state. Hence, the underlying payoff function that defines the dynamic is $\pi : \Delta^n \to \mathbf{R}^{n+1}$ with $\pi_i(x)$ defined in (1). The discussion that follows is, however, more general and applies to any population game. The derivation of the dynamics can also be readily extended to multipopulation cases.

An evolutionary dynamic is an ordinary differential equation $\dot{x} = V(x)$[17]where $x \in \Delta^n$ and $V(x)$ is the vector of change in social state x. To be admissible as an evolutionary dynamic, we

[16]We can provide microfoundations to this general model by appealing to the model of revision protocols in Sandholm(2006c). In this model, agents myopically change their behavior in response to the present social state whenever they receive opportunities to revise their strategies. The resulting process of social change can then be summarized using an evolutionary dynamic.

[17]To be strictly accurate, we should write $V(x)$ as $V_\pi(x)$ to indicate the dependence of the dynamic on the payoff function. However, since the underlying game is usually clear from the context, we will dispense with the extra notation.

require that from each initial condition $x_0 \in \Delta^n$, there must exist a unique solution trajectory $\{x_t\}_{t\in[0,\infty)}$ with $x_t \in \Delta^n$, for all $t \in [0, \infty)$.

PBR dynamics[18]are generated by requiring agents to optimize against payoffs after they have been subjected to some perturbations. These shocks can be interpreted to mean actual payoff noise, or mistakes agents make in perceiving payoffs or in implementing pure best responses.[19]

We call $v : int\,(\Delta^n) \to \mathbf{R}$ an admissible deterministic perturbation if the second derivative of v at x, $D^2 v\,(x)$ is positive definite for all $x \in int\,(\Delta^n)$ and if $|\nabla v\,(x)| \to \infty$ whenever $x \to bd\,(\Delta^n)$. In words, v is admissible if it is convex and becomes infinitely steep at the boundary of the simplex. We may interpret v as a "control cost function" associated with implementing any particular mixed strategy. The cost becomes large whenever the agent puts too little probability on any pure strategy.

Given the payoff function π and population state x, we define the perturbed payoff to mixed strategy $q \in int\,(\Delta^n)$ as $q'\pi(x) - \eta v\,(q)$.

The perturbed best response to x, $\tilde{B}\,(x)$ is the solution to the maximization exercise

$$\tilde{B}\,(x) = \underset{q\in int(\Delta^n)}{\operatorname{argmax}}\; q'\pi - \eta v\,(q)\,. \tag{5}$$

Convexity of the cost function ensures that the perturbed best response to every population state is unique. Steepness implies that the perturbed best response is a fully mixed strategy.[20] Moreover, $\tilde{B}\,(x)$ is differentiable with respect to x. In terms of these three properties- uniqueness, complete mixture and smoothness- the perturbed best response differs critically from the actual best response. Nevertheless, if the perturbation factor η is small, then $\tilde{B}\,(x)$ puts most of the weight on the actual best response to x.

State x is a *perturbed equilibrium* of the population game π if it is a fixed point of the perturbed best response function, i.e. if $x = \tilde{B}\,(x)$. Given a particular η, the set of perturbed equilibria and Nash equilibria will differ for most games. However, if x^* is a Nash equilibrium, then, typically, for small η, there will be an associated perturbed equilibrium $\tilde{x}_\eta$ such that $\lim_{\eta\to 0} \tilde{x}_\eta = x^*$.

If all agents revise strategies according to $\tilde{B}(x)$, then the evolution of social behaviour can be summarized using the PBR dynamic

$$\dot{x} = \tilde{B}\,(x) - x.$$

[18]The prototypical perturbed best response dynamic, the logit dynamic, was introduced by Fudenberg and Levine (1998). Since then, a number of authors including Benaim and Hirsch (1999), Hofbauer and Hopkins (2005), and Hofbauer and Sandholm (2002, 2005) have studied these dynamics in more general form. Fudenberg and Levine (1998) introduced the logit dynamic in the context of the learning literature. In the learning literature, this dynamic is generated by players playing a perturbed best response response to the state variable that is the history of opponents' play. In the population games context, the state variable is the current population state (Hofbauer and Sandholm, 2005). Nevertheless, the functional form of the dynamic and the nature of the perturbation that generates the dynamic in the two situations are identical.

[19]Due to its appealing behavioural properties, perturbed best response has been widely used in the experimental literature as a tool to rationalize noisy experimental data (Cheung and Friedman (1997), Camerer and Ho (1999), Battalio et al. (2001)).

[20]The method we described generates $\tilde{B}(x)$ using deterministic perturbation of the payoffs. The traditional method of deriving the perturbed best response function is by adding stochastic perturbations to the payoffs. However, Hofbauer and Sandholm (2002) show that the deterministic perturbation method is the more general technique.

Clearly, rest points of the dynamic coincide with the set of perturbed equilibria. For most evolutionary dynamics, a Nash equilibrium is a rest point. However, for perturbed best response dynamics, rest points are not Nash equilibria. Hence, any stability result for these dynamics will refer to stability of perturbed equilibria rather than Nash equilibria. Our interest, however, is primarily on the dynamics when η is small. Since typically, in such a situation, a perturbed equilibrium lies very close to a Nash equilibrium, stability of perturbed equilibria is sufficient to inform us whether the corresponding Nash equilibrium is a credible long run prediction.

The prototypical perturbed best response dynamic is the logit dynamic obtained from the logit best response function. The logit best response function can be obtained by specifying $v(q) = \sum\limits_{x_i \in S^n} q_i \log q_i$. This gives us the function

$$\tilde{B}_i(x) = \frac{\exp(\eta^{-1}\pi_i(x))}{\sum\limits_{x_j \in S^n} \exp(\eta^{-1}\pi_j(x))}.$$

4 Simulation: A Game with Six Prices

In Section 5, we prove that dispersed equilibria under the PBR dynamics are unstable. In this section, we present one numerical simulation that motivate the results of Section 5. We simulate solution trajectories and show that these trajectories converge to limit cycles. These simulations therefore imply that we need to regard observed price dispersion as a phenomenon of persistent disequilibrium. We then evaluate the time average of these limit cycles to see whether the credibility of the Nash equilibria prediction can be partially restored.

We note at this point that in general, it is very difficult to prove the identity or the existence of limit cycles. Hence, unless we make further simplifications in our model, these simulations are perhaps the only feasible way to verify the presence of limit cycles.[21] In Section 6, we simplify our model and impose some additional assumptions which allow us to prove the existence of limit cycles.

We consider the game in Example 2.6 in which $n = 5$, $r = 3$ and $c = 0.07$. The particular dynamic under which we run our simulation is the logit dynamic.

In figures 1 and 2, we plot solution trajectories for the logit dynamic with $\eta = 0.001$ starting from two different initial points. In Figure 1, we plot trajectories with the initial point being the Nash equilibrium (x^*, y^*) where $x^* = (0, 0, 0, 0.4684, 0.4176, 0.1140)$ and $y^* = (0.6680, 0.3319, 0)$. In Figure 2, the initial point is $x(0) = (\frac{1}{6}, \frac{1}{6}, \cdots, \frac{1}{6})$ and $y(0) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Note that we have only plotted the trajectories for some of the variables since the other variables fluctuate at levels that are positive but indistinguishable from zero.

It is clear that from both initial points, trajectories converge to price cycles. However, the two limit cycles are different. Let us look at the limit cycle in Figure 1, which we call **LC1**. Here, if

[21] It is even possible that long run disequilibrium behavior may take even more complicated forms like strange or chaotic attractor, where trajectories would exhibit elaborate dependence on initial conditions.

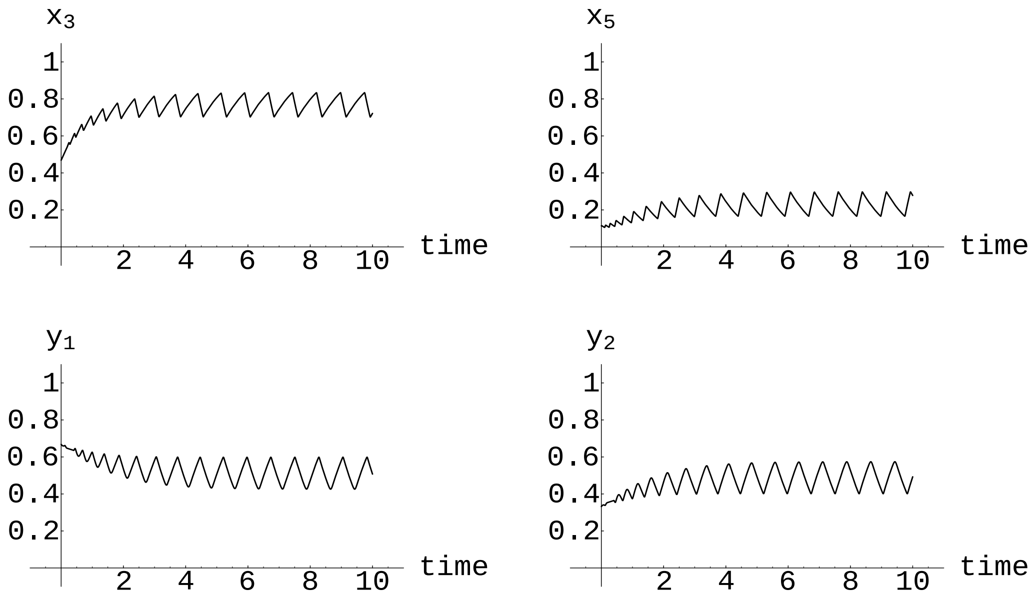


Figure 1: Solution trajectories under the logit (0.001) dynamic in the game with 6 prices, $r = 3$, and $c = 0.07$. The initial point is a Nash equilibrium.

we consider seller behavior, it is only prices $\frac{3}{5}$ and 1 that are charged by significant proportions of the population: x_3 fluctuates between about 0.7 and 0.8, while x_5 fluctuates between about 0.2 and 0.3. The population shares of the prices remain at levels very close to zero, but nevertheless positive. Hence, we can loosely term the support of the cycle to be prices $\{\frac{3}{5}, 1\}$. For the cycle in Figure 2, which we call **LC2**, the support consists of $\{\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$.

