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Deterministic Single Object Auctions with Private Values

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Deterministic Single Object Auctions
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Abstract

We study deterministic single object auctions in private values environments. We show that a deterministic allocation rule is implementable (in dominant strategies) if and only if it is a *generalized utility* maximizer. With a mild continuity condition, we show that a deterministic allocation rule is implementable and non-bossy if and only if it is a *virtual utility* maximizer (with appropriate tie-breaking). Both our results extend the seminal result of Roberts (1979) from unrestricted domain to the restricted domain of single object auctions.

JEL Classification Codes: D44

Keywords: single object auction; implementation in dominant strategies; rationalizability; non-bossiness.

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1 Introduction

We revisit a classical model of auction theory - single object auctions when agents have private values. We restrict attention to deterministic single object auctions, i.e., auctions where the probability of allocating the object to any agent is either zero or one. An allocation rule for single object auction is implementable if we can find payments such that truth-telling is a dominant strategy for every agent. We provide new characterizations of deterministic implementable allocation rules in this setting.

The underlying theme of our characterizations is the following: implementability involves a form of maximization at every valuation profile involving valuations at that profile. First, we show that implementability alone is equivalent to *generalized utility maximization*. A generalized utility function maps the set of valuation *profiles* of agents to the set of real numbers. A generalized utility maximizer allocation rule chooses a generalized utility function for every agent which satisfies a form of the *single-crossing* condition. Then, at every valuation profile (a) it does not allocate the object if every agent has negative generalized utility and (b) if at least one agent has positive generalized utility, then it allocates the object to the agent with the highest generalized utility.

For our second characterization, we generalize the virtual utility idea in Myerson (1981). A virtual utility function is any monotone function that maps the set of possible valuations of an agent to the set of real numbers. Contrast this with a generalized utility function which maps the set of valuation *profiles* to the set of real numbers. Hence, a virtual utility function is a simpler generalized utility function. A virtual utility maximizer is a generalized utility maximizer where every agent's generalized utility function is a virtual utility function. We show that if an allocation rule satisfies a mild continuity condition, then it is implementable and *non-bossy* if and only if it is a virtual utility maximizer allocation rule with an appropriate tie-breaking. We discuss non-bossiness and our continuity condition in detail later.

Our characterization of virtual utility maximizers bears resemblance to the virtual utility maximizing optimal auction in Myerson (1981), but has no direct relation. Myerson (1981) shows that if the auctioneer wants to choose a mechanism that maximizes his expected revenue, then, under independence, he must use a **specific** virtual utility maximizer. In Myerson's optimal auction, agent i with valuation v_i is assigned a virtual utility of $v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$, where F_i is the cumulative distribution function of the valuation of agent i and f_i is his density function [1]. For us, a virtual utility function of agent i is any monotone map of the form $U_i : V_i \to \mathbb{R}$, where V_i is the set of valuations of agent i. Notice that none of our characterizations require any distributional information. By focusing on expected revenue maximization, Myerson (1981) derives one particular virtual maximizer. On the other hand,

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[1] This result requires that F_is are independent and hazard rate of each distribution is non-decreasing.

by focusing solely on implementability (and some additional conditions), we characterize a class of allocation rules.

Our axiomatization of virtual utility maximizers using implementability and familiar conditions like non-bossiness and continuity captures the widely prevalent virtual utility maximizers in practice and theory - like the efficient allocation rule (Vickