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Abstract

We study mechanism design in quasi-linear environments when there are two alternatives. We show that under a mild range condition, every implementable deterministic allocation rule is a *generalized utility function maximizer*. In unbounded domains, if we replace our range condition by an *independence* condition, then every implementable deterministic allocation rule is an affine maximizer. Our results extend Roberts' affine maximizer theorem (Roberts, 1979) to the case of two alternatives.

Keywords: Roberts theorem; dominant strategy mechanism design; affine maximizer; generalized utility function maximizer

JEL Classifications: D02, D04, D44, D71

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1 Introduction

This paper considers dominant strategy implementation in quasi-linear environments with two alternatives, e.g. bilateral trading, provision of a public good, choosing one out of two locations for locating a facility or any situation with a status-quo alternative and a new alternative. The private information of each agent is a two dimensional vector, representing the valuation (a real number) for each alternative. Given the reported valuations of agents, an allocation rule chooses an alternative and a payment rule determines the payments of each agent. The net utility of each agent is quasi-linear in the payment he makes. An allocation rule is implementable (in dominant strategies) if there is a payment rule which makes truth-telling a weakly dominant strategy for each agent. We answer the following fundamental question in our model.

Which allocation rules are implementable?

We offer three main results [1].

1. Under a mild range condition, we show that a deterministic (no randomization) allocation rule is implementable if and only if it is a *generalized utility function (GUF) maximizer*. At every valuation profile, a GUF of an agent translates his valuation vector to a pair of real numbers, which we call his *generalized utilities* for these two alternatives at this valuation profile. At every valuation profile, a GUF maximizer allocation rule chooses an alternative that maximizes the sum of generalized utilities of agents.

2. Our second result shows that an implementable deterministic allocation rule satisfying an *independence* condition is an *affine maximizer*. Affine maximizer allocation rules, introduced in Roberts (1979), are generalizations of the efficient allocation rule. They can be thought of as *linearized* GUF maximizer allocation rules.

   Conversely, every affine maximizer satisfies our independence condition. It is well known that under a mild condition, an affine maximizer is implementable.

   To prove this result, we prove another result, which is of independent interest. We show that if a deterministic implementable allocation rule satisfies *unanimity* and *transitivity* [2] in our model, then it must be a *weighted efficient allocation rule*. Weighted efficient

[1] Most of our results require some richness of the domain. We discuss these specifics of the domain restrictions later in the paper.

[2] Unanimity requires that if valuation of every agent for an alternative is larger than the other alternative, then the higher valuation alternative must be the outcome of the allocation rule. Transitivity requires that

allocation rules are a special class of affine maximizer allocation rules.

Conversely, every weighted efficient allocation rule satisfies unanimity and transitivity.

3. Finally, we explore the implication of randomization. We show that every implementable randomized allocation rule is a convex combination of two distinct implementable allocation rules, whereas every implementable deterministic allocation rule cannot be expressed as a convex combination of two distinct implementable allocation rules. Mathematically, the set of implementable deterministic allocation rules constitute the set of "extreme points" of the set of implementable allocation rules.

Though a mechanism design problem with two alternatives seem far-fetched, many well-studied problems fall into this category. First, the bilateral trading problem is a problem with two agents (a buyer and a seller) and two alternatives - trade or no trade. Second, the non-excludable public good provision problem is a problem with two alternatives - whether to provide the public good or not. Since the valuation for the status-quo alternative (no trade in the case of bilateral trading problem and not providing the public good in case of public good provision problem) is zero in these problems, the private information of each agent is uni-dimensional here. However, there are two-dimensional problems where our results can be applied. For instance, consider the problem of locating a facility in one of two locations. Each agent has a two-dimensional valuation vector representing his valuation for each location. With sufficient richness in domain, all our results can be applied to this problem to identify the set of implementable allocation rules. Our results can also be applied to some extensions of classical bilateral trading problem and public good provision problem. The classical versions of these problems assume that the "status-quo" alternative (no trade in the case of bilateral trading and not providing the public good in the case of public good provision) has zero valuation for all the agents. Our model of two alternatives can allow agents to have non-zero private valuation for such a status-quo alternative.

1.1 Relation to the Literature

The pursuit of identifying the set of implementable allocation rules in voting models goes back to the seminal work of Gibbard (1973) and Satterthwaite (1975), who establish that dictatorship is the only implementable deterministic allocation rule under a mild range condition with unrestricted domain, when there are at least three alternatives. In quasi-linear environments, the analogue of the Gibbard-Satterthwaite theorem is due to Roberts (1979). In a remarkable result, Roberts (1979) showed that under a mild range condition, every

outcomes at three valuation profiles which are linked in a certain way must be transitive in some sense.

implementable deterministic allocation rule is an *affine maximizer* if there are at least three alternatives and the domain of valuations for each alternative is unrestricted. It is well known that an affine maximizer is implementable using generalized Groves payment rules (Vickrey, 1961; Clarke, 1971; Groves, 1973) if it satisfies a mild tie-breaking condition.

When the domain of valuations is restricted or the number of alternatives is two, Roberts' affine maximizer theorem is no longer true, and the set of implementable deterministic allocation rules is significantly enlarged. However, there has been very little progress in understanding the extensions of Roberts' theorem in restricted domains of valuations or in problems with two alternatives. We note some exceptions. Jehiel et al. (2008) show that Roberts' theorem extends to certain environments with interdependent valuations. Mishra and Sen (2012) show that in multidimensional open interval domains, every *neutral* and implementable deterministic allocation rule is a weighted efficient allocation rule if the number of alternatives is at least three.

Carbajal et al. (2012) show that if the domain of valuation profiles is restricted to the space of continuous functions defined on a topological space, or the space of piecewise linear functions defined on an affine space, or the space of smooth functions defined on a compact differentiable manifold, then a deterministic allocation rule is implementable if and only if it is a *lexicographic affine maximizer*. Their results do not require the set of alternatives to be finite. Lexicographic affine maximizers, which are defined recursively, are generalizations of affine maximizer allocation rules. Thus, they generalize Roberts' theorem to a restricted environment. Lexicographic affine maximization does not require the number of alternatives to be at least three. However, when the number of alternatives is two, lexicographic affine maximization in Carbajal et al. (2012) is a *monotonicity* condition (or equivalently a cutoff in differences condition). This monotonicity is similar to the monotonicity condition used to characterize implementability in the single object auction setting (Myerson, 1981). Further, this is equivalent to the 2-cycle monotonicity condition widely used in the multidimensional mechanism design literature (Bikhchandani et al., 2006; Saks and Yu, 2005; Ashlagi et al., 2010).

We show that such a monotonicity condition on allocation rule is necessary and sufficient for implementability even when the allocation rule is a randomized allocation rule, and in *any* arbitrary domain of valuation vectors. Our main results use this result as a building block.

The difference between the "monotonicity" characterizations and the "maximization" characterizations (a la Roberts (1979) and our GUF maximization) is significant. A monotonicity characterization will say that for every agent and for every valuation vector of other agents, the allocation rule must be "monotone" in some sense when the valuation vector of this agent is changed. On the other hand, a maximization characterization is more explicit. It tells you the exact parameters that define an implementable deterministic allocation rule. Thus, it is a direct prescription for designing a dominant strategy mechanism.

Because of this reason, there have been several attempts at simplifying the proof in Roberts' theorem - Lavi et al. (2009); Dobzinski and Nisan (2009); Vohra (2011). Dobzinski and Nisan (2011) show that in combinatorial auction domains (a restricted domain) involving two agents, there are non-affine maximizer deterministic allocation rules which give good approximation to efficiency. However, they do not provide any general characterization result (except for a specific case of auction of two goods among two agents).

Our result on showing that the set of *extreme points* of implementable allocation rules is the set of deterministic implementable allocation rules is analogous to a result proved in Manelli and Vincent (2007). They show that in the single object auction with *one* agent, every randomized implementable allocation rule is a convex combination of two distinct implementable allocation rules - they state their results in terms of net utility of the agent instead of allocation probabilities. We show that this result holds in our model with two alternatives and any (finite) number of agents. Notice that a single object auction with one agent is also a model of two alternatives (the agent gets the object or does not get the object), where the valuation for one of the alternatives (where the agent does not get the object) is always zero. Hence, we generalize the result in Manelli and Vincent (2007) to a two-dimensional model with arbitrary number of agents.

Some specific models with two alternatives have been studied extensively in the literature. We review them below.

- One such model is the bilateral trading model, where there is one buyer and one seller who want to trade a good (owned by the seller). Myerson and Satterthwaite (1983) showed that Bayes-Nash implementation, budget-balance, efficiency, and individual rationality are incompatible in bilateral trading. Hagerty and Rogerson (1987) showed that the only mechanisms which are *dominant strategy incentive compatible, budget-balanced, and individually rational* are *posted-price* mechanisms.

  Our GUF maximizer result applies to the bilateral trading model. Indeed, our results can be applied to the bilateral trading models where the no-trade alternative (outside option) also has some non-zero value (which can be a private information of the agents). Further, our characterizations are of implementable allocation rules and not mechanisms (allocation rule and payments). Thus, we do not impose additional properties like budget-balance and individually rationality, which are all properties of payments.

- Another model with two alternatives is the public good provision problem, where a planner is deciding whether to provide the public good or not. An excellent treatment of this problem is given in Borgers (2010) - see also Güth and Hellwig (1986). Like in the bilateral trading problem, our results can be applied to this problem. Our results are applicable even if agents have private valuation for the status quo alternative. Unlike the literature, where the focus has been to find incentive compatible *mechanisms* satisfying additional properties like budget-balance, individual rationality etc., our results characterize implementable *allocation rules*.

We will like to note that in the voting model of Gibbard (1973) and Satterthwaite (1975), the implications of having only two alternatives on *strategy-proofness* is well known (Fishburn and Gehrlein, 1977) - see also the surveys of Moulin (1983) and Barbera (2011). The strategy-proof rules identified in this voting model continue to be implementable in our model. However, these allocation rules are *ordinal rules* - the ordinal ranking of alternatives, and not their cardinal valuations, matter. The range condition we use in our main characterization and the independence condition we use in our affine maximizer characterization excludes such ordinal allocation rules. Hence, the allocation rules we identify in this paper do not capture the strategy-proof allocation rules identified in the voting model.

Finally, though we characterize implementable allocation rules, we can use revenue equivalence to pin down the class of payments in our model. This allows us to describe the entire class of incentive compatible *mechanisms*.

2 The Model and a Preliminary Result

The set of agents is $N := \{1, \ldots, n\}$. There are exactly two alternatives: a_1 and a_2. The set of alternatives is denoted by $A := \{a_1, a_2\}$. Each agent $i \in N$ has a valuation for each alternative, and this is denoted as $v_i(a_j)$ for every $j \in \{1, 2\}$. A valuation vector for agent i is denoted as v_i. For any agent $i \in N$, let V_i denote the set of all valuation vectors for agent i. A valuation profile is denoted as $v := (v_1, \ldots, v_n)$ and the set of all valuation profiles is $V := V_1 \times \ldots \times V_n$. We will use the standard notations v_{-i} to denote a valuation profile of agents other than agent i and V_{-i} to denote the set of all such valuation profiles.