There are eight other mixed equilibria in this game, all listed in the appendix. We ran simulations of the solution trajectories of the same dynamic from each of these equilibria as initial points. We found that trajectories from all but one of the mixed equilibria[22]exhibit the same cycle as in Figure 2. From the one equilibrium that is the exception, the limit cycle is the one in Figure 1. Similarly, simulations from other points reveal that most trajectories converge to the cycle in Figure 2. The simulations could not detect any other limit cycle. Hence, it appears that the likelihood of the emergence of cycle in Figure 2 is much higher than that in Figure 1. We also note that even though some of the mixed equilibria puts positive probability on price $\frac{1}{5}$, the weight on this price declines to zero in both the limit cycles. Hence, even the fact that these equilibria put positive weight on price $\frac{1}{5}$ is misleading as a prediction. These simulations show that the monopoly equilibrium is not globally stable. However, being a strict equilibrium, it is locally stable.

These simulations suggest that it is limit cycles rather than mixed equilibria that provide a more credible explanation of observed price dispersion. From arbitrary social states, the two populations move to the limit cycles instead of to a mixed equilibria. It is, however, possible that over time,

[22]This particular Nash equilibrium is $x^9 = (0, 0, 0, 0.7739, 0, 0.2261)$; $y^9 = (0.5522, 0.4398)$

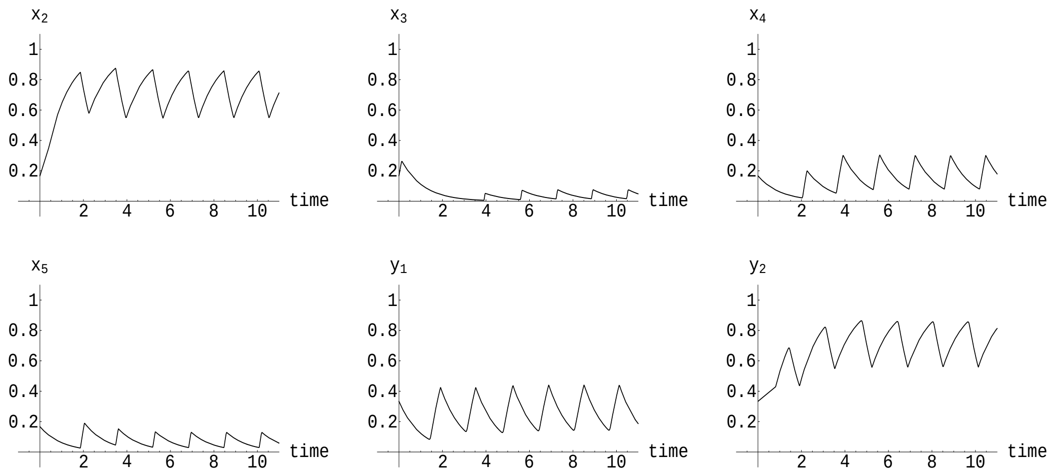


Figure 2: Solution trajectories under the logit (0.001) dynamic in the game with 6 prices, $r = 3$, and $c = 0.07$. $x(0) = (\frac{1}{6}, \frac{1}{6}, \cdots, \frac{1}{6})$ and $y(0) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

the time average of a limit cycle approaches a mixed equilibria. If so, that would partially restore the credibility of the equilibrium prediction. To test this possibility, we now calculate the time averages of the two limit cycles.

The time average of a limit cycle may be computed as

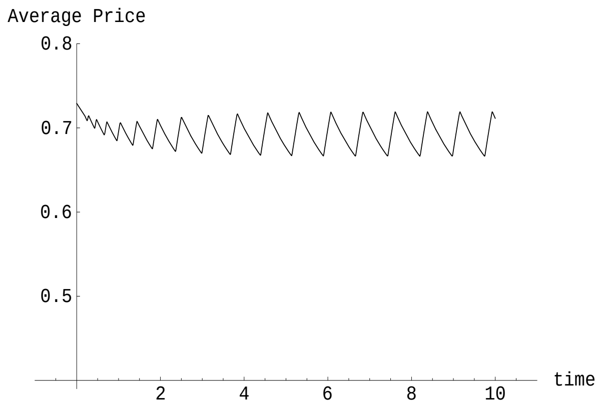
$$\overline{x} = \lim_{T\to\infty} \int_0^T x(t)dt$$

where $x(t)$ is any solution trajectory that converges to the limit cycle. The time average corresponding to **LC1** is $(\overline{x}_1, \overline{y}_1)$ where $\overline{x}_1 = (\sim 0, \sim 0, \sim 0, 0.7726, \sim 0, 0.2274)$ and $\overline{y}_1 = (0.5085, 0.4915, \sim 0)$. The time average corresponding to **LC2** is $(\overline{x}_2, \overline{y}_2)$ with $\overline{x}_2 = (\sim 0, \sim 0, 0.7205, 0.0393, 0.1718, 0.0684)$ and $\overline{y}_2 = (0.2712, 0.7287, \sim 0)$.

How do these time averages compare to the mixed equilibria of our example? It is of interest to note that $\overline{x}_1$ is very close to x^9, where x^9 is the distribution in the population of sellers at the Nash equilibrium (x^9, y^9) listed in footnote 22. However, $\overline{y}_1$ is still very significantly different from y^9. Hence, none of the mixed Nash equilibrium prediction is borne out by the time average of **LC1**. The time average of **LC2** is even more drastically different from any of the mixed Nash equilibria. None of the Nash equilibria even have a support of four prices, unlike the time average of **LC2**.

The cyclical fluctuation in the distribution x imply a cycle in the average price over time. The average price at a particular time is $\overline{p}(t) = \sum_{i=0}^{n} p_i x_i(t)$. Since solution trajectories converge to a limit cycle, $\bar{p}(t)$ also exhibits a regular cycle. We interpret this cycle as the Edgeworth cycle noted by Eckert (2003) and Noel (2003) in their analysis of retail gasoline markets in Canada.

An important characteristic of these Edgeworth cycles is that the upward phase of the cycles is



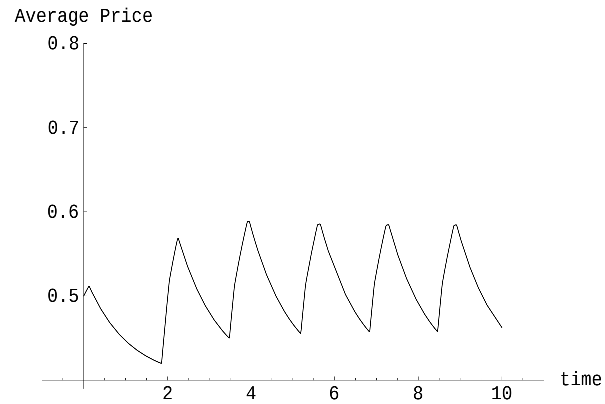


Figure 3: In the left panel, we plot $\overline{x}(t)$ with initial point same as in figure 1. In the right panel, the initial point is the one in figure 2.

much steeper than the downward phase. The following story of firm behavior might explain this feature of these cycles. Upon reaching the highest level of the cycle, firms slowly start undercutting their rivals. Hence, the proportion of firms charging lower prices increases. This phase continues till a certain minimum level is reached. Upon reaching that level, most firms then increase their prices very significantly. This raises the average price sharply and the cycle continues. Such a pattern of firm behavior has been noted by Noel (2003) in his analysis of the Toronto retail gasoline market.

These limit cycles are of importance not only in the context of this example but also more generally. To our knowledge, these are the first examples of evolutionary limit cycles in an economic model. Conceptually, they provide a new way of thinking about the long run consequences of economic interaction.

5 Instability of Dispersed Price Equilibria

We characterized the dispersed equilibria of our model in Theorem 2.5. The simulation in Section 4 suggest that the perturbed equilibria corresponding to these mixed equilibria are unstable under perturbed best response dynamics. In this section, we provide a rigorous proof of this result. Our broad strategy is to first fix consumers into two types—those that sample only one price and those that sample two prices. We show that mixed equilibria in this special case is unstable. We then use this result to prove the instability of mixed equilibria in the more general case where consumer behavior is endogenous.