An **allocation rule** is a mapping $f : V \to [0, 1]$, where for every $v \in V$, $f(v)$ denotes the probability with which alternative a_1 is chosen and $1 - f(v)$ denotes the probability with which alternative a_2 is chosen. For convenience, we will denote $f(v)$ as $f_1(v)$ and $(1 - f(v))$ as $f_2(v)$. A **deterministic allocation rule** f is an allocation rule such that $f(v) \in \{0, 1\}$ for every $v \in V$. A **payment rule** of agent i is a mapping $p_i : V \to \mathbb{R}$.

2.1 Implementable Allocation Rules

An allocation rule f is **(dominant strategy) implementable** if there exists payment rules $p_1, \dots, p_n$ such that for every agent $i \in N$ and for every $v_{-i} \in V_{-i}$ the following inequality holds for every $v_i, v'_i \in V_i$,

$$\sum_{k=1,2} v_i(a_k) f_k(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq \sum_{k=1,2} v_i(a_k) f_k(v'_i, v_{-i}) - p_i(v'_i, v_{-i}).$$

In this case, we say that the payment rules $p_1, \dots, p_n$ implement f. A **mechanism** is an allocation rule f and payment rules $(p_1, \dots, p_n)$. A mechanism $M \equiv (f, p_1, \dots, p_n)$ is **incentive compatible** if $(p_1, \dots, p_n)$ implement f.

For every agent $i \in N$ and for any valuation vector $v_i \in V_i$, define $\partial v_i := v_i(a_1) - v_i(a_2)$.

Definition 1 *An allocation rule f is monotone if for every $i \in N$, for every $v_{-i} \in V_{-i}$, and for every $v_i, v'_i \in V_i$, if $\partial v_i > \partial v'_i$, then $f_1(v_i, v_{-i}) \geq f_1(v'_i, v_{-i})$.*

The following preliminary result characterizes implementable allocation rules. We use this result to prove our main results. The proof is in the Appendix.

Proposition 1 *An allocation rule is implementable if and only if it is monotone.*

The monotonicity condition we use to characterize implementability in Proposition 1 is equivalent to the well-known *2-cycle monotonicity*. It is a folklore that such monotonicity is necessary and sufficient for implementability in one-dimensional value models such as single object auctions (Myerson, 1981). Though agents have two-dimensional values in our model, what matters for implementability is their difference of value between the two alternatives. This ensures that monotonicity is still necessary and sufficient in our model.

Carbajal et al. (2012) show that for *deterministic* allocation rules, monotonicity is equivalent to implementability in our model. Proposition 1 shows that it holds for non-deterministic allocation rules as well (without any restriction on valuations).

3 Deterministic Implementation

We present our main result in this section. Our focus is on deterministic allocation rules. We give a characterization of implementable deterministic allocation rules under a mild condition. Before presenting this characterization, we discuss Roberts' affine maximizer theorem (Roberts, 1979).

3.1 Roberts' Affine Maximizers

In this subsection we let A to be any finite set of alternatives, and do not put the restriction that $|A| = 2$. An allocation rule f is an **affine maximizer** if there exists non-negative real numbers $\lambda_1, \ldots, \lambda_n$ and a mapping $\gamma : A \to \mathbb{R}$ such that at every valuation profile v, we have

$$f(v) \in \arg\max_{a \in A} \big[\sum_{i \in N} \lambda_i v_i(a) + \gamma(a)\big]$$

An affine maximizer allocation rule f with weights $\lambda_1, \ldots, \lambda_n \geq 0$ and $\gamma : A \to \mathbb{R}$ satisfies **unresponsiveness to irrelevant agents (UIA)** if for every $i \in N$ such that $\lambda_i = 0$, we have $f(v_i, v_{-i}) = f(v'_i, v_{-i})$ for every $v_{-i} \in V_{-i}$ and for every $v_i, v'_i \in V_i$. It is well known that an affine maximizer which satisfies UIA can be implemented using generalized Groves (Groves, 1973) payment rules - see for instance Mishra and Sen (2012).

Note that in the definition of an affine maximizer, we can choose, without loss of generality, λ_i for all $i \in N$ such that $\sum_{i \in N} \lambda_i = 1$ if $\lambda_i > 0$ for some $i \in N$. We call such an affine maximizer a *responsive* affine maximizer. Roberts (1979) showed that if $|A| \geq 3$ and $V_i = \mathbb{R}^{|A|}$ for all $i \in N$, then every *onto* deterministic implementable allocation rule is a responsive affine maximizer. To remind, an allocation rule f is onto if for every $a \in A$, there exists a valuation profile $v \in V$ such that $f(v) = a$.

Hence, Roberts (1979) almost characterizes the set of deterministic implementable allocation rules in unrestricted domains (i.e., when $V_i = \mathbb{R}^{|A|}$ for all $i \in N$) and when $|A| \geq 3$.

Example **1**

Roberts' affine maximizer theorem is no longer true if $|A| = 2$. For instance, consider the following allocation rule $\bar{f}$ with two agents $\{1, 2\}$ and $V_1 = V_2 = \mathbb{R}^2$. For every $v \in V$,

$$\bar{f}(v) = \begin{cases} a_1 & \text{if } (\partial v_1)^3 + \partial v_2 \geq 0 \\ a_2 & \text{if } (\partial v_1)^3 + \partial v_2 < 0. \end{cases}$$

It is easy to verify that $\bar{f}$ is monotone, and hence, implementable by Proposition 1. But $\bar{f}$ is not an affine maximizer. Next, we provide a characterization of deterministic implementable allocation rules extending Roberts' affine maximizer theorem. Our characterization covers allocation rules of the form $\bar{f}$.

3.2 Generalized Utility Function Maximizers

The main tool of our characterization is the notion of a *generalized utility function*.

Definition 2 *A* **generalized utility function (GUF)** *is a mapping* $u : A \times V \to \mathbb{R}$ *for all* $v \in V$.

We associate a GUF with every agent. The GUF associated with agent i is denoted by u_i. At any $v \in V$, let

$$\partial u_i(v) = u_i(a_1, v) - u_i(a_2, v).$$

In other words, $\partial u_i(v)$ denotes the difference in "generalized utility" of agent i at valuation profile v. We concentrate on a particular class of GUFs.

Definition 3 *A GUF* u_i *of agent* i *is* **strictly monotone** *if*

1. *for every* $v_{-i} \in V_{-i}$, *for every* $v_i, v'_i \in V_i$ *with* $\partial v_i > \partial v'_i$, *we have*

$$\partial u_i(v_i, v_{-i}) > \partial u_i(v'_i, v_{-i}).$$

2. *for every* $j \neq i$, *for every* $v_{-j} \in V_{-j}$, *and every* $v_j, v'_j \in V_j$ *with* $\partial v_j > \partial v'_j$, *we have*

$$\partial u_i(v_j, v_{-j}) \geq \partial u_i(v'_j, v_{-j}).$$

Using the notion of GUFs, we define a broad class of allocation rules. Abusing notation, we will now let a deterministic allocation rule be a map $f : V \to A$, i.e., $f(v) \in A$ for all $v \in V$.

Definition 4 *An allocation rule* f *is a GUF maximizer if there exist strictly monotone GUFs* $(u_1, \ldots, u_n)$ *such that for all* $v \in V$, *we have*

$$f(v) \in \arg\max_{a \in A} \sum_{i \in N} u_i(a, v).$$

In this case, we say that f *is representable by* $(u_1, \ldots, u_n)$.

We now show that every GUF maximizer is implementable.

Lemma 1 *Every GUF maximizer allocation rule is implementable.*

Proof: Consider a GUF maximizer allocation rule f, and suppose f is representable by $(u_1, \ldots, u_n)$. Fix an agent i and $v_{-i} \in V_{-i}$. Consider $v_i, v'_i \in V_i$ such that $\partial v_i > \partial v'_i$. Suppose $f(v'_i, v_{-i}) = a_1$. Then, by definition of f, we have

$$\sum_{j \in N} u_j(a_1, v'_i, v_{-i}) \geq \sum_{j \in N} u_j(a_2, v'_i, v_{-i}).$$

Hence, we get that

$$\sum_{j \in N} \partial u_j(v_i', v_{-i}) \geq 0.$$

By strict monotonicity, $\partial u_i(v_i, v_{-i}) > \partial u_i(v_i', v_{-i})$ and $\partial u_j(v_i, v_{-i}) \geq \partial u_j(v_i', v_{-i})$ for all $j \neq i$. Hence, we get

$$\sum_{j \in N} \partial u_j(v_i, v_{-i}) > 0.$$

This implies that

$$\sum_{j \in N} u_j(a_1, v_i, v_{-i}) > \sum_{j \in N} u_j(a_2, v_i, v_{-i}).$$

By the definition of GUF maximizer, $f(v_i, v_{-i}) = a_1$. Since f is deterministic, this shows that f is monotone, and by Proposition 1, f is implementable. ■

Our main result shows that under a mild range condition, GUF maximizers are the only implementable deterministic allocation rules.

Definition 5 *A deterministic allocation rule f satisfies* **agent sovereignty** *if for every agent $i \in N$, every $v_{-i} \in V_{-i}$, and every $a \in A$, there is a $v_i \in V_i$ such that $f(v_i, v_{-i}) = a$. A deterministic allocation rule f satisfies* **weak agent sovereignty** *if for every agent $i \in N$, every $v_{-i} \in V_{-i}$, there is a $v_i \in V_i$ such that $f(v_i, v_{-i}) = a_1$.*

Agent sovereignty requires every agent to have some decisive power irrespective of the values of other agents. It has been used extensively in public good provision problems (Moulin, 1999; Moulin and Shenker, 2001). Lavi et al. (2009) use agent sovereignty [3] to give a clean proof of Roberts' affine maximizer theorem (Roberts, 1979). In many settings, agent sovereignty is a consequence of optimizing payments. For instance, Masso et al. (2011) consider the model of choosing a binary excludable public good. They show that any mechanism that minimizes the *maximal welfare loss* in their model must involve an allocation rule which satisfies agent sovereignty.

For every $i \in N$, define $D_i := \{\partial v_i : v_i \in V_i\}$. Note that $D_i \subseteq \mathbb{R}$. Throughout, we will make the assumption that D_i is an interval.

Theorem 1 *Let f be a deterministic allocation rule. Suppose one of the following conditions hold:*

Ca *f satisfies agent sovereignty and for every $i \in N$, D_i is an interval.*

[3] What we call agent sovereignty, Lavi et al. (2009) refer to it as *player decisiveness.*

CB f *satisfies weak agent sovereignty and for every* $i \in N$, D_i *is an interval bounded from below.*

Then, f *is implementable if and only if it is a GUF maximizer.*

The natural domains where condition CA and CB can be satisfied are *product interval domains*. Denote by V_i^a the set of possible valuations on alternative $a \in A$ for agent $i \in N$. Let $V_i = V_i^{a_1} \times V_i^{a_2}$. Condition CA holds if f satisfies agent sovereignty and V_i^a is an interval for every $a \in A$. Condition CB holds if f satisfies weak agent sovereignty and $V_i^{a_1}$ and $V_i^{a_2}$ are intervals, and $V_i^{a_1}$ is bounded from below (for instance $\mathbb{R}_+$) and $V_i^{a_2}$ is bounded from above (for instance any compact interval). These domain restrictions cover the classical problems of bilateral trading, public good provision, and their extensions.

3.3 Proof of Theorem 1

Before proving Theorem 1, we establish some claims. Suppose f is an implementable deterministic allocation rule. Then, for every $i \in N$ and every $v_{-i} \in V_{-i}$, define $d_i^f(v_{-i})$ as follows:

$$d_i^f(v_{-i}) = \inf\{\partial v_i \in D_i : f(v_i, v_{-i}) = a_1\}.$$

We prove a series of claims. In each claim, we assume that f is a deterministic implementable allocation rule. Further, f satisfies agent sovereignty and for every $i \in N$, D_i is an interval.

The first claim shows when $d_i^f(v_{-i})$ is well defined for every $i \in N$ and for every $v_{-i} \in V_{-i}$.

CLAIM **1** *For every* $i \in N$ *and for every* $v_{-i} \in V_{-i}$, $d_i^f(v_{-i})$ *is a real number.*

Proof: Fix agent i and $v_{-i} \in V_{-i}$. Under conditions (CA) or (CB), there is some value $v_i \in V_i$ such that $f(v_i, v_{-i}) = a_1$.

If condition (CA) holds, then for some v_i', $f(v_i', v_{-i}) = a_2$. Since f is implementable, it is monotone (Proposition 1). Hence, $\partial v_i' \leq \partial v_i$. Since D_i is an interval, we get that $\inf\{\partial v_i \in D_i : f(v_i, v_{-i}) = a_1\}$ is a real number.

If condition (CB) holds, then since D_i is an interval bounded below, $\inf\{\partial v_i \in D_i : f(v_i, v_{-i}) = a_1\}$ is a real number. ■

We now define a payment rule. For every agent $i \in N$, define p_i^f as follows:

$$p_i^f(v_i, v_{-i}) = \begin{cases} 0 & \text{if } f(v_i, v_{-i}) = a_2 \\ d_i^f(v_{-i}) & \text{if } f(v_i, v_{-i}) = a_1. \end{cases}$$

These payments are counterparts of Myerson's cutoff-based payments for single object auction (Myerson, 1981).

Claim **2** *The payment rule $(p^f_1,\dots,p^f_n)$ implements f.*

Proof: Fix an agent $i \in N$ and $v_{-i} \in V_{-i}$. Consider $v_i, v'_i \in V_i$. We will show that

$$v_i(f(v_i,v_{-i}) - p^f_i(v_i,v_{-i}) \geq v_i(f(v'_i,v_{-i})) - p^f_i(v'_i,v_{-i}).$$

If $f(v_i,v_{-i}) = f(v'_i,v_{-i})$, we are done. So, assume that $f(v_i,v_{-i}) \neq f(v'_i,v_{-i})$. We consider two cases.

Case 1. Suppose $f(v_i,v_{-i}) = a_1$ and $f(v'_i,v_{-i}) = a_2$. Then, $v_i(f(v_i,v_{-i})) - p^f_i(v_i,v_{-i}) = v_i(a_1) - d^f_i(v_{-i})$. Since $d^f_i(v_{-i}) \leq \partial v_i$, we get that $v_i(a_1) - d^f_i(v_{-i}) \geq v_i(a_2) = v_i(f(v'_i,v_{-i})) - p^f_i(v'_i,v_{-i})$, where we used the fact that $p^f_i(v'_i,v_{-i}) = 0$ since $f(v'_i,v_{-i}) = a_2$.

Case 2. Suppose $f(v_i,v_{-i}) = a_2$ and $f(v'_i,v_{-i}) = a_1$. We argue that $\partial v_i \leq d^f_i(v_{-i})$. Assume for contradiction that $\partial v_i > d^f_i(v_{-i})$. By definition of $d^f_i(v_{-i})$, there is v''_i such that $f(v''_i,v_{-i}) = a_1$ and $\partial v''_i$ is arbitrarily close to $d^f_i(v_{-i})$. Hence, $\partial v_i > \partial v''_i$. Then, since f is monotone, $f(v_i,v_{-i}) = a_1$, which is a contradiction.

Hence, $v_i(a_2) \geq v_i(a_1) - d^f_i(v_{-i})$. Using the fact that $p^f_i(v_i,v_{-i}) = 0$ since $f(v_i,v_{-i}) = a_2$ and $p^f_i(v'_i,v_{-i}) = d^f_i(v_{-i})$ since $f(v'_i,v_{-i}) = a_1$, we get that $v_i(f(v_i,v_{-i})) - p^f_i(v_i,v_{-i}) = v_i(a_2) \geq v_i(a_1) - d^f_i(v_{-i}) = v_i(f(v'_i,v_{-i})) - p^f_i(v'_i,v_{-i})$. ■

Claim 2 has other implications. If V_i is connected for each $i \in N$, then by well-known results on revenue equivalence (Heydenreich et al., 2009), we can conclude that any other payment rule p_i of agent i must look as follows: $p_i(v) = p^f_i(v) + h_i(v_{-i})$ for all $v \in V$, where $h_i : V_{-i} \to \mathbb{R}$ is any function.

The next claim shows a monotonicity property of $d^f_i(\cdot)$ for every $i \in N$.

Claim **3** *For every i, for every $j \neq i$, for every $v_j, v'_j \in V_j$ such that $\partial v_j < \partial v'_j$, we have that $d^f_i(v_j,v_{-ij}) \geq d^f_i(v'_j,v_{-ij})$ for all $v_{-ij} \in V_{-ij}$.*

Proof: Fix agents i and $j \neq i$, and consider $v_j, v'_j \in V_j$ such that $\partial v_j < \partial v'_j$. Assume for contradiction that $d^f_i(v_j,v_{-ij}) < d^f_i(v'_j,v_{-ij})$ for some $v_{-ij} \in V_{-ij}$. Let v_i be such that $\partial v_i = d^f_i(v_j,v_{-ij}) + \epsilon < d^f_i(v'_j,v_{-ij})$ for some sufficiently small $\epsilon > 0$. Since D_i is an interval, such a v_i exists. By definition, $f(v_i,v_j,v_{-ij}) = a_1$ and $f(v_i,v'_j,v_{-ij}) = a_2$. But $\partial v_j < \partial v'_j$ means $f(v_i,v'_j,v_{-ij}) = a_1$ by monotonicity. This is a contradiction. ■

This leads us to the proof of Theorem 1.

Proof: Lemma 1 shows that every GUF maximizer is implementable. We prove the converse. Let f be an implementable deterministic allocation rule, and suppose (sc Ca) or (Cb) holds. For every $i \in N$, define the GUF of agent i as follows: for every $(v_i, v_{-i}) \in V$ let

$$u_i(a, v_i, v_{-i}) = \begin{cases} 0 & \text{if } a = a_2 \\ \partial v_i - d_i^f(v_{-i}) & \text{if } a = a_1. \end{cases}$$

By Claim 1, the GUFs are well-defined. We show that for any $i \in N$, u_i is strictly monotone. By definition, for every v_{-i}, $u_i(a_1, v) = \partial v_i - d_i^f(v_{-i}) > \partial v_i' - d_i^f(v_{-i})$ if $\partial v_i > \partial v_i'$. Now, fix any $j \neq i$ and consider v_j, v_j' such that $\partial v_j > \partial v_j'$. By Claim 3, $u_i(a_1, v_j, v_{-j}) = \partial v_i - d_i^f(v_j, v_{-ij}) \geq \partial v_i - d_i^f(v_j', v_{-ij}) = u_i(a_1, v_j', v_{-j})$.

Now, consider any $v \in V$ and suppose $f(v) = a_1$. Then, by definition, for every $i \in N$, $d_i^f(v_{-i}) \leq \partial v_i$. Hence, $u_i(a_1, v) \geq u_i(a_2, v)$, which implies that $\sum_{j \in N} u_j(a_1, v) \geq \sum_{j \in N} u_j(a_2, v)$. Similarly, suppose $f(v) = a_2$. Then, for every $i \in N$, since f satisfies agent sovereignty, there is a v_i' such that $f(v_i', v_{-i}) = a_1$. But, by Claim 2, $v_i(a_2) - 0 \geq v_i(a_1) - d_i^f(v_{-i})$. Hence, $u_i(a_2, v) \geq u_i(a_1, v)$, which implies that $\sum_{j \in N} u_j(a_2, v) \geq \sum_{j \in N} u_j(a_1, v)$. This shows that the f is representable by GUFs $(u_1, \ldots, u_n)$. ■

4 An Axiomatization of Affine Maximizers

Theorem 1 shows the rich class of "maximizers" that can be implemented when there are two alternatives. However, when we have more than two alternatives, we only get affine maximizers in unrestricted domains. Then, a natural question to ask is: *what extra condition(s) besides implementability are needed to pin down the affine maximizers when there are two alternatives?* This will help us understand the case of two alternatives even further.

The aim of this section is to *axiomatize* the affine maximizers for the case of $|A| = 2$ using implementability and some additional condition(s). It turns out, we only need one new condition besides implementability. To introduce the new condition, we will need some notation.

We will assume that the set of possible valuations of each for each alternative is an open interval. Hence, throughout this section, we will assume that for every $i \in N$, $V_i = L_i \times L_i$, where L_i is an open interval. We will call this the **open interval domain**. Notice that the valuation for every alternative lies in the same interval.

Given a profile of valuations $(v_1, \ldots, v_n)$, we will often be interested in the vector of valuations associated with each alternative. In particular, for $j \in \{1, 2\}$, let $v(a_j) \in \mathbb{R}^n$ denote the valuation vector associated with alternative a_j. Let U be the set of all valuation vectors for alternatives given our open interval domain assumption. Note that U is an open *rectangle* in $\mathbb{R}^n$. A profile of valuations contains exactly two valuation vectors from U, one denoting the valuations for alternative a_1 and the other denoting the valuations for alternative a_2. For convenience, we will denote the profile of valuations as $(v(a_1), v(a_2))$ instead of $(v_1, \ldots, v_n)$. Further, for every $a \in A$, we will sometimes write $(v(a), v(-a))$ to denote the profile of valuations $(v(a_1), v(a_2))$.

We are now ready to state our new condition.