5.1 Instability with Two Exogenous Consumer Types

We fix the strategy set size n and the distribution of consumer types $\{y_1, y_2\}$. We further assume that $0 < y_1 < 1$, and that n is large enough, to ensure the existence of mixed Nash equilibria. Since we are considering the one population case, our state space is Δ_1^n and the tangent space is

$T\Delta_1^n$.[23] However, for the rest of this subsection, we will dispense with the subscript and superscript in referring to the state space and the tangent space.

To determine the stability properties of rest points, we use the standard techniques of linearizing the dynamic around the rest points. Given the control cost function $v(x)$ and the dynamic $\dot{x} = V(x)$, let $\tilde{x}$ be a perturbed equilibrium, and hence a rest point of the dynamic. By Taylor's theorem, if we consider the dynamic at a point $\tilde{x} + z$ in the neighborhood of $\tilde{x}$, then $V(\tilde{x}+z) \approx V(\tilde{x}) + DV(\tilde{x})z = DV(\tilde{x})z$ where $DV(\tilde{x})$ is the Jacobian $DV(x) : T\Delta \to T\Delta$ evaluated at $\tilde{x}$. The non-linear dynamic $V(\tilde{x}+z)$ can therefore be approximated by a linear differential equation $DV(\tilde{x})z$ in a neighborhood of the rest point. Standard results then imply that if even a single eigenvalue of $DV(\tilde{x})$ has a positive real part, then the rest point is unstable. Now, since $V(x) = \tilde{B}(x) - x$

$$DV(\tilde{x}) = D\tilde{B}(\tilde{x}) - I,$$

where $D\tilde{B}(\tilde{x}) : T\Delta \to T\Delta$ is the Jacobian of $\tilde{B}$ evaluated at $\tilde{x}$, and I is the $(n+1)$ dimensional identity matrix. So, to determine stability of $\tilde{x}$, it is sufficient to determine the eigenvalues of $D\tilde{B}(\tilde{x})$. If the real parts of all the eigenvalues of $D\tilde{B}(\tilde{x})$ are less than one, then the rest point $\tilde{x}$ is locally stable. If, on the other hand, even one eigenvalue of $D\tilde{B}(\tilde{x})$ has real part greater than one, then $\tilde{x}$ is unstable.

While determining eigenvalues, we need to bear in mind that any change in population state must leave the total population mass unchanged. Hence, from a given state x, the only possible directions in which the population can move are those that are in the tangent space.[24] So the stability of an equilibrium $\tilde{x}$ is determined by the n eigenvalues of $DV(\tilde{x})$ or $D\tilde{B}(\tilde{x})$ that refer to the tangent space.

To summarize the above discussion, the operators $DV(\tilde{x})$ and $D\tilde{B}(\tilde{x})$ are defined from $T\Delta$ to $T\Delta$ and hence have n eigenvalues. If even one eigenvalue of $DV(\tilde{x})$ has a positive real part, or equivalently, if any eigenvalue of $D\tilde{B}(\tilde{x})$ has real part greater than 1, then the equilibrium $\tilde{x}$ is unstable.

A result by Hopkins (1999) simplifies the task determining the eigenvalues of $D\tilde{B}(x)$ considerably. Hopkins' result shows that the matrix $D\tilde{B}(x)$ may be written as the product of two matrices Q and $D\pi(x)$.

Before stating this result, we define the notion of a positive definite game.

Definition 5.1 *A population game with payoff function $\pi : \Delta_1^n \to \mathbf{R}^{n+1}$ is positive definite on $T\Delta_1^n$ if*

$$zD\pi(x)z > 0, \forall x \in \Delta_1^n, z \in T\Delta^n, z \neq 0 \tag{6}$$

Positive definiteness of a game implies that if a small group of players switch from strategy i to strategy j, then the marginal improvement in the payoff of strategy j resulting from the switch

[23] The tangent space is the set of feasible directions of motion of the population. Formally, $T\Delta_1^n = \left\{ z \in \mathbf{R}^{n+1} : \sum_{i=0}^{1} z_i = 0 \right\}$

[24] This is the reason why the domain of $DV(x)$ and $D\tilde{B}(\tilde{x})$ is $T\Delta$ and not $\mathbf{R}^{n+1}$.

exceeds the improvement in the payoff of i. This property is known as "self-improving externalities". We discuss positive definiteness in greater detail in the appendix.

Similarly, we say a matrix Q is positive definite with respect to $T\Delta_1^n$ if

$$zQz > 0, \forall z \in T\Delta_1^n, z \neq 0.$$

We now state the result from Hopkins (1999) in the following lemma.

Lemma 5.2 *(Hopkins, 1999): We may write the operator $D\tilde{B}\left(x\right) : T\Delta \rightarrow T\Delta$ as*

$$D\tilde{B}\left(x\right) = \frac{1}{\eta} Q\left(x\right) D\pi\left(x\right)$$

where $Q\left(x\right)$ is a symmetric matrix positive definite with respect to $T\Delta(x)$. Furthermore, $Q\mathbf{1} = 0$.

For example, in the logit dynamic, $Q\left(x\right)$ is a $(n+1)\times(n+1)$ matrix where $Q_{ii} = \tilde{B}_i(x)(1-\tilde{B}_i(x))$ and $Q_{ij} = -\tilde{B}_i(x)\tilde{B}_j(x)$, $i \in \{0, 1, \cdots, n\}$.

The following lemma then permits us to determine the sign of the eigenvalues of $D\tilde{B}\left(x\right)$ if the game is positive or negative definite at x. This lemma appears in Hofbauer and Sigmund (1988, p. 129) as an exercise. Sandholm (2006a, Lemma A.4) provides a proof. The proof is actually for positive definiteness on $T\Delta$. But it can be readily adapted to positive definiteness on $T\Delta(x)_0$.[25]

Lemma 5.3 *(Hofbauer and Sigmund ,1988) Suppose $Q\left(x\right)$ is a symmetric positive definite matrix with respect to $T\Delta(x)_0$, $Q\mathbf{1} = 0$, and π is a positive definite game. Then all eigenvalues of $Q\left(x\right) D\pi\left(x\right) : T\Delta(x)_0 \rightarrow T\Delta(x)_0$ have positive real parts. If, on the other hand, $D\pi\left(x\right)$ is negative definite, then all the eigenvalues have negative real parts.*

Before going to the instability results, we define a regular equilibrium as in Hofbauer and Hopkins (2005) (Van Damme (1987) calls this a quasi-strict equilibria).

Definition 5.4 *Let x^* be a partially mixed equilibrium. We say that x^* is a regular equilibrium if $\pi_i\left(x^*\right) > \pi_j\left(x^*\right)$, for all $i \in \mathrm{supp}(x^*)$, $j \notin \mathrm{supp}(x^*)$.*

Hence, x^* is a regular equilibrium if the Nash equilibrium payoff is strictly greater than the payoff of any pure strategy not in the support of the Nash equilibrium. It is well known that almost all equilibria in generic simultaneous move games are regular.

Our stability results apply only for a class of perturbed best response dynamics that satisfy a certain technical condition stated in Assumption 5.5 below. The condition relates to the limiting behavior of the Q matrix and is necessary to define the operator $\lim_{\eta\rightarrow 0} Q(\tilde{x}_\eta)D\pi(\tilde{x}_\eta)$. This assumption is necessary because the mixed Nash equilibria of our game do not have complete support.

[25] $T\Delta(x)_0$ is the subspace of the tangent space in which movement is restricted to the support of x. Formally, $T\Delta^n(x)_0 = \{z \in T\Delta^n : z_i = 0 \text{ if } x_i = 0\}$. Since π is a positive definite game, $D\pi\left(x\right)$ is positive definite on $T\Delta(x)_0$.

While it would be very difficult to prove that this condition holds in general, it is not very stringent and is satisfied by the logit dynamic. Moreover, we conjecture that our results can be proved even without using the assumption. However, the condition greatly simplifies the proofs.

We now formally state the assumptions under which our stability results will be based. With a little abuse of notation, we write $\lim_{\eta\to 0} Q(\tilde{x}_\eta)D\pi(\tilde{x}_\eta)$ as $Q(x^*)D\pi(x^*)$. Hence, for the logit dynamic, $Q_{ii}(x^*) = x_i^*(1 - x_i^*)$ and $Q_{ij}(x^*) = -x_i^* x_j^*$.

Assumption 5.5 *We assume the following.*

1. x^* *is a regular equilibrium.*

2. *Let* x^* *be a partially mixed equilibrium and let* $\{\tilde{x}_\eta\}$ *be the sequence of corresponding perturbed equilibria. Then* $\lim_{\eta\to 0} Q(\tilde{x}_\eta) = Q(x^*)$ *exists.*

Part 2 of the assumption ensures that the operator $Q(x^*)D\pi(x^*)$ is well defined.