DEFINITION **6** *An allocation rule f satisfies* **independence** *if for every $a \in A$ and for every pair of valuation profiles v, v' such that $f(v) = f(v') = a$, we have $f(v(a) + \epsilon, v'(-a)) = a$ or $f(v'(a) + \epsilon, v(-a)) = a$ for all $\epsilon \in \mathbb{R}^n_{++}$.*

An allocation rule can be thought of as *evaluating* valuation vectors for alternatives at every profile of valuations, and selecting one of the valuation vectors. The independence condition requires some consistency in such evaluations. To understand independence better, suppose the independence condition is not satisfied by an allocation rule f. Then, there is a pair of valuation profiles v and v' such that

$$f(v(a), v(-a)) = a \tag{1}$$

$$f(v'(a), v'(-a)) = a \tag{2}$$

and for some $\epsilon \in \mathbb{R}^n_{++}$

$$f(v(a) + \epsilon, v'(-a)) = b \tag{3}$$

$$f(v'(a) + \epsilon, v(-a)) = b. \tag{4}$$

Now, consider Equations 1 and 4. These equations show that in the presence of $v(-a)$, valuation vector $v(a)$ tilts the outcome in favor of a more than valuation vector $v'(a)$. However, Equations 2 and 3 show that in the presence of $v'(-a)$, $v'(a)$ tilts the outcome in favor of a more than $v(a)$. This is counterintuitive: the way $v(a)$ and $v'(a)$ compare to each other depends on the second argument of the allocation rule [4]. So, the role of Independence is precisely to make the comparison of $v(a)$ and $v'(a)$ independent of the second argument.

A similar condition is used in Debreu's theorem on the additive representation of a binary relation over a Cartesian product (see Theorem 3 in Debreu (1960)).

We show that an affine maximizer allocation rule satisfies independence.

[4] If we assume f to be implementable, then this is even more counterintuitive. To see, this if f is implementable, then it is monotone, and Equations 1 and 2 can be strengthened to say: $f(v(a)+\epsilon, v(-a)) = f(v'(a) + \epsilon, v'(-a)) = a$, where ϵ is as in Equations 3 and 4. This can now be directly compared with Equations 3 and 4 to see that the comparison of $v(a) + \epsilon$ and $v'(a) + \epsilon$ depends on the second argument of the allocation rule.

Lemma 2 *Every affine maximizer allocation rule satisfies independence.*

Proof: Let f be an affine maximizer allocation rule with weights $\lambda_1, \ldots, \lambda_n \geq 0$ and $\gamma : A \rightarrow \mathbb{R}$. Consider a pair of valuation profiles v, v' such that $f(v) = f(v') = a_1$ (the other case where $f(v) = f(v') = a_2$ can be dealt similarly). Then, affine maximization gives us

$$\sum_{i \in N} \lambda_i v_i(a_1) + \gamma(a_1) \geq \sum_{i \in N} \lambda_i v_i(a_2) + \gamma(a_2)$$
$$\sum_{i \in N} \lambda_i v'_i(a_1) + \gamma(a_1) \geq \sum_{i \in N} \lambda_i v'_i(a_2) + \gamma(a_2).$$

Adding these two inequalities gives us

$$\sum_{i \in N} \lambda_i \big[v_i(a_1) + v'_i(a_1)\big] + 2\gamma(a_1) \geq \sum_{i \in N} \lambda_i \big[v_i(a_2) + v'_i(a_2)\big] + 2\gamma(a_2). \tag{5}$$

$$\tag{6}$$

Now, assume for contradiction $f(v(a_1) + \epsilon, v'(a_2)) = a_2$ and $f(v'(a_1) + \epsilon', v(a_2)) = a_2$ for some $\epsilon, \epsilon' \in \mathbb{R}^n_{++}$. Then, f is a non-constant affine maximizer. Since $\epsilon, \epsilon' \in \mathbb{R}^n_{++}$, this implies that

$$\sum_{i \in N} \lambda_i v'_i(a_2) + \gamma(a_2) > \sum_{i \in N} \lambda_i v_i(a_1) + \gamma(a_1)$$
$$\sum_{i \in N} \lambda_i v_i(a_2) + \gamma(a_2) > \sum_{i \in N} \lambda_i v'_i(a_1) + \gamma(a_1).$$

Adding these two inequalities gives a contradiction to Inequality 5. ■

There are non-affine maximizer allocation rules which are implementable but do not satisfy independence. For instance, consider the implementable allocation rule $\bar{f}$ in Example 1. Suppose v is the valuation profile where $v_1(a_1) = 2, v_1(a_2) = 0, v_2(a_1) = 1, v_2(a_2) = 4$ and v' is the valuation profile where $v'_1(a_1) = 0, v'_1(a_2) = 1, v'_2(a_1) = v'_2(a_2) = 3$. By definition, $\bar{f}(v) = \bar{f}(v') = a_1$. Now, for sufficiently small $\epsilon \in \mathbb{R}^2_{++}$, it is easily verified that $\bar{f}(v(a_1) + \epsilon, v'(a_2)) = \bar{f}(v'(a_1) + \epsilon, v(a_2)) = a_2$. Hence, $\bar{f}$ does not satisfy independence.

We show that amongst the implementable deterministic allocation rules, only affine maximizers satisfy independence.

Theorem 2 *Suppose for every $i \in N$, L_i is an open interval unbounded from above. If a deterministic allocation rule is implementable and satisfies independence, then it is an affine maximizer. Conversely, if f is an affine maximizer, then it satisfies independence, and further, if it satisfies UIA, then it is implementable.*

The proof of Theorem 2 is in the Appendix. The proof uses another interesting result on axiomatizing *weighted efficiency*, which we state next.

4.1 An Axiomatization of Weighted Efficiency

Weighted efficiency is a particular form of affine maximizer. An allocation rule f is a **weighted efficient allocation rule** if there exists weights $\lambda_1, \ldots, \lambda_n \geq 0$ with $\lambda_i > 0$ for some $i \in N$, such that for every valuation profile $v \in V$, we have $f(v) \in \arg\max_{a \in A} \sum_{i \in N} \lambda_i v_i(a)$.

Among the class of affine maximizer allocation rules, weighted efficient allocation rules do not discriminate between alternatives. We will show that under some additional conditions, implementability will imply weighted efficiency. To define the additional conditions, we need some preparation. First, we introduce a well known monotonicity condition due to Roberts (1979).

Definition 7 *A deterministic allocation rule f satisfies* **positive association of differences (PAD)** *if for every pair of profile of valuations $v, v' \in V$ such that $\partial v_i > \partial v'_i$ for every $i \in N$ and $f(v') = a_1$ we have $f(v) = a_1$.*

Consider a pair of profile of valuations $v, v' \in V$ such that $\partial v_i < \partial v'_i$ for every $i \in N$ and $f(v') = a_2$. Note that PAD implies that $f(v) = a_2$. To see this, assume for contradiction $f(v) = a_1$. Then, applying PAD (interchanging the role of v and v' in above definition), we get that $f(v') = a_1$, a contradiction.

Roberts (1979) showed that PAD is a necessary condition for implementability.

Lemma 3 (**Roberts (1979)**) *If a deterministic allocation rule is implementable, then it satisfies PAD.*

It can be shown that monotonicity implies PAD, and hence, Lemma 3 is a direct consequence of Proposition 1.

Given a deterministic allocation rule f, define the **choice set** at a profile of valuations v as

$$C^f(v) = \{a \in A : f(v(a) + \epsilon, v(-a)) = a \;\forall\; \epsilon \in \mathbb{R}^n_{++}\}.$$

Since U is open, $C^f(v)$ is well defined for every profile of valuations v. Using PAD, one notices that if f is implementable, then $f(v) \in C^f(v)$ for every valuation profile $v \in V$. Hence, the choice set is non-empty. The choice set allows us to look at potential "candidates" other than $f(v)$ which could have been selected by the allocation rule f at valuation profile v.

We now introduce two new conditions on deterministic allocation rules. The first one is a transitivity requirement.

Definition 8 *A deterministic allocation rule f is* **transitive** *if for every $x, y, z \in U$, and every $v, v', v'' \in V$ such that $v(a_1) = x = v''(a_1)$, $v(a_2) = y = v'(a_1)$, and $v'(a_2) = z = v''(a_2)$, we have,*

- *if* $C^f(v) = \{a_1\}$ *and* $C^f(v') = \{a_1\}$, *then* $f(v'') = a_1$ *and*
- *if* $C^f(v) = \{a_2\}$ *and* $C^f(v') = \{a_2\}$, *then* $f(v'') = a_2$.

The next condition is unanimity, which is very similar in flavor to the unanimity axiom used in the social choice theory literature.

Definition 9 *A deterministic allocation rule* f *is* **unanimous** *if for every* $x, y \in U$ *such that* $x_i > y_i$ *for all* $i \in N$, *we have* $f(v) = a$ *if* $v(a) = x$ *and* $v(-a) = y$.

Theorem 3 *Suppose for every* $i \in N$, L_i *is an open interval. If* f *is a deterministic implementable allocation rule that is unanimous and transitive, then it is a weighted efficient allocation rule. Conversely, a weighted efficient allocation rule* f *is unanimous and transitive, and further, if it satisfies UIA, then it is implementable.*

The proof of Theorem 3 is in the Appendix. In Mishra and Sen (2012), it was shown that if the number of alternatives is at least three, then in open interval domains, every *neutral* and deterministic implementable allocation rule is a weighted efficient allocation rule. Neutrality requires that the allocation rule does not discriminate between alternatives. Theorem 3 is the counterpart of this result for the two alternatives case. The proof of Theorem 3 reveals that in the presence of transitivity, unanimity is equivalent to neutrality in our model. Hence, compared to Mishra and Sen (2012), the extra axiom required to characterize weighted efficiency in our two alternatives model is transitivity.

5 Randomization

In the previous section, we only discussed deterministic implementable allocation rules. However, the consequence of randomization is completely unexplored in Roberts' theorem - even when the number of alternatives is at least three, nothing is known about the set of implementable allocation rules when one allows for randomization. In this section, we make some progress in our model with two alternatives.

The following straightforward lemma establishes that the set of implementable allocation rules is a convex set.

Lemma 4 *Suppose* f *and* f' *are two implementable allocation rules, and let* $\lambda \in (0, 1)$. *Define another allocation rule* f'' *as* $f''(v) = \lambda f(v) + (1 - \lambda)f'(v)$ *for all* $v \in V$. *The allocation rule* f'' *is also implementable.*

Proof: Suppose $(p_1, \ldots, p_n)$ implement the allocation rule f and $(q_1, \ldots, q_n)$ implement f'. Define for every $i \in N$, $r_i(v) = \lambda p_i(v) + (1-\lambda) q_i(v)$ for all $v \in V$. Then, for every $i \in N$ and every v_{-i} we have for every $v_i, v_i' \in V$,

$$
\begin{aligned}
\sum_{k=1,2} f''_k(v_i, v_{-i}) v_i(a_k) + r_i(v_i, v_{-i}) &= \sum_{k=1,2} [\lambda f_k(v_i, v_{-i}) + (1-\lambda) f'_k(v_i, v_{-i})] v_i(a_k) \\
&+ \lambda p_i(v_i, v_{-i}) + (1-\lambda) q_i(v_i, v_{-i})) \\
&= \lambda \big[\sum_{k=1,2} f_k(v_i, v_{-i}) v_i(a_k) + p_i(v_i, v_{-i}) \big] \\
&+ (1-\lambda) \big[\sum_{k=1,2} f'_k(v_i, v_{-i}) v_i(a_k) + q_i(v_i, v_{-i}) \big] \\
&\geq \lambda \big[\sum_{k=1,2} f_k(v'_i, v_{-i}) v_i(a_k) + p_i(v'_i, v_{-i}) \big] \\
&+ (1-\lambda) \big[\sum_{k=1,2} f'_k(v'_i, v_{-i}) v_i(a_k) + q_i(v'_i, v_{-i}) \big] \\
&= \sum_{k=1,2} f''_k(v'_i, v_{-i}) v_i(a_k) + r_i(v'_i, v_{-i}).
\end{aligned}
$$

Hence, $(r_1, \ldots, r_n)$ implement f''. ∎

This leads to a natural definition of extreme point.

Definition 10 *An implementable allocation rule f'' is an* **extreme point allocation rule** *if there does not exist two distinct implementable allocation rules f and f', and $\lambda \in (0,1)$ such that $f''(v) = \lambda f(v) + (1-\lambda) f'(v)$ for all $v \in V$.*

It is clear that every implementable deterministic allocation rule is an extreme point allocation rule.

Lemma **5** *Every implementable deterministic allocation rule is an extreme point allocation rule.*

Proof: Let f'' be an implementable deterministic allocation rule. Assume for contradiction there exist distinct implementable allocation rules f and f', and $\lambda \in (0,1)$ such that $f''(v) = \lambda f(v) + (1-\lambda) f'(v)$ for all $v \in V$. Since f and f' are distinct, for some $v \in V$ and for some $k \in \{1,2\}$, we have $f_k(v) \neq f'_k(v)$. Then, $f''_k(v)$ lies between $f_k(v)$ and $f'_k(v)$. This contradicts the fact that f'' is deterministic. ∎

The main result of this section is that the converse of Lemma 5 is true in our model.

Theorem **4** *An implementable allocation rule is an extreme point allocation rule if and only if it is an implementable deterministic allocation rule.*

Proof: By Lemma 5, every implementable deterministic allocation rule is an extreme point allocation rule. Now, consider an implementable allocation rule f which is an extreme point allocation rule. Assume for contradiction that f is not deterministic. Consider the following function $g : V \rightarrow [0,1]$ defined as follows. For every $v \in V$,

$$g(v) = \begin{cases} f(v) & \text{if } f(v) \leq 0.5 \\ 1 - f(v) & \text{if } f(v) > 0.5. \end{cases}$$

Let f' and f'' be two allocation rules defined as follows. For all $v \in V$, let

$$f'(v) = f(v) + g(v)$$
$$f''(v) = f(v) - g(v).$$

Note that f' and f'' are well-defined. Further, since f is not deterministic, for some $v \in V$, we have $f(v) \in (0,1)$, and hence, $g(v) \neq 0$. This further implies that $f'(v) = f(v) + g(v) \neq f(v) - g(v) = f''(v)$. Hence, f' and f'' are two distinct allocation rules, and their *convex combination* yields f.

We will show that f' and f'' are implementable, and this will give us a contradiction to the fact that f is an extreme point. Fix an agent i and $v_{-i} \in V_{-i}$. Consider v_i and v'_i such that $\partial v_i > \partial v'_i$. Since f is implementable $f(v_i, v_{-i}) \geq f(v'_i, v_{-i})$ (by Proposition 1).

Consider the allocation rule f'. Suppose $f(v_i, v_{-i}) \leq 0.5$. Then, $f(v'_i, v_{-i}) \leq 0.5$. As a consequence, $f'(v_i, v_{-i}) = 2f(v_i, v_{-i}) \geq 2f(v'_i, v_{-i}) = f'(v'_i, v_{-i})$. Suppose $f(v_i, v_{-i}) > 0.5$. Then, consider the two possible cases. Suppose $f(v'_i, v_{-i}) > 0.5$. In this case, $f'(v_i, v_{-i}) = 1 = f'(v'_i, v_{-i})$. Suppose $f(v'_i, v_{-i}) \leq 0.5$. Then, $f'(v_i, v_{-i}) = 1 \geq 2f(v'_i, v_{-i}) = f'(v'_i, v_{-i})$. Hence, in all cases, we have $f'(v_i, v_{-i}) \geq f'(v'_i, v_{-i})$. This shows that f' is monotone, and hence, implementable by Proposition 1.

Now, consider the allocation rule f''. Suppose $f(v_i, v_{-i}) \leq 0.5$. Then, since f is monotone, $f(v'_i, v_{-i}) \leq 0.5$. As a consequence, $f''(v_i, v_{-i}) = 0 = f''(v'_i, v_{-i})$. Suppose $f(v_i, v_{-i}) > 0.5$. Then, consider the two possible cases. Suppose $f(v'_i, v_{-i}) > 0.5$. In this case, $f''(v_i, v_{-i}) = 2f(v_i, v_{-i}) - 1 \geq 2f(v'_i, v_{-i}) - 1 = f''(v'_i, v_{-i})$. Suppose $f(v'_i, v_{-i}) \leq 0.5$. In this case, $f''(v_i, v_{-i}) = 2f(v_i, v_{-i}) - 1 \geq 0 = f''(v'_i, v_{-i})$. This shows that f'' is monotone, and hence, implementable by Proposition 1. ■

The question we ask in Theorem 4 can be asked in general mechanism design setting (with arbitrary number of alternatives): When is an implementable allocation rule an extreme point allocation rule? Manelli and Vincent (2007) show the usefulness of such a result in multi-object auction setting. In particular, they show that finding revenue maximizing mechanisms subject to individual rationality constraint boils down to searching over extreme points of allocation rules, which they characterize in their model (for one agent and one object auction, these extreme points are deterministic allocation rules in their model). We leave such an analysis in our model for future research.

6 Conclusion

In quasi-linear private values environment, Roberts' affine maximizer theorem is a seminal contribution. Two crucial assumptions of this theorem are (a) there are at least three alternatives and (b) the domain of valuations is unrestricted. We extend this theorem by considering the case of two alternatives. Unlike the three or more alternatives result of Roberts (1979), which requires the domain of valuations to be unrestricted, our results for two alternatives hold in various restricted domains of valuations. An interesting future research direction will be to apply these results to specific problems with two alternatives, and do some optimization - for instance, revenue maximization or budget-balancing with minimal efficiency loss etc.

We consider implementation in dominant strategies. One can think of weakening the notion of equilibrium to Bayes-Nash. Gershkov et al. (2012) show that in one dimensional models, for every allocation rule that can be Bayes-Nash implemented, there is a corresponding allocation rule that can be implemented in dominant strategies such that the *interim allocation probabilities* of each alternative is the same in both the allocation rules. This result is true in models with two alternatives also - see Gershkov et al. (2011). This shows that our restriction to dominant strategy implementation is not a significant restriction.

Finally, the notion of a GUF maximizer can be extended to environments with more than two alternatives also. With suitable restrictions on GUFs, one can make a GUF maximizer implementable in such environments. However, an open question remains whether such GUF maximizers are the only implementable allocation rules (under some additional mild conditions) in such environments.

References

Ashlagi, I., M. Braverman, A. Hassidim, and D. Monderer (2010): "Monotonicity and Implementability," *Econometrica*, 78, 1749–1772.

Barbera, S. (2011): *Handbook of Social Choice and Welfare*, Elsevier, vol. 2, chap. 25, Strategyproof Social Choice, 731–831, editors: K. J. Arrow and Amartya Sen and Kotaro Suzumura.

Bikhchandani, S., S. Chatterji, R. Lavi, A. Mualem, N. Nisan, and A. Sen

(2006): "Weak Monotonicity Characterizes Deterministic Dominant Strategy Implementation," *Econometrica*, 74, 1109–1132.

Borgers, T. (2010): "An Introduction to the Theory of Mechanism Design," Teaching Notes, University of Michigan (author's personal website).

Carbajal, J. C., A. McLennan, and R. Tourky (2012): "Truthful Implementation and Aggregation in Restricted Domains," Working Paper, University of Queensland.

Clarke, E. (1971): "Multipart Pricing of Public Goods," *Public Choice*, 11, 17–33.

Debreu, G. (1960): *Mathematical Methods in the Social Sciences*, Stanford University Press, chap. Topological Methods in Cardinal Utility Theory, 16–26, editors: K. J. Arrow and S. Karlin and P. Suppes.

Dobzinski, S. and N. Nisan (2009): "A Modular Approach to Roberts' Theorem," in *In Proceedings of the 2nd International Symposium on Algorithmic Game Theory (SAGT 2009)*, Springer (Lecture Notes in Computer Science).

——— (2011): "Multi-Unit Auctions: Beyond Roberts," in *Forthcoming, Proceedings of* 12^{th} *ACM Conference on Electronic Commerce*, ACM Press.

Fishburn, P. C. and W. V. Gehrlein (1977): "Collective Rationality versus Distribution of Power for Binary Social Choice Functions," *Journal of Economic Theory*, 15, 72–91.

Gershkov, A., J. P. Goeree, A. Kushnir, B. Moldovanu, and X. Shi (2012): "On the Equivalence of Bayesian and Dominant Strategy Implementation," Working Paper, University of Bonn.

Gershkov, A., B. Moldovanu, and X. Shi (2011): "Bayesian and Dominant Strategy Implementation Revisited," Working Paper, Hebrew University of Jerusalem.

Gibbard, A. (1973): "Manipulation of Voting Schemes: A General Result," *Econometrica*, 41, 587–602.

Groves, T. (1973): "Incentives in Teams," *Econometrica*, 41, 617–631.

Güth, W. and M. Hellwig (1986): "The Private Supply of a Public good," *Journal of Economics*, 5, 121–159.

Hagerty, K. M. and W. P. Rogerson (1987): "Robust Trading Mechanisms," *Journal of Economic Theory*, 42, 94–107.

Heydenreich, B., R. Muller, M. Uetz, and R. V. Vohra (2009): "Characterization of Revenue Equivalence," *Econometrica*, 77, 307–316.

Jehiel, P., M. M. ter Vehn, and B. Moldovanu (2008): "Ex-Post Implementation and Preference Aggregation via Potentials," *Economic Theory*, 37, 469–490.

Lavi, R., A. Mualem, and N. Nisan (2009): "Two Simplified Proofs of Roberts' Theorem," *Social Choice and Welfare*, 32, 407–423.

Manelli, A. M. and D. Vincent (2007): "Multidimensional Mechanism Design: Revenue Maximization and the Multiple Good Monopoly," *Journal of Economic Theory*, 137, 153–185.

Masso, J., A. Nicolo, and A. Sen (2011): "Strategy-proofness and Equal-cost Sharing for Binary and Excludable Public Goods with Fixed Cost," Working Paper, University of Padua.

Mishra, D. and A. Sen (2012): "Roberts' Theorem with Neutrality: A Social Welfare Ordering Approach," *Games and Economic Behavior*, forthcoming.

Moulin, H. (1983): *The Strategy of Social Choice*, North-Holland.