We now consider the equilibria in the price dispersion games with exogenous consumer behavior $\{y_1, y_2\}$, $0 < y_1 < 1$. Since the game has dominated strategies, any mixed equilibrium has less than complete support. We show that all mixed equilibria in this game are unstable. This result is based on the following lemma about the positive definiteness of the game.

Lemma 5.6 *Consider the price dispersion game 1 with exogenous consumer types* $\{y_1, y_2\}, 0 < y_1 < 1$. *The resulting finite game is positive definite.*

Proof. In the Appendix. ■

We can now state our result about the stability of dispersed equilibria in this game.

Proposition 5.7 *Let* $\pi(x)$ *be the price dispersion game with* $n+1$ *prices. Let* $\{y_1, y_2\}, 0 < y_1 < 1$ *be the exogenous distribution of consumer types. Let* x^* *be a mixed equilibrium. Let* $\tilde{x}_\eta$ *be the perturbed equilibrium corresponding to* x^* *with perturbation level* η. *If the perturbed best response dynamic satisfies part 2 of Assumption 5.5, then there exists* $\eta^* > 0$ *such that for all* $\eta < \eta^*$, *the equilibrium* $\tilde{x}_\eta$ *is unstable.*

Proof. In the Appendix. ■

The intuition behind this result is as follows. The property of "self-improving externalities" implies that near a mixed equilibrium, if a small group of sellers deviate to another strategy, then this creates an incentive for other sellers to do likewise. The population, therefore, tends to move away from an equilibrium. It is this tendency that leads to the instability of equilibria.

Hence, for each mixed Nash equilibrium, we can find an η small enough such that the corresponding perturbed equilibria is unstable. Since the number of Nash equilibria, and hence perturbed equilibria, is generically finite, we can find an η small enough such that all the perturbed equilibria are unstable.

We should also note that this proposition is not saying that there exists some η^* such that for all $\eta < \eta^*$, perturbed equilibria will be unstable for all n. Whether this is true or not remains an open question. For the purposes of the above proposition, it is critical that we fix n beforehand and then look at the equilibria of the game corresponding to that particular n.

5.2 Instability with Endogenous Types

We now consider the general model of price dispersion with endogenous consumer behavior. Since this is a two population game, the evolutionary dynamic must specify motion in both populations. Given $(x, y) \in \Delta = \Delta_1 \times \Delta_2$, we denote the corresponding vector of change in social state as $V(x, y) \in T\Delta = T\Delta_1 \times T\Delta_2$ where $T\Delta_2$ is the tangent space of population 2. The payoff function for population 1 is given by (1) and of population 2 by the negative of (3). For simplicity, we assume that both populations face the same perturbation factor η. The control cost function v can, however, differ between the two populations. Hence, the perturbed best response dynamics at a population state $(x, y) \in \Delta$ are given by

$$\begin{aligned} V^1(x, y) &= \dot{x} = \tilde{B}^1(x, y) - x \\ V^2(x, y) &= \dot{y} = \tilde{B}^2(x, y) - y \end{aligned}$$

where $\tilde{B}^1(x, y)$ and $\tilde{B}^2(x, y)$ are the perturbed best response functions of populations 1 and 2 respectively. Clearly, a perturbed equilibrium $(\tilde{x}, \tilde{y})$ of the game is a rest point of the dynamic.

The stability of an equilibrium is once again determined by the eigenvalues of $DV(x, y)$ evaluated at the rest point $(\tilde{x}, \tilde{y})$. As in the single population case, $DV(\tilde{x}, \tilde{y}) = D\tilde{B}(\tilde{x}, \tilde{y}) - I$ where I is now the $(n + 1 + r) \times (n + 1 + r)$ identity matrix. Since any change in the social state must leave the mass in both populations unchanged, we need to view both $DV(x, y)$ and $D\tilde{B}(x, y)$ as operators from $T\Delta$ to $T\Delta$. Hence, the stability properties of $(\tilde{x}, \tilde{y})$ is determined by the $n+(r-1)$ eigenvalues that refer to $T\Delta$. The equilibrium $(\tilde{x}, \tilde{y})$ is unstable if at least one eigenvalue of $D\tilde{B}(\tilde{x}, \tilde{y})$ is greater than one.

The results of Hopkins (1999) apply to multipopulation game. Hence, in order to determine the eigenvalues of the Jacobian $D\tilde{B}(x, y)$, we apply Lemma 5.2 and write $D\tilde{B}(x, y)$ as

$$\begin{aligned} D\tilde{B}(x, y) &= \frac{1}{\eta} \begin{pmatrix} Q^1(x) & 0 \\ 0 & Q^2(y) \end{pmatrix} \begin{pmatrix} D_x\pi(x, y) & D_y\pi(x, y) \\ -D_xC(x) & -D_yC(x) \end{pmatrix} \\ &= \frac{1}{\eta} Q(x, y) D(x, y) \end{aligned}$$

Let us now look at each of the two matrices on the right hand side. The first matrix is a block diagonal matrix with $Q^1(x, y)$ and $Q^2(x, y)$ being both square matrices of dimensions $n + 1$ and r respectively, and both being symmetric and positive definite with respect to $T\Delta_1(x)_0$ and $T\Delta_2(x)_0$ respectively.

Two characteristics of the second matrix are of importance in determining the stability properties of perturbed equilibria. The first is that at a mixed Nash equilibrium, consumers sample either only one price or two prices. Hence $D_x\pi\left(x, y^*\right)$ is positive definite on $T\Delta_1$ by Lemma 5.6. The second critical fact is that consumers payoffs are independent of the distribution y. Hence, $D_y C\left(x, y\right) = 0$, at all population states (x, y).

We now show that given the strategy size n, a perturbed equilibrium $(\tilde{x}, \tilde{y})$ corresponding to a mixed equilibrium will be unstable. Since a mixed equilibrium has less than complete support, we continue to assume that Assumption 5.5 holds. We assume that both parts of the assumption holds separately for Q^1 and Q^2, so that in particular

$$\lim_{\eta\to 0} Q^1(\tilde{x}_\eta) = Q^1(x^*), \;\; \lim_{\eta\to 0} Q^2(\tilde{x}_\eta) = Q^2(x^*). \tag{7}$$

Proposition 5.8 *Consider the two population price dispersion game. Let $(\tilde{x}_\eta, \tilde{y}_\eta)$ be a perturbed equilibrium of this game corresponding to a regular mixed strategy Nash equilibrium (x^*, y^*) with perturbation level η. If the perturbed best response dynamic satisfies (7), then there exists $\eta^* > 0$ such that for all $\eta < \eta^*$, the equilibrium is unstable.*

Proof. In the Appendix. ■

Since the number of Nash equilibria is generically finite, we can find an η small enough that all perturbed equilibria corresponding to dispersed equilibria are unstable for perturbation levels smaller than that η^*.

Example 5.9 *We illustrate Proposition 5.8 with Example 2.6. We consider the mixed Nash equilibrium (x^*, y^*) where $x^* = (0, 0, 0, 0.4684, 0.4176, 0.1140)$ and $y^* = (0.6680, 0.3319, 0)$. We demonstrate instability under the logit dynamic. Let $\eta = 0.001$. The corresponding perturbed equilibrium is $(\tilde{x}, \tilde{y})$ with $\tilde{x} = (\sim 0, \sim 0, \sim 0, 0.4621, 0.4278,\ 0.1100)$, and $\tilde{y} = (0.6689, 0.3310, \sim 0)$.*

First, we consider the operator $Q^1(x^*)D_x\pi(x^*, y^*)$ restricted to $T\Delta^1(x^*)_0$. Holding y^* fixed, $\pi(x, y^*)$ is a positive definite game. Hence, $D_x\pi(x^*, y^*)$ is positive definite with respect to $T\Delta_1$. $Q^1(x^*)$ is positive definite on $T\Delta^1(x^*)_0$. By Lemma 5.3, the eigenvalues of $Q^1(x^*)D_x\pi(x^*, y^*) : T\Delta^1(x^*)_0 \to T\Delta^1(x^*)_0$ have positive real parts. Since x^* has three strategies in its support, $Q^1(x^*)D_x\pi(x^*, y^*)$ restricted to $T\Delta^1(x^*)_0$ has two eigenvalues, namely, $0.0116 \pm 0.0392i$. Hence, its trace is 0.0232.

Since $D_yC(x^*) = 0$, the trace of $Q\left(x^*, y^*\right) D\left(x^*, y^*\right) : T\Delta^1(x^*)_0\times\ T\Delta^2(y^*)_0 \to T\Delta^1(x^*)_0\times T\Delta^2(y^*)_0$ is also 0.0232. Hence, at least one of its three eigenvalues has positive real part. The three eigenvalues are $\{-0.0054 \pm 0.0684i, 0.0340\}$. As an operator from $T\Delta$ to $T\Delta$, $Q\left(x^*, y^*\right) D\left(x^*, y^*\right)$ has the same three eigenvalues along with five zero eigenvalues corresponding to the five unused strategies at the Nash equilibrium. The three corresponding eigenvalues of $Q(\tilde{x}_\eta, \tilde{y}_\eta)D(\tilde{x}_\eta, \tilde{y}_\eta)$ are then $\{-0.0060 \pm 0.0672i,\ 0.0349\}$, which also has five other eigenvalues close to zero. Hence, $DB\left(\tilde{x}, \tilde{y}\right)$ has eigenvalues with real parts $\{-6.0269, 34.9928\}$. The real parts of the corresponding eigenvalues of $DV\left(\tilde{x}, \tilde{y}\right)$ are $\{-7.0269, 33.9928\}$. Hence, $(\tilde{x}, \tilde{y})$ is an unstable rest point.

Proposition 5.8 rules out mixed equilibria as a credible explanation of observed price dispersion. Since all dispersed equilibria are unstable, solution trajectories will either settle down around a pure strategy Nash equilibria, or exhibit some form of long run disequilibrium behavior. The unique pure strategy equilibrium of our general model—the monopoly equilibrium—is not globally stable, as can be seen from the simulations in the two-population game in Section 4.

Hence, we need to invoke disequilibrium attractors like limit cycles or chaotic attractors to explain long run price dispersion. Due to the intractability of the problem, we do not attempt to prove the existence of such attractors in the general case. However, in the next section, we are able to prove such an existence result for a simplified version of our model.

6 Cycling in the Exogenous Game with Two Consumer Types

We now simplify our model by focusing on the case of exogenous consumer type distribution $\{y_1, y_2\}$, $0 < y_1 < 1$. Our objective is to analytically prove the existence of disequilibrium attractors, a task of infeasible difficulty in the general model. We show that if the game satisfies a further condition called *quasi-monocyclicity*, then there exists a globally attracting limit cycle under the *best response dynamic*. We can then show that for small η, a perturbed best response dynamic also has an attractor near the best response limit cycle. The attractor under the perturbed best response dynamic can be a limit cycle or a strange attractor.

Let us consider the game with strategy space S^n. The strategy size n is assumed to be sufficiently large that there is no pure strategy Nash equilibrium. For the purpose of this section, it will be helpful to express the payoff function (1) in the following equivalent form. Given the population state p, the payoff to price x_i is

$$\widehat{\pi}_i(x) = \pi_i(x) - \sum_{j=1}^{n} p_j x_j = p_i(y_1 + 2y_2(\frac{x_i}{2} + \sum_{j>i} x_j)) - \sum_{j=1}^{n} p_j x_j \tag{8}$$

where $\pi_i(x)$ is given by (1) with $m = 2$. The best response dynamic (Gilboa and Matsui (1991)) takes the form of the following differential inclusion.

$$\dot{x} \in BR(x) - x, \tag{9}$$

where $BR(x)$ is the best response to population state x. Hofbauer (1995) studies these dynamics and proves that at least one solution from each initial point is guaranteed. Any rest point of the best response dynamic is a Nash equilibrium.

It has been shown in Benaim, Hofbauer, and Hopkins (2005) (Proposition 2) that all mixed Nash equilibria of a positive definite game are unstable under the best response dynamic. We now consider the existence of a limit cycle. Formally, a limit cycle is a locally attracting closed or invariant solution trajectory without a rest point.

Given the finite game with n prices, we define the function $W : \Delta^n \to \mathbf{R}$ by

$$W(x) = \max_{x_i \in S^n} \widehat{\pi}_i(x) \tag{10}$$

as a Lyapunov function. Gaunersdorfer and Hofbauer (1995) use this function to identify the limit cycle of the bad Rock-Paper-Scissor game, a positive definite game. Here, we show that if (8) satisfies a condition called *quasi-monocyclicity*, then the game contains a unique almost globally attracting limit cycle with characteristic $W(x) = 0$ for any x in the limit cycle.

First, we define *monocyclic games* (Hofbauer, 1995). A two-player symmetric normal form game A with n strategies is called monocyclic if

1. $a_{ii} = 0$
2. $a_{ij} > 0$ for $i \equiv j + 1 \pmod N$ and $a_{ij} < 0$ otherwise.

To see the relevance of monocyclicity for our model, we note that the game with payoff function (8) has an equivalent two-player normal form representation

$$\begin{array}{c} p_i y_1 - p_j, \text{ if } i > j; \\ p_i(y_1 + y_2) - p_j = 0, \text{ if } i = j \\ p_i(y_1 + 2y_2) - p_j, \text{ if } i < j \end{array} \tag{11}$$

The normal form representation expresses the idea that the seller charging the lower price acquires all the consumers who sample twice. The subtraction by p_j ensures that the diagonal elements of the normal form are zeros.

Since our model has dominated strategies, the monocyclicity condition is not satisfied for the entire game. Let $S^{ud} \subset S^n$ be the set of strategies that are undominated by strategy 1. We now define a restricted notion of monocyclicity we call *quasi-monocyclicity*.

Definition 6.1 *We call the game defined by strategy set S^n and payoff function (8) quasi-monocyclic if its equivalent normal form (11) satisfies the monocyclicity condition on the strategy set S^{ud}.*

The intuition behind the quasi-monocyclicity condition is as follows. Let $p_j \in S^{ud}$. Suppose the population state is e_j, that is, the entire population is playing strategy p_j. The quasi-monotonicity condition requires that the payoff to strategy p_j should be less than the strategy that immediately precedes it in S^{ud}, but be more than the payoffs of all the other strategies in S^{ud} . Here precedence is in the modular sense. The strategy immediately preceding the lowest undominated strategy is 1. It is, however, not easy to provide a condition that ensures quasi-monocyclicity in this game.

Example 6.2 *The game with $S = \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$, $y_1 = 0.45$ and $y_2 = 0.55$ is a quasi-monocyclic game. Prices 0 and $\frac{1}{5}$ are dominated by 1. The normal form equivalent of the game satisfies the monocyclicity conditions on $S^{ud} = \{\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$. This game has three Nash equilibria: (0, 0, 0.6364, 0.0909, 0.2727, 0); (0, 0, 0, 3388, 0.4876, 0, 0.1736); (0, 0, 0, 0.5105, 0.2587, 0.1259, 0.1049).*

To prove the existence of a limit cycle in a quasi-monocyclic game. we need the following lemma.

Lemma 6.3 *Consider the finite dimensional game with strategy space S^n with consumer types $\{y_1, y_2\}$, $0 < y_1 < 1$. Let x^* be a Nash equilibrium of the game. Then $W(x^*) < 0$. Let δ_i be the pure strategy that puts probability 1 on the pure strategy x_i. Then, for n sufficiently large, $W(\delta_i) > 0$.*

Proof. In the Appendix. ■

We now assume that n is sufficiently large so that Lemma 6.3 is satisfied. By the convexity of Δ^{ud} and the continuity of $W(x)$ we can be assured that there exists a set $W^0 = \left\{x \in \Delta^{ud} : W(x) = 0\right\}$. Moreover, by Lemma 6.3, W^0 is disjoint from the set of Nash equilibria.

The following proposition establishes the existence of an almost globally attracting limit cycle in our model under the best response dynamic. The proof relies on a result in Benaim, Hofbauer and Hopkins (2005).

Proposition 6.4 *Consider the finite price dispersion game with consumer types given exogenously by the distribution $\{y_1, y_2\}$, $0 < y_1 < 1$. Suppose the game satisfies the condition of quasi-monotonicity. Then Δ^n contains a closed orbit under the best response dynamic. Furthermore, from a dense, open and full measure set of initial conditions, the best response dynamics converge to this closed orbit. Moreover, for any state x in the limit cycle, $W(x) = 0$.*

Proof. The set $\Delta^{ud} = \{x \in \Delta^n : x_i = 0 \text{ if } p_i \notin S^{ud}\}$ is invariant under the best response dynamic. We now consider an initial point $x(0) \in \Delta^{ud}$. Proposition 1 in Benaim, Hofbauer and Hopkins (2005) then implies the existence of a limit cycle in Δ^{ud} that attracts trajectories from a dense, open and full measure set of initial conditions in Δ^{ud}. Moreover, since $W^0 \in \Delta^{ud}$, the same proposition in Benaim, Hofbauer and Hopkins (2005) implies $W(x) = 0$, for any x in the limit cycle. To complete the argument, we note that if $x(0) \notin \Delta^{ud}$, then solution trajectories will converge to Δ^{ud}. ■

In terms of the original payoff function π, $W^0 = \{x \in \Delta^n : \max_{x_i \in S^n} \pi_i(x) = \sum_{j=1}^{n} p_j x_j\}$. In order to understand W^0, we invoke the intuition provided by Gaunersdorfer and Hofbauer (1995) in explaining the emergence of a limit cycle in the bad Rock-Paper-Scissors game. We note that in terms of the original payoff function, $\pi_j(e_j) = p_j$. At any Nash equilibrium x^*, we have, by Lemma 6.3

$$\pi(x^*) < \sum_{j \in S^n} x_j^* \pi_j(e_j)$$

This condition means that at the Nash equilibrium, the population benefits from splitting itself into a number of different subpopulations, this number being equal to the number of strategies in the support of the equilibrium. This causes the population to move away from the equilibrium towards W^0.