——— (1999): "Incremental Cost Sharing: Characterization by Coalition Strategy-proofness," *Social Choice and Welfare*, 16, 279–320.

Moulin, H. and S. Shenker (2001): "Strategyproof Sharing of Submodular Costs: Budget Balance versus Efficiency," *Economic Theory*, 18, 511–533.

Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

Myerson, R. B. and M. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 29, 265–281.

Roberts, K. (1979): *The Characterization of Implementable Choice Rules*, North Holland Publishing, chap. Aggregation and Revelation of Preferences, 321–348, editor: J-J. Laffont.

Rochet, J. C. (1987): "A Necessary and Sufficient Condition for Rationalizability in a Quasi-linear Context," *Journal of Mathematical Economics*, 16, 191–200.

Saks, M. E. and L. Yu (2005): "Weak Monotonicity Suffices for Truthfulness on Convex Domains," in *Proceedings of* 7^{th} *ACM Conference on Electronic Commerce*, ACM Press, 286–293.

Satterthwaite, M. (1975): “Strategy-proofness and Arrow’s Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions,” *Journal of Economic Theory*, 10, 187–217.

Vickrey, W. (1961): “Counterspeculation, Auctions, and Competitive Sealed Tenders,” *Journal of Finance*, 16, 8–37.

Vohra, R. V. (2011): *Mechanism Design: A Linear Programming Approach*, Cambridge University Press.

Appendix

Proof of Proposition 1

Proof: Suppose f is implementable. Let $p_1, \ldots, p_n$ implement f. Fix an agent $i \in N$, $v_{-i} \in V_{-i}$, and $v_i, v'_i \in V_i$. Note that the incentive constraint for agent i, when his true value is v_i and he deviates to v'_i, can be written as (by using the fact that $f_1(v_i, v_{-i}) = 1 - f_2(v_i, v_{-i})$),

$$\partial v_i[f_1(v_i, v_{-i}) - f_1(v'_i, v_{-i})] \geq p_i(v_i, v_{-i}) - p_i(v'_i, v_{-i}).$$

Suppose $\partial v_i > \partial v'_i$. Writing the pair of incentive constraints for v_i and v'_i, we get

$$\begin{aligned} \partial v_i[f_1(v_i, v_{-i}) - f_1(v'_i, v_{-i})] &\geq p_i(v_i, v_{-i}) - p_i(v'_i, v_{-i}) \\ \partial v'_i[f_1(v'_i, v_{-i}) - f_1(v_i, v_{-i})] &\geq p_i(v'_i, v_{-i}) - p_i(v_i, v_{-i}). \end{aligned}$$

Adding the constraints, we get

$$(\partial v_i - \partial v'_i)[f_1(v_i, v_{-i}) - f_1(v'_i, v_{-i})] \geq 0.$$

Using the fact that $\partial v_i > \partial v'_i$, we immediately conclude that $f_1(v_i, v_{-i}) \geq f_1(v'_i, v_{-i})$.

Now, assume that f is monotone. To show that f is implementable, we will show that f is *cyclically monotone*, and by Rochet (1987), we will be done. Cycle monotonicity in our setting is the following requirement. Fix agent i and $v_{-i} \in V_{-i}$. For any pair of values $v_i, v'_i \in V_i$, let

$$\ell(v'_i, v_i) = \partial v_i[f_1(v_i, v_{-i}) - f_1(v'_i, v_{-i})].$$

Consider any finite sequence of values $v_i^1, v_i^2, \ldots, v_i^h$. The allocation rule f is cyclically monotone if

$$\ell(v_i^1, v_i^2) + \ell(v_i^2, v_i^3) + \ldots + \ell(v_i^{h-1}, v_i^h) + \ell(v_i^h, v_i^1) \geq 0. \tag{7}$$

We show this using induction on h. If $h = 2$, then we need to show that $\ell(v_i^1, v_i^2) + \ell(v_i^2, v_i^1) \geq 0$. This is equivalent to showing $(\partial v_i^1 - \partial v_i^2)[f_1(v_i^1, v_{-i}) - f_1(v_i^2, v_{-i})] \geq 0$. This follows from the fact that f is monotone. Now, suppose Inequality 7 holds for all $h < r$ and consider $h = r$. If $\partial v_i^1 = \partial v_i^2 = \ldots = \partial v_i^r$, then

$$\ell(v_i^1, v_i^2) + \ell(v_i^2, v_i^3) + \ldots + \ell(v_i^{r-1}, v_i^r) + \ell(v_i^r, v_i^1) = 0.$$

Otherwise, there is some $q \leq r$ such that $\partial v_i^q > \partial v_i^{q-1}$ (where $0 \equiv r$) and $\partial v_i^q \geq \partial v_i^j$ for all $j \in \{1, \ldots, r\}$. Note that by monotonicity,

$$f_1(v_i^q, v_{-i}) \geq f_1(v_i^{q-1}, v_{-i}). \tag{8}$$

Consider $\ell(v^{q-1}, v^q) + \ell(v^q, r^{q+1}) - \ell(v^{q-1}, v^{q+1})$, where $r + 1 \equiv 1$. By substituting, we get

$$\begin{aligned}
\ell(v_i^{q-1}, v_i^q) + \ell(v_i^q, v_i^{q+1}) - \ell(v_i^{q-1}, v_i^{q+1}) &= \partial v_i^q [f_1(v_i^q, v_{-i}) - f_1(v_i^{q-1}, v_{-i})] \\
&\quad + \partial v_i^{q+1} [f_1(v_i^{q+1}, v_{-i}) - f_1(v_i^q, v_{-i})] \\
&\quad - \partial v_i^{q+1} [f_1(v_i^{q+1}, v_{-i}) - f_1(v_i^{q-1}, v_{-i})] \\
&= (\partial v_i^q - \partial v_i^{q+1})[f_1(v_i^q, v_{-i}) - f_1(v_i^{q-1}, v_{-i})] \\
&\geq 0,
\end{aligned}$$

where the last inequality comes from Inequality 8 and from the fact that $\partial v_i^q \geq \partial v_i^{q+1}$. This means that $\ell(v_i^{q-1}, v_i^q) + \ell(v_i^q, v_i^{q+1}) \geq \ell(v_i^{q-1}, v_i^{q+1})$. Then,

$$\begin{aligned}
&\ell(v_i^1, v_i^2) + \ldots + \ell(v_i^{q-1}, v_i^q) + \ell(v_i^q, v_i^{q+1}) + \ell(v_i^{q+1}, v_i^{q+2}) + \ldots + \ell(v_i^r, v_i^1) \\
&\geq \ell(v_i^1, v_i^2) + \ldots + \ell(v_i^{q-1}, v_i^{q+1}) + \ell(v_i^{q+1}, v_i^{q+2}) + \ldots + \ell(v_i^r, v_i^1) \\
&\geq 0,
\end{aligned}$$

where the last inequality follows from our induction hypothesis. This concludes the proof. ■

Proofs of Theorems 2 and 3

We prove Theorems 2 and 3 in this sections. Before we do so, we comment on the methodology of the proof. The proof methodology is based on an *ordering based approach* of Mishra and Sen (2012) (M&S from now on). M&S provide an alternate proof of Roberts' theorem when there are at least three alternatives. The general idea of their proof is to characterize weighted efficiency using *neutrality* and implementability. In the unrestricted domain, for every implementable allocation rule, there is another implementable allocation rule that satisfies neutrality. This new allocation rule can be obtained by translating the original allocation rule. One can then leverage the weighted efficiency characterization to get a characterization of affine maximization in the unrestricted domain.

Although, we employ this methodology, our proof is different in many aspects from M&S. This is mainly because we have two alternatives. Our characterization of weighted efficiency requires stronger condition than the neutrality condition of M&S. Further, our affine maximization characterization requires implementability and a new condition called (*independence*), which M&S do not require if there are more than two alternatives.

Proof of Theorem 3

Like in M&S, we start by proving the characterization of weighted efficiency first, and then use this result to prove the affine maximizer characterization.

Fix an implementable deterministic allocation rule f. Consider the binary relation R^f over U defined by xR^fy iff $a_1 \in C^f(v)$, with $v(a_1) = x$ and $v(a_2) = y$. Let P^f and I^f respectively denote the asymmetric and symmetric part of R^f. They are well-behaved (in a sense made precise in Lemma 6) if f satisfies a neutrality condition.

Definition **11** *An allocation rule f is neutral if for every pair of valuations $v, v' \in V$ such that $v(a_1) = v'(a_2)$ and $v(a_2) = v'(a_1)$ we have*

$$C^f(v) = \begin{cases} C^f(v') & \textit{if } C^f(v) = A \\ A \setminus C^f(v) & \textit{otherwise.} \end{cases}$$

The usual definition of neutrality will require that for every pair of valuations $v, v' \in V$ such that $v(a) = v'(-a)$ and $v(-a) = v'(a)$ with $v \neq v'$ we have $\{f(v')\} = A \setminus \{f(v)\}$. One can verify that this version of neutrality implies our version of neutrality if the allocation rule is implementable - see Mishra and Sen (2012) for a proof.

Lemma **6** *Suppose f is neutral and implementable. Then R^f is reflexive and complete. Further, if $v(a_1) = x$ and $v(a_2) = y$, then*

- *$C^f(v) = \{a_1\}$ implies xP^fy and $C^f(v) = \{a_2\}$ implies yP^fx, and*
- *$C^f(v) = A$ implies xI^fy.*

Proof: R^f is reflexive. For any $x \in U$, consider the valuation profile v where $v(a_1) = v(a_2) = x$. Since $C^f(v)$ is non-empty and f is neutral, $C^f(v) = A$. Hence, xR^fx.

R^f is complete. For every $x, y \in U$, we can construct a valuation profile v with $v(a_1) = x$ and $v(a_2) = y$. If $a_1 \in C^f(v)$, then xR^fy. If $a_1 \notin C^f(v)$, then $a_2 \in C^f(v)$. Then, by neutrality, $a_1 \in C^f(v')$, with $v'(a_1) = y$ and $v'(a_2) = x$. Therefore, yR^fx.

We now show that $C^f(v) = \{a_1\}$ implies xP^fy. Suppose $C^f(v) = \{a_1\}$. This clearly implies xR^fy. Assume for contradiction that we also have yR^fx. This implies that $a_1 \in C^f(v')$, with $v'(a_1) = y$ and $v'(a_2) = x$. Then, by neutrality, $a_2 \in C^f(v)$, which gives us a contradiction.

A similar reasoning ensures that $C^f(v) = \{a_2\}$ implies yP^fx.

Finally, we show that $C^f(v) = A$ implies xI^fy. Suppose $C^f(v) = A$. This clearly implies xR^fy. Neutrality implies that $C^f(v') = A$, with $v'(a_1) = y$ and $v'(a_2) = x$. So, yR^fx, and hence, xI^fy. ■

Lemma **7** *Suppose f is a deterministic implementable and transitive allocation rule. Then, f is unanimous if and only if it is neutral.*

Proof: Suppose f is neutral and implementable. Consider $x, y \in U$ such that $x_i > y_i$ for all $i \in N$. Then, due to neutrality, $C^f(v) = A$ if $v(a) = y$ for all $a \in A$. By PAD, $C^f(v') = \{a\}$ if $v'(a) = x$ and $v'(-a) = y$. Hence, f is unanimous.