We now use results from Benaim, Hofbauer and Sorin (2005) to establish the existence of a limit cycle in our model under perturbed best response dynamics.[26] As the level of perturbation η tends to zero, the perturbed best response function puts weight tending to 1 on the absolute best response. Hence, if η is sufficiently small, solution trajectories under PBR dynamics approximate arbitrarily well trajectories under the best response dynamic over any time interval $[0, T]$. Benaim, Hofbauer and Sorin (2005) call such trajectories *asymptotic pseudo trajectories* of the best response dynamic. Hence, if we translate the conclusion of Benaim, Hofbauer and Sorin (2005) to our context, we can conclude that any set of chain recurrent points under PBR dynamics is upper hemicontinuous in η. This implies that for η close to zero, there will exist an attractor of the PBR dynamic close to the attractor of the best response dynamic. Hence, near the attracting limit cycle, there will exist an attractor (either a limit cycle or a chaotic attractor) under the a PBR dynamic for sufficiently small η.

We can summarize the above discussion in the following proposition.

Proposition 6.5 *Consider the finite price dispersion game with consumer types given exogenously by the distribution* $\{y_1, y_2\}$, $0 < y_1 < 1$. *Suppose the game satisfies the condition of quasi-monotonicity. Let* SP *be the globally attracting closed orbit under the best response dynamic. Then, for* η *sufficiently small, a perturbed best response dynamic will contain a global attractor near* SP. *From an open, dense and full measure set of initial conditions, the perturbed best response dynamic converge to this attractor.*

We ran numerical simulations for the game with six prices in Example 6.2 under the logit dynamic with $\eta = 0.001$. Simulations suggest the presence of a unique limit cycle that is globally attracting. In Figure 4 we plot the trajectory converging to the limit cycle from the initial point $(0, 0, 0.5105, 0.2587, 0.1259, 0.1049)$ which is a Nash equilibrium. The corresponding perturbed equilibrium is $(\sim 0, \sim 0, 0.5150, 0.2548, 0.1301, 0.1001)$ We plot only the support of the limit cycle, (x_2, x_3, x_4, x_5). The time average of the limit cycle is $(\sim 0, \sim 0, 0.3292, 0.4736, 0.1256, 0.0716)$ which is very different from even the equilibrium that has the same support.

7 Conclusion

In this paper, we have analyzed the question of price dispersion from an evolutionary standpoint. In order to avoid technical complications, we have constructed a finite dimensional model of price dispersion based on the original Burdett and Judd (1983) model. We have focused our attention on the mixed equilibria of the model and have analyzed their stability properties under perturbed best response dynamics. Building on the theoretical work of Hopkins (1999), we have found that mixed equilibria in our model are unstable under these dynamics. Intuitively, instability arises due to positive definiteness of the game around a mixed Nash equilibrium.

[26]These results have been used in Fudenberg and Takahashi (2007) to establish the existence of a limit cycle in a rock-paper-scissor game under stochastic fictitious play.

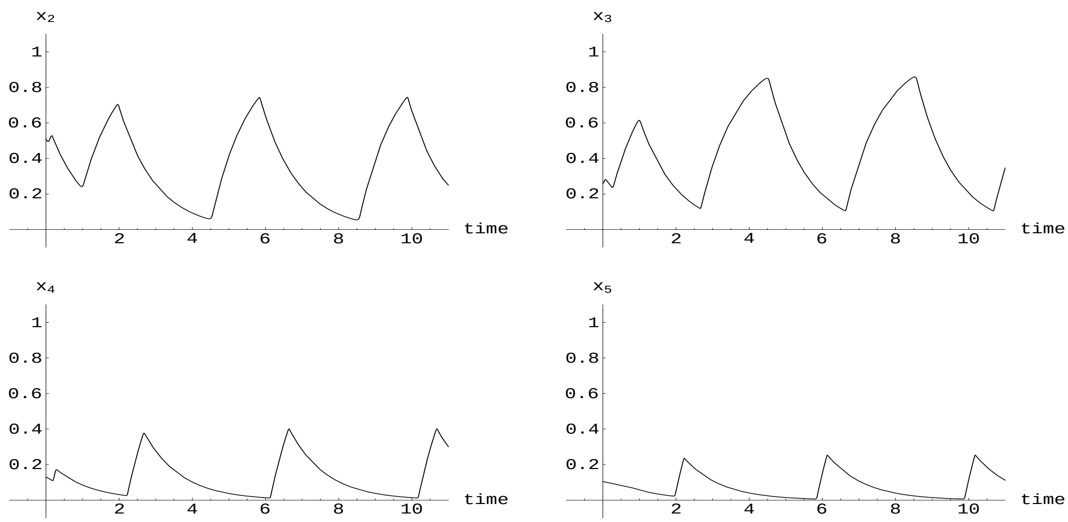


Figure 4: Solution trajectories under the logit (0.001) dynamic in the game with 6 prices, $y_1 = 0.45$. The initial point is a Nash equilibrium.

Given these instability results, we view observed price dispersion as a long run disequilibrium phenomenon. We show through numerical simulation that the disequilibrium phenomenon can be expected to take the form of limit cycles that attract solution trajectories of the perturbed best response dynamics. In these cycles, both the proportion of firms charging a particular price and the average market price keeps fluctuating in a regular manner. Such attractors may be limit cycles or chaotic. In general, it is difficult to prove the existence of such disequilibrium attractors. But for a simple case, we have established the existence such a long run disequilibrium state. The detailed investigation of such attractors for the more general model can be a potentially rich avenue for research.

This paper illustrates the general principle that certain economic phenomenon may not be explained by invoking traditional equilibrium concepts, particularly when empirical and experimental evidence runs contrary to the equilibrium prediction of economic theory. Perpetual disequilibrium is captured naturally by evolutionary game theory. By exploiting this aspect of the evolutionary approach, we believe that this paper has made a major methodological contribution that should lead to further work on the application of evolutionary game theory in economics.

In a companion paper (Lahkar, 2007b), we extend the analysis to the original continuous strategy game of Burdett and Judd. We have shown that the infinite dimensional logit dynamic is well defined in the Burdett and Judd model. Establishing stability results in the infinite dimensional context, however, remains a challenge. The interest in this question is not merely technical, but also practical since most economic situations of interest are naturally modeled as having continuous strategy spaces.

Among other research questions in this area, we can try to generalize the results established here to other evolutionary dynamics and other price dispersion models. Finally, one can use evolutionary game theory to try and investigate other possible economic issues. For example, perturbed best response can be a attractive way to study markets which are subject to rapid change, either due to changes in technology or in consumer tastes. In such situations, firms may not have exact knowledge of demand and supply conditions, and so would be prone to making mistakes in recognizing payoffs or in implementing best response. Perturbed best response can take into account such possibilities.

Appendix

A Nash equilibria

Proof. **Proposition 2.2**

Suppose there exists some price such that $\overline{x}_i^n > \varepsilon > 0$. By making n sufficiently large, the price that is immediately lower than p_i can be brought arbitrarily close to p_i. Let us denote this price by p_{i-}. The payoff from p_{i-} is at least

$$p_{i-}\left[y_1 + \sum_{m=2}^{r} m y_m (\sum_{j\geq i} \overline{x}_j^n)^{m-1}\right]$$

On the other hand, the payoff from p_i is given by (1). Let us denote $\sum\limits_{j>i} \overline{x}_j^n$ by G. By applying the binomial formula, it can be established than,

$$\begin{aligned} \pi_{i-}(x) - \pi_i(x) &\geq y_1 (p_{i-} - p_i) \\ &+ \sum_{m=2}^{r} \left[\sum_{k=0}^{m-1} \binom{m-1}{k} (\overline{x}_i^n)^k G^{m-1-k} \left(p_{i-} - \frac{p_i}{k+1}\right)\right] \end{aligned} \tag{12}$$

As $n \to \infty$, $y_1 (p_{i-} - p_i) \nearrow 0$. Now, for all prices and for all n, $p_{i-} > \frac{p_i}{3}$ except when $i = 1$. Also, $p_{i-} > \frac{p_i}{2}$ for all prices except the zero price and the first two positive prices. However, the first two positive prices are dominated for all n sufficiently large and hence, can be ignored. Hence, for all m, the part of the above expression inside the square bracket will be positive if

$$\left({}^{m-1}C_1\right) (\overline{x}_i^n) G^{m-2} \left(p_{i-} - \frac{p_i}{2}\right) \geq G^{m-1} (p_{i-} - p_i)$$

Since $G \leq 1$, $\overline{x}_i^n > \varepsilon$, and $m \geq 2$, for the above expression to hold, it is sufficient that

$$\left(p_{i-} - \frac{p_i}{2}\right) \varepsilon \geq (p_{i-} - p_i) = \frac{1}{n} \tag{13}$$

Now, let $\underline{p}$ be the lowest price in the support of the corresponding continuous game. This price is greater than zero since $0 < y_1 < 1$. Hence, $\lim_{n\to\infty} \left(p_{i-} - \frac{p_i}{2}\right) \varepsilon \geq \left(\underline{x} - \frac{x}{2}\right) \varepsilon > c > 0$, for some c. So, for n sufficiently large (13) holds. Hence, as $n \to \infty$, the expression inside the square bracket in (12) remains bounded away from zero whereas $y_1 (p_{i-} - p_i)$, while being negative, goes to zero. So, for n sufficiently large, $\pi_{i-}(x) - \pi_i(x)$ will be positive. Hence, $\overline{x}_i^n$ cannot be a Nash equilibrium.