Now, suppose f is unanimous and transitive. Assume for contradiction that f is not neutral. Then, for some $x, y \in U$, we consider v and v' such that $v(a_1) = x = v'(a_2)$ and $v(a_2) = y = v'(a_1)$. We consider two cases.

Case 1. Assume for contradiction $C^f(v) = A$ but $C^f(v') = \{a_1\}$ (the argument does not change if $C^f(v') = \{a_2\}$). Since $a_2 \notin C^f(v')$, there is some $\epsilon \in \mathbb{R}^n_{++}$ such that $f(v'(a_1), v'(a_2) + 2\epsilon) = a_1$. This implies that $C^f(v'(a_1), v'(a_2) + \epsilon) = \{a_1\}$. Choose $\epsilon' \in \mathbb{R}^n_{++}$ such that $\epsilon'_i < \epsilon_i$ for all $i \in N$. Since $C^f(v) = A$, by PAD, $f(v(a_1) + \epsilon, v(a_2) + \epsilon') = a_1$. Moreover, by PAD, $C^f(v(a_1) + \epsilon, v(a_2) + \epsilon') = \{a_1\}$. Now, consider the valuation profile v'' such that $v''(a_1) = v'(a_1) = y$ and $v''(a_2) = v(a_2) + \epsilon' = y + \epsilon'$. By transitivity, $f(v'') = a_1$. But this contradicts the fact that f is unanimous.

Case 2. Assume for contradiction $C^f(v) = \{a_1\}$ but $C^f(v') \neq \{a_2\}$ (the argument is unchanged if $C^f(v) = \{a_2\}$). If $C^f(v') = A$, then we can apply the argument in Case 1 to reach a contradiction (by interchanging the roles of v and v'). Now, assume for contradiction $C^f(v') = \{a_1\}$. Since $a_2 \notin C^f(v)$, there is some sufficiently small $\epsilon \in \mathbb{R}^n_{++}$ such that $C^f(v(a_1), v(a_2) + \epsilon) = \{a_1\}$. Also, there is some ϵ' such that $\epsilon'_i < \epsilon_i$ for all $i \in N$ such that $f(v'(a_1) + \epsilon, v'(a_2) + \epsilon') = a_1$. Moreover, by PAD, $C^f(v'(a_1) + \epsilon, v'(a_2) + \epsilon') = \{a_1\}$. Consider a valuation profile v'' such that $v''(a_1) = v(a_1) = x$ and $v''(a_2) = v'(a_2) + \epsilon' = x + \epsilon$. By transitivity, $f(v'') = a_1$. But this contradicts the fact that f is unanimous. ■

Finally, we show that if f is implementable, transitive, and unanimous, then R^f is transitive.

Lemma 8 *If a deterministic implementable allocation rule f is transitive and unanimous, then R^f is an ordering.*

Proof: By Lemmas 6 and 7, if f is a deterministic implementable allocation rule that is transitive and unanimous, then R^f is a well-behaved binary relation. We need to show that R^f is transitive. We will show that P^f and I^f are each transitive, and this in turn will imply that R^f is transitive.

P^f is transitive. Consider $x, y, z \in U$ such that $xP^f y$ and $yP^f z$. Fix any $\epsilon \in \mathbb{R}^n_{++}$. By definition, if $v(a_1) = x$ and $v(a_2) = y$, then $f(v(a_1) + 2\epsilon, v(a_2) + \epsilon) = a_1$. Moreover, by PAD, $C^f(v(a_1) + 2\epsilon, v(a_2) + \epsilon) = \{a_1\}$. Similarly, if $v'(a_1) = y$ and $v'(a_2) = z$, then $C^f(v'(a_1) + \epsilon, v'(a_2)) = \{a_1\}$. Consider the valuation profile v'' such that $v''(a_1) = x$ and $v''(a_2) = z$. By transitivity, $f(v''(a_1) + 2\epsilon, v''(a_2)) = a_1$. Hence, $a_1 \in C^f(v'')$.

Also, for some $\epsilon \in \mathbb{R}^n_{++}$, we have $C^f(v(a_1), v(a_2) + \epsilon) = \{a_1\}$ and for some $\epsilon' \in \mathbb{R}^n_{++}$, we have $C^f(v'(a_1) + \epsilon, v'(a_2) + \epsilon') = \{a_1\}$. Again, by transitivity, $f(v''(a_1), v''(a_2) + \epsilon') = a_1$. Hence, $a_2 \notin C^f(v'')$. This shows that $xP^f z$.

I^f is transitive. Consider $x, y, z \in U$ such that $xI^f y$ and $yI^f z$. Fix some $\epsilon \in \mathbb{R}^n_{++}$. By definition, if $v(a_1) = x$ and $v(a_2) = y$, then (as in the earlier paragraph) $C^f(v(a_1)+2\epsilon, v(a_2)+\epsilon) = \{a_1\}$. Similarly, if $v'(a_1) = y$ and $v'(a_2) = z$, then $C^f(v'(a_1) + \epsilon, v'(a_2)) = \{a_1\}$. Consider the valuation profile v'' such that $v''(a_1) = x$ and $v''(a_2) = z$. By transitivity, $f(v''(a_1) + 2\epsilon, v''(a_2)) = a_1$. Hence, $a_1 \in C^f(v'')$. A similar argument shows $a_2 \in C^f(v'')$. Hence, $xI^f z$. ■

An ordering R on U satisfies **weak Pareto** if for any $x, y \in U$ if $x_i > y_i$ for all $i \in N$, then xPy.

An ordering R on U satisfies **translation invariance (tr-invariance)** if for any $x, y \in U$ and $z \in \mathbb{R}^n$ such that $x + z, y + z \in U$, we have xPy implies $(x + z)P(y + z)$ and xIy implies $(x + z)I(y + z)$.

An ordering R on U satisfies **continuity** if for every $x \in U$, the sets $\{y \in U : xRy\}$ and $\{y \in U : yRx\}$ are closed in U.

Lemma 9 *If f is a deterministic implementable allocation rule such that R^f is an ordering, then R^f satisfies weak Pareto, tr-invariance, and continuity.*

Proof: Since f is unanimous, it is clear that R^f satisfies weak Pareto.

Further, since f satisfies PAD (by Lemma 3), R^f satisfies tr-invariance. To see this, pick $x, y \in U$ and $z \in \mathbb{R}^n$ such that $x + z, y + z \in U$. Suppose $xP^f y$. Then, if $v(a_1) = x$ and

$v(a_2) = y$ for every $\epsilon \in \mathbb{R}^n_{++}$, $f(v(a_1) + \epsilon, v(a_2)) = a_1$. Choose such an ϵ. By PAD, for every $\epsilon' \in \mathbb{R}^n_{++}$ such that $\epsilon'_i > \epsilon_i$ for all $i \in N$, we have $f(v(a_1) + z + \epsilon', v(a_2) + z) = a_1$. Hence, $a_1 \in C^f(v(a_1)+z, v(a_2)+z)$. We also know that for some $\epsilon \in \mathbb{R}^n_{++}$, we have $f(v(a_1), v(a_2) + 2\epsilon) = a_1$. By PAD, $f(v(a_1)+z, v(a_2)+z+\epsilon) = a_1$. Hence, $a_2 \notin C^f(v(a_1)+z, v(a_2)+z)$. This shows that $(x+z)P^f(y+z)$. A similar argument shows that xI^fy implies $(x+z)I^f(y+z)$. Hence, R^f satisfies tr-invariance.

We now show that R^f satisfies continuity. To see this consider $x \in U$. We will first show that $\{y \in U : yR^fx\}$ is closed. Consider a sequence of points $\{x^k\}_k$ such that x^kR^fx and the limit of this sequence is $z \in U$. Assume for contradiction that xP^fz. Hence, if $v(a_1) = x$ and $v(a_2) = z$, then $f(v(a_1), v(a_2) + \epsilon) = a_1$ for some $\epsilon \in \mathbb{R}^n_{++}$. Hence, $xR^f(z+\epsilon)$. Since the sequence converges to z, there is a point z' in the sequence arbitrarily close to z such that $z'R^fx$. Since z' is arbitrarily close to z, we know that $(z+\epsilon)P^fz'$. Hence, by transitivity of R^f, $(z+\epsilon)P^fx$. This is a contradiction.

Next, we show that $\{y \in U : xR^fy\}$ is closed. Consider a sequence of points $\{x^k\}_k$ such that xR^fx^k and the limit of this sequence is $z \in U$. Assume for contradiction that zP^fx. Interchanging the role of x and z in the previous argument, we get that $zR^f(x+\epsilon)$ for some $\epsilon \in \mathbb{R}^n_{++}$. Since the sequence converges to z, there is a point in the sequence z' arbitrarily close to z such that xR^fz'. Since z' is arbitrarily close to z, by weak Pareto, $(x+\epsilon)P^fz'$. This is a contradiction. ■

Proof of Theorem 3

Proof: Suppose f is a deterministic implementable allocation rule that is unanimous and transitive. By Lemmas 8 and 9, the relation R^f is an ordering on U satisfying weak Pareto, tr-invariance, and continuity. Since U is open and convex, by Mishra and Sen (2012), there exists $\lambda_1, \ldots, \lambda_n \geq 0$ with $\lambda_i > 0$ for some $i \in N$, such that for every $x, y \in U$, xR^fy if and only if $\sum_{i \in N} \lambda_i x_i \geq \sum_{i \in N} \lambda_i y_i$.

Now, consider any valuation profile v. Since $f(v) \in C^f(v)$, we know that $v(f(v))R^fv(a)$ for all $a \in A$. Hence, $\sum_{i \in N} \lambda_i v_i(f(v)) \geq \sum_{i \in N} \lambda_i v_i(a)$. So, f is a weighted efficient allocation rule.

Clearly, a weighted efficient allocation rule is transitive and unanimous. It is well known that if a weighted efficient allocation rule satisfies UIA, then it is implementable (Mishra and Sen, 2012). ■

Proof of Theorem 2

We now use Theorem 3 to give a proof of Theorem 2. Before, we go into the details of the proof, we highlight the richness assumption of our domain. We assume that for every $i \in N$, the range of values for every alternative lies in an open interval L_i, which is unbounded from above. This implies that for every $i \in N$, $D_i = \mathbb{R}$ [5], a fact which we will use extensively in our proofs. Denote by $D = D_1 \times \ldots \times D_n$, and note that $D = \mathbb{R}^n$.

We will use the standard range condition of Roberts (1979) for the proof.

Definition 12 *An allocation rule f satisfies* **non-imposition** *if for every $a \in A$, there exists $v \in V$ such that $f(v) = a$.*

Fix an implementable deterministic allocation rule f. Suppose f satisfies independence. We first observe that the choice set only depends on differences of valuations.

Lemma 10 *Suppose f is implementable. Then, for every pair of valuation profiles, v, v' such that $\partial v_i = \partial v'_i$ for all $i \in N$, we have $C^f(v) = C^f(v')$.*

Proof: Choose v, v' such that $\partial v_i = \partial v'_i$ for all $i \in N$. Pick $a \in C^f(v)$ and $\epsilon \in \mathbb{R}^n_{++}$. By definition, $f(v(a) + \frac{\epsilon}{2}, v(-a)) = a$. By PAD and using the fact that $\partial v_i = \partial v'_i$ for all $i \in N$, we have $f(v'(a) + \epsilon, v'(-a)) = a$. Hence, $a \in C^f(v')$. Switching the role of v and v', we can show that if $a \in C^f(v')$, then $a \in C^f(v)$. As a result, $C^f(v) = C^f(v')$. ■

As a consequence of Lemma 10, we will define a mapping $c^f : D \to \{S \subseteq A : S \neq \emptyset\}$, such that for every $x \in D \subseteq \mathbb{R}^n$, $c^f(x) = C^f(v)$, where v is such that $\partial v_i = x_i$ for all $i \in N$.

Now, define κ^f as follows. For every $\alpha \in \mathbb{R}$, denote by 1_α the vector in $\mathbb{R}^n$ such that each component of 1_α has value α. By our assumption on D, $1_0 \in D$. If $a_1 \in c^f(1_0)$, then let

$$\kappa^f = -\sup\{\alpha \in \mathbb{R}_+ : a_1 \in c^f(1_{-\alpha})\}.$$

If $a_1 \notin c^f(1_0)$, then let

$$\kappa^f = \inf\{\alpha \in \mathbb{R}_+ : a_1 \in c^f(1_\alpha)\}.$$

Lemma 11 *If f is a deterministic implementable allocation rule satisfying non-imposition, then κ^f is a well defined real number.*

Proof: Suppose $a_1 \in c^f(1_0)$. By non-imposition (and using Lemma 10), we get that there is some $\beta \in \mathbb{R}$ such that $a_2 \in c^f(1_{-\beta})$. Since $a_1 \in c^f(1_0)$, by PAD, $\beta > \sup\{\alpha \in \mathbb{R}_+ : a_1 \in$

[5] To remind, $D_i = \{\partial v_i : v_i \in V_i\}$.

$c^f(1_{-\alpha})\} \geq 0$. This shows that κ^f exists since the set $\{\alpha \in \mathbb{R}_+ : a_1 \in c^f(1_{-\alpha})\}$ is bounded. So, κ^f is a real number. A similar proof works if $a_1 \notin c^f(1_0)$. ■

The next lemma proves another property of c^f.

LEMMA **12** *If f is a deterministic implementable allocation rule satisfying non-imposition, then $c^f(1_{\kappa^f}) = A$.*

Proof: By our assumption on D, $1_{\kappa^f} \in D$. First, we show that $a_1 \in c^f(1_{\kappa^f})$. Assume for contradiction that $a_1 \notin c^f(1_{\kappa^f})$. In that case, for all $v \in V$ with $\partial v_i = \kappa^f$, we have $a_1 \notin C^f(v)$. This implies that there is some $\epsilon \in \mathbb{R}^n_{++}$ such that $f(v(a_1) + \epsilon, v(a_2)) \neq a_1$. Hence, $a_1 \notin c^f(1_{\kappa^f} + \frac{\epsilon}{2})$. But, by definition of κ^f, for any $\epsilon' \in \mathbb{R}^n_{++}$, $a_1 \in c^f(1_{\kappa^f} + \epsilon')$, and this is a contradiction.

Next, we show that $a_2 \in c^f(1_{\kappa^f})$. Again, assume for contradiction that $a_2 \notin c^f(1_{\kappa^f})$. As in the previous case, there is some $\epsilon \in \mathbb{R}^n_{++}$ and $v \in V$ such that $\partial v_i = \kappa^f - \epsilon$ and $f(v) \neq a_2$. Hence, $a_2 \notin c^f(1_{\kappa^f} - \frac{\epsilon}{2})$. But, by definition of κ^f, for any $\epsilon' \in \mathbb{R}^n_{++}$, $a_1 \notin c^f(1_{\kappa^f} - \epsilon')$. Since for any $\epsilon' \in \mathbb{R}^n_{++}$, $c^f(1_{\kappa^f} - \epsilon')$ is non-empty, $a_2 \in c^f(1_{\kappa^f} - \epsilon')$. This is a contradiction. ■

Now, let f be a deterministic implementable allocation rule satisfying non-imposition. Define a new allocation rule $\bar{f}$ as follows. For every $v \in V$, define the valuation profile v^{tr} as follows: $\partial v_i^{tr} = \partial v_i + \kappa^f$ for all $i \in N$. Note that by our assumption of D, $v^{tr} \in V$. Now, the allocation rule $\bar{f}$ is defined as:

$$\bar{f}(v) = f(v^{tr}).$$

We now establish an important lemma.

LEMMA **13** *If f is a deterministic implementable allocation rule satisfying independence and non-imposition, then $\bar{f}$ is implementable, unanimous, and transitive.*

Proof: Suppose f is a deterministic implementable allocation rule satisfying independence and non-imposition. Let $(p_1, \ldots, p_n)$ be the payments that implement f. For every $i \in N$ and for every v_{-i}, let $\bar{p}_i(v_i, v_{-i}) = p_i(v_i^{tr}, v_{-i}^{tr}) - \kappa^f$ if $f(v_i, v_{-i}) = a_1$ and $\bar{p}_i(v_i, v_{-i}) = p_i(v_i^{tr}, v_{-i}^{tr})$ if $f(v_i, v_{-i}) = a_2$. We will show that $(\bar{p}_1, \ldots, \bar{p}_n)$ implement $\bar{f}$. To see this, consider $i \in N$ and v_{-i}. Also, consider v_i, v_i' such that $\bar{f}(v_i, v_{-i}) = a_1$ and $\bar{f}(v_i', v_{-i}) = a_2$ (a similar proof works if $\bar{f}(v_i, v_{-i}) = a_2$ and $\bar{f}(v_i', v_{-i}) = a_1$). Now,

$$\begin{aligned} v_i(a_1) - \bar{p}_i(v_i, v_{-i}) &= v_i^{tr}(f(v_i^{tr}, v_{-i}^{tr})) - p_i(v_i^{tr}, v_{-i}^{tr}) \\ &\geq v_i^{tr}(f(v_i'^{tr}, v_{-i}^{tr})) - p_i(v_i'^{tr}, v_{-i}^{tr}) \\ &= v_i(\bar{f}(v_i', v_{-i})) - \bar{p}_i(v_i', v_{-i}). \end{aligned}$$

Hence, $(\bar{p}_1, \ldots, \bar{p}_n)$ implement $\bar{f}$.

We show that $\bar{f}$ is unanimous. Consider a valuation profile v such that $v(a_1) = x$, $v(a_2) = y$, and $x_i > y_i$ for all $i \in N$. We need to show that $\bar{f}(v) = a_1$. To see this, consider the valuation profile v' such that $v'(a_1) = y = v'(a_2)$. But $c^{\bar{f}}(1_0) = c^f(1_{\kappa^f}) = A$. Hence, $C^{\bar{f}}(v') = A$, and using PAD, we get that $\bar{f}(v) = a_1$.

Finally, we show that $\bar{f}$ is transitive. For this, we consider $x, y, z \in D$ and v, v', v'' such that $v(a_1) = x = v''(a_1)$, $v(a_2) = y = v'(a_1)$, and $v'(a_2) = z = v''(a_2)$.

Suppose $C^{\bar{f}}(v) = \{a_1\}$ and $C^{\bar{f}}(v') = \{a_1\}$. We will show that $\bar{f}(v'') = a_1$. Note that since $C^{\bar{f}}(v') = \{a_1\}$, there is some $\epsilon \in \mathbb{R}^n_{++}$ such that $\bar{f}(v'(a_1) - \epsilon, v'(a_2)) = a_1$. To see this, suppose for all $\epsilon \in \mathbb{R}^n_{++}$, we have $\bar{f}(v'(a_1) - \epsilon, v'(a_2)) = a_2$. We know that for some $\epsilon' \in \mathbb{R}^n_{++}$, we have $\bar{f}(v'(a_1), v'(a_2) + \epsilon') = a_1$ (since $a_2 \notin C^{\bar{f}}(v')$). By PAD, $\bar{f}(v'(a_1) - \frac{\epsilon'}{2}, v'(a_2)) = a_1$. This is a contradiction. Similarly, there is an $\epsilon' \in \mathbb{R}^n_{++}$ such that $f(v(a_1) - \epsilon', v(a_2)) = a_1$.

Now, choose an $\epsilon'' \in \mathbb{R}^n_{++}$ such that $\bar{f}(v'(a_1) - \epsilon'', v'(a_2)) = a_1$ and $\bar{f}(v(a_1) - \frac{\epsilon''}{2}, v(a_2)) = a_1$ - note that such an ϵ'' can be chosen. In that case, by independence, either $\bar{f}(v(a_1), v'(a_2)) = a_1$ or $\bar{f}(v'(a_1) - \frac{\epsilon''}{2}, v(a_2)) = a_1$. Since $v'(a_1) = v(a_2) = y$ and $\bar{f}$ is unanimous, the latter is not possible. Hence, $\bar{f}(v'') = \bar{f}(v(a_1), v'(a_2)) = a_1$.

A similar argument shows if $C^{\bar{f}}(v) = \{a_2\}$ and $C^{\bar{f}}(v') = \{a_2\}$, then $\bar{f}(v'') = a_2$. ■

This leads to the proof of Theorem 2.

Proof of Theorem 2.

Proof: Suppose f is a deterministic implementable allocation rule. If f does not satisfy non-imposition, then clearly it is an affine maximizer. Now, suppose f satisfies non-imposition and independence. Then, by Lemma 13, $\bar{f}$ is a deterministic implementable allocation rule which is unanimous and transitive. By Theorem 3, there exists non-negative weights $\lambda_1, \ldots, \lambda_n$ such that for all v, if $\sum_{i \in N} \lambda_i \partial v_i > 0$, then $\bar{f}(v) = a_1$ and if $\sum_{i \in N} \lambda_i \partial v_i < 0$, then $\bar{f}(v) = a_2$. Furthermore, we can choose these weights, without loss of generality, such that $\sum_{i \in N} \lambda_i = 1$.

Now, using the definition of $\bar{f}$, we get that if $\sum_{i \in N} \lambda_i \partial v_i > \kappa^f$, then $f(v) = a_1$ and if $\sum_{i \in N} \lambda_i \partial v_i < \kappa^f$, then $f(v) = a_2$. Setting $\gamma(a_1) = \kappa^f$ and $\gamma(a_2) = 0$, we get that f is an affine maximizer.

For the converse, Lemma 2 shows that an affine maximizer satisfies independence. It is well known that an affine maximizer is implementable by generalized Groves payments if it is UIA. ■