■

Proof. **Lemma 2.3**

1. This is obvious. The payoff to any strategy p_i is $p_i y_1$. Hence, the highest price dominates all other prices.

2. If $x_0^n = 1$, then $\pi_i^n = 0$ for all prices. If $x_1^n = 1$, then $\pi_1^n = p_1^n$. But since $y_1 = 0$, $\pi_i^n = 0$, for all other prices p_i^n. Hence, these two pure strategy Nash equilibria always exist. For any price p_i^n greater than p_1^n, if $x_i^n = 1$, then

$$\pi_i^n = \sum_{m=2}^{r} p_i^n y_m = p_i^n, \text{ whereas}$$
$$\pi_{i-}^n = p_{i-}^n \sum_{m=2}^{r} m y_m = \sum_{m=2}^{r} m p_{i-}^n y_m$$

where p_{i-}^n is the price immediately lower than p_i^n. If $p_i^n > p_1^n$, then $m p_{i-}^n \geq p_i^n$ with the equality holding only for $m = 2$ and $p_i^n = p_2^n$. Hence, if $r \geq 3$, then $\pi_{i-}^n > \pi_i^n$ for all $i > 1$ and so, there can be no other pure equilibria. Now, consider the special case where $r = 2$. Hence, $y_2 = 1$. Then, if $x_2^n = 1$, $\pi_2^n = p_2^n$, $\pi_1^n = 2p_1^n$ and $\pi_i^n = 0$, for all other i. Since $p_2^n = 2p_1^n$, $x_2^n = 1$ is a Nash equilibrium for the case $r = 2$.

Next, we rule out the possibility of any mixed equilibria. Suppose p_H^n is the highest price in the support of a mixed equilibrium $\overline{x}^n$. The payoff to p_H^n is

$$\pi_H^n = p_H^n \left(\sum_{m=2}^{r} m y_m \frac{(\overline{x}_H^n)^{m-1}}{m} \right) = \sum_{m=2}^{r} y_m p_H^n (\overline{x}_H^n)^{m-1}$$

Let p_{H-}^n be the price that is immediately lower than p_H^n. Since price 0 cannot be a part of a mixed strategy, $p_{H-}^n \geq \frac{1}{n}$. The payoff to p_{H-}^n is

$$\begin{aligned}
\pi_{H-}^n &= p_{H-}^n \left[\sum_{m=2}^{r} m y_m \left\{ \sum_{k=0}^{m-1} \binom{m-1}{k} \left(\overline{x}_{H-}^n\right)^k \left(\overline{x}_H^n\right)^{m-1-k} \frac{1}{k+1} \right\} \right] \\
&\geq p_{H-}^n \left[\sum_{m=2}^{r} m y_m (\overline{x}_H^n)^{m-1} \right] = \sum_{m=2}^{r} m y_m p_{H-}^n (\overline{x}_H^n)^{m-1}
\end{aligned}$$

$m p_{H-}^n > p_H^n$ except for the case where $m = 2$ and $p_H^n = p_2^n$. Hence, if $r \geq 3$, $\pi_{H-}^n > \pi_H^n$ which shows that $\overline{x}^n$ cannot be a Nash equilibrium. If $r = 2$ and $p_H^n > p_2^n$, then too $\pi_{H-}^n > \pi_H^n$. The only case that remains is where $r = 2$ and $p_H^n = p_2^n$. Hence, $y_2 = 1$,

$$\pi_1^n = p_1^n \left(2 \left(\frac{\overline{x}_1^n}{2} + \overline{x}_2^n \right) \right), \; \pi_2^n = p_2^n \left(2 \left(\frac{\overline{x}_2^n}{2} \right) \right)$$

Since $p_2^n = 2p_1^n$, these payoffs can only be equal if $\overline{x}_2^n = 1$. Hence, this special case reduces to the case of the pure strategy equilibrium $x_2^n = 1$ when $y_2 = 1$. ■

B Mixed Equilibria of Example 2.6.

The mixed equilibria of the game are listed below. The first set of numbers refer to seller behavior while the second set refers to consumer behavior.

$$\begin{array}{ll}
x^1 = (0, 0, 0, 0.4684, 0.4176, 0.1140) & y^1 = (0.6680, 0.3319, 0) \\
x^2 = (0, 0, 0.8084, 0, 0.1496, 0.0416) & y^2 = (0.4201, 0.5799, 0) \\
x^3 = (0, 0, 0.2673, 0.6485, 0, 0.084) & y^3 = (0.5037, 0.4963, 0) \\
x^4 = (0, 0.42202, 0.5413, 0, 0, 0.0367) & y^4 = (0.2585, 0.7415, 0) \\
x^5 = (0, 0.8035, 0, 0.179, 0, 0.0174) & y^5 = (0.2171, 0.7829, 0) \\
x^6 = (0, 0.7738, 0, 0.2262, 0, 0) & y^6 = (0.215, 0.785, 0) \\
x^7 = (0, 0, 0.7738, 0, 0.2262, 0) & y^7 = (0.4363, 0.5637, 0) \\
x^8 = (0, 0, 0.8651, 0, 0, 0.1349) & y^8 = (0.3472, 0.6528, 0) \\
x^9 = (0, 0, 0, 0.7739, 0, 0.2261) & y^9 = (0.5622, 0.4398, 0)
\end{array}$$

C Positive Definite Games

We now discuss the notion of a positive definite game which has been crucial to us in determining the stability properties of mixed equilibria. For our purpose, it is enough to define the concept for a one population game. Let us have a game with state space Δ^n, tangent space $T\Delta^n$ and payoff function $\pi : \Delta^n \to \mathbf{R}^{n+1}$ is positive definite at $x \in \Delta^n$ if

$$zD\pi(x)\,z > 0, \text{ for all } z \in T\Delta^n, z \neq 0. \tag{14}$$

If (14) is satisfied for all $x \in \Delta^n$, we say that the game is *positive definite.*

Example C.1 *The canonical example of a positive definite game is a symmetric two player coordination game with positive diagonal elements and zero non-diagonal elements. Let us consider the following three strategy coordination game with strategy set* $S = \{1, 2, 3\}$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The tangent space for this game is $T\Delta = \{z \in \mathbf{R}^3 : \sum_{i=0}^{3} z_i = 0\}$. *Given the population state* $x, \pi_i(x) = ix_i$. $D\pi(x) = C$. *Hence,* $zD\pi(x)\,z = \sum_{i=1}^{3} iz_i^2 > 0$ *if* $z \neq 0$. *Thus, this game is positive definite.*

One way to interpret the positive definiteness condition is through the notion of "self-improving externalities" which is analogous to the notion of "self-defeating externalities" introduced in Hofbauer and Sandholm (2006a) in connection with negative definite games, or "stable" games. Condition (14) is equivalent to the condition $\sum_{i \in S^n} z_i(D\pi_i(x)z) > 0$ where $D\pi_i(x)z$ is the directional derivative of $\pi_i(x)$ in the direction z. The vector z describes the change in population state when a small group of agents revise strategies at state x. $D\pi_i(x)z$ then represents the marginal impact of this strategy revision on the payoffs of those agents currently playing i. If we weigh these payoff changes with the changes in the population weight of each strategy, then condition (14) says that the aggregate effect should be positive. Intuitively, self improving externalities mean that if a small group of players are switching from strategy i to strategy j, then the marginal improvement of the payoff of strategy j resulting from the switch exceeds the improvement of the payoff of i.

We have also used a restricted notion of positive definiteness in which the game is positive definite at some p only with respect to some subspace of $T\Delta^n$. Let us denote the support of p by supp(p). Let $T\Delta^n(p)_0$ be a subspace of the tangent space defined as

$$T\Delta^n(p)_0 = \{z \in T\Delta^n : z_i = 0 \text{ if } i \notin \text{supp}(p)\} \tag{15}$$

Then, we say that the game is positive definite with respect to $T\Delta^n(p)_0$ at p if $zD\pi\left(p\right)z > 0$ for all $z \in T\Delta^n(p)_0$. Note that if p has full support, then $T\Delta^n(p)_0 = T\Delta^n$

D Positive Definiteness in the Finite Game

Proof. **Proposition 5.6**,

The payoff to price p_i is

$$\pi_i\left(x\right) = p_i\left[y_1 + 2y_2\left\{\frac{x_i}{2}_i + \sum_{j>i} x_j\right\}\right]$$

Hence, for $z \neq 0$,

$$D\pi_i\left(x\right)z = p_i\left[2y_2\left\{\frac{z_i}{2}_i + \sum_{j>i} z_j\right\}\right]$$

Hence,

$$zD\pi\left(x\right)z = -2p_0y_2Z_0^2 + p_0y_2z_0^2 - 2y_2\sum_{i=1}^{n} p_iZ_i(Z_i - Z_{i-1}) + y_2\sum_{i=1}^{n} p_iz_i^2$$

because $z_0 = Z_0$ and $z_i = (Z_i - Z_{i-1})$ for all $i > 0$.

Now

$$Z_i(Z_i - Z_{i-1}) = \frac{1}{2}(Z_i^2 - Z_{i-1}^2) + \frac{z_i^2}{2}$$

and $Z_0^2 = \frac{1}{2}Z_0^2 + \frac{1}{2}z_0^2$. So, we can rewrite $zD\pi(x)$ as

$$
\begin{aligned}
zD\pi(x)\,z &= -2p_0y_2(\frac{1}{2}Z_0^2 + \frac{1}{2}z_0^2) + p_0y_2z_0^2 \\
&- 2y_2\sum_{i=1}^{n}p_i(\frac{1}{2}(Z_i^2 - Z_{i-1}^2) + \frac{z_i^2}{2}) + y_2\sum_{i=1}^{n}p_iz_i^2 \\
&= -p_0y_2Z_0^2 - y_2\sum_{i=1}^{n}p_i(Z_i^2 - Z_{i-1}^2) = -y_2\sum_{i=0}^{n}Z_i^2(p_i - p_{i+1})
\end{aligned}
$$

where we make use of the fact that $Z_n^2 = 0$. Since $p_i - p_{i+1} < 0$ and $Z_i^2 > 0$, we conclude $zD\pi(x)\,z > 0$.

E Instability of Dispersed Price Equilibria

Proof. **Proposition 5.7** We first consider the operator $Q(x^*)D\pi(x^*)$. By assumption, x^* is a regular equilibrium. Hence, we can regard $Q(x^*)D\pi(x^*)$ as an operator from $T\Delta(x^*)_0$ to $T\Delta(x^*)_0$. Since π is a positive definite game, $D\pi(x^*)$ is positive definite on $T\Delta(x^*)_0$. As an operator on $T\Delta(x^*)_0$, $Q(x^*)$ is positive definite. Let the cardinality of supp(x^*) be k. Hence, by Lemma 5.3, all the $(k-1)$ eigenvalues of $Q(x^*)D\pi(x^*) : T\Delta(x^*)_0 \to T\Delta(x^*)_0$ have positive real parts. Now, we consider the eigenvalues of $Q(x^*)D\pi(x^*) : T\Delta \to T\Delta$. If λ_1 is an eigenvalue of $Q(x^*)D\pi(x^*) : T\Delta(x^*)_0 \to T\Delta(x^*)_0$, then it is also an eigenvalue of $Q(x^*)D\pi(x^*) : T\Delta \to T\Delta$. Hence, at least one eigenvalue of $Q(x^*)D\pi(x^*) : T\Delta \to T\Delta$ has a positive real part. Let this eigenvalue be $\overline{\lambda}$ with real part $\overline{\lambda}^R > 0$.

Now, we consider $Q(\tilde{x}_\eta)D\pi(\tilde{x}_\eta) : T\Delta \to T\Delta$. Part 2 of Assumption 5.5 implies that for small η, the eigenvalues of $Q(\tilde{x}_\eta)D\pi(\tilde{x}_\eta)$ are close to the eigenvalues of $Q(x^*)D\pi(x^*)$. Hence, $Q(\tilde{x}_\eta)D\pi(\tilde{x}_\eta)$ has an eigenvalue $\tilde{\lambda}_\eta$ such that $\lim_{\eta\to 0}\tilde{\lambda}_\eta = \overline{\lambda}$. Denoting the real part of $\tilde{\lambda}_\eta$ by $\tilde{\lambda}_\eta^R$, we conclude that sufficiently small η, $\frac{\tilde{\lambda}_\eta^R}{\eta} > 1$. But $\frac{\tilde{\lambda}_\eta^R}{\eta}$ is the real part of an eigenvalue of $\frac{1}{\eta}Q(\tilde{x}_\eta)D\pi(\tilde{x}_\eta)$. This completes the proof. ■

Proof. **Proposition 5.8** We first consider the operator $Q(x^*, y^*)\,D(x^*, y^*)$ and show that it has at least one positive eigenvalue.

By our discussion preceding the statement of this proposition, $D_x\pi(x, y^*)$ is equal to the Jacobian of the payoff function of the one population game with an exogenous consumer type distribution being $\{y_1^*, y_2^*\}$. Hence, $D_x\pi(x, y^*)$ is positive definite with respect to $T\Delta_1$. By assumption, (x^*, y^*) is a regular equilibrium. Hence, we can regard $Q^1(x^*)D_x\pi(x^*, y^*)$ as an operator from $T\Delta^1(x^*)_0$ to $T\Delta^1(x^*)_0$. As an operator on $T\Delta^1(x^*)_0$, $Q^1(x^*)$ is positive definite. Let the cardinality of supp(x^*) be k. Hence, by lemma 5.3, all the $(k-1)$ eigenvalues of $Q^1(x^*)D_x\pi(x^*, y^*)$ will have positive real parts. Hence, the trace of $Q^1(x^*)D_x\pi(x^*, y^*)$ is positive.

Next, we consider $Q(x^*, y^*)\,D(x^*, y^*) : T\Delta^1(x^*)_0\times T\Delta^2(y^*)_0 \to T\Delta^1(x^*)_0\times T\Delta^2(y^*)_0$. Since $D_yC(x^*, y^*) = 0$, the trace of $Q(x^*, y^*)\,D(x^*, y^*)$ is equal to the trace of $Q^1(x^*)\,D_x\pi(x^*, y^*)$, the latter regarded as an operator on $T\Delta^1(x^*)_0$. Hence, the trace of $Q(x^*, y^*)\,D(x^*, y^*)$ must also be positive. But this means that $Q(x^*,y^*)\,D(x^*,y^*)$ has at least one eigenvalue with positive real part. Let this eigenvalue be $\overline{\lambda}$ with real part $\overline{\lambda}^R > 0$.

If λ_1 is an eigenvalue of $Q(x^*,y^*)\,D(x^*,y^*)$ as an operator on $T\Delta^1(x^*)_0 \times T\Delta^2(y^*)_0$, then it is also an eigenvalue of $Q(x^*,y^*)\,D(x^*,y^*) : T\Delta \to T\Delta$, which therefore has an eigenvalue with positive real part.

$Q(x^*,y^*)\,D(x^*,y^*)$, as an operator on $T\Delta$, therefore has an eigenvalue $\overline{\lambda}$ with real part $\overline{\lambda}^R > 0$. Part 2 of Assumption 5.5 implies that for small η, the eigenvalues of $Q(\tilde{x}_\eta,\tilde{y}_\eta)D(\tilde{x}_\eta,\tilde{y}_\eta) : T\Delta \to T\Delta$ are close to the eigenvalues of $Q(x^*,y^*)\,D(x^*,y^*)$. Hence, by an argument similar to that in proposition 5.7, we can conclude that $\frac{1}{\eta}Q(\tilde{x}_\eta,\tilde{y}_\eta)D(\tilde{x}_\eta,\tilde{y}_\eta)$ has an eigenvalue greater than one if η is sufficiently small. This completes the proof. ■

F Cycling

To prove lemma 6.3, we denote the normal form representation of $\widehat{\pi}(x)$ as $\widehat{A}$. Thus, $\widehat{\pi}(x) = \widehat{A}x$

Proof. **Lemma 6.3**

Consider a mixed equilibrium x^*. Let $\widehat{A}$ be the normal form matrix of the game. By positive definiteness of the game, $(x-x^*)\widehat{A}\,(x-x^*) > 0$, $\forall x \neq x^*$. Let x be such that if $x_i^* = 0$, then $x_i = 0$. Then, $(x-x^*)\widehat{A}\,(x-x^*) = (x-x^*)\widehat{A}\,x > 0$. Take $x = e_j$ for some j in the support of x^*. Since the diagonal elements of $\widehat{A}$ are zero, $e_j\widehat{A}e_j = 0$ which implies $x^*\widehat{A}e_j < 0$. This implies $x^*\widehat{A}x^* < 0$. Hence, $W(x^*) < 0$.

Let p_i be price $\frac{i}{n}$. The payoff from δ_i is 0. On the other hand, the payoff from p_{i-1} given δ_i is $p_{i-1} + p_{i-1}y_2 - p_i$. Since $p_i - p_{i-1} = \frac{1}{n}$, it can easily be shown that if $i > \frac{1}{y2} + 1$, then $\widehat{\pi}_{i-1}(\delta_i) > \widehat{\pi}_i(\delta_i)$. For such prices, $W(\delta_i) > 0$. For prices less than $\frac{1}{y_2} + 1$, we need to make n sufficiently large such that $\frac{\frac{1}{y_2}+1}{n} < \underline{p}$. This ensures that such prices are dominated by 1. Then, $W(\delta_i) = y_1 > 0$. ■

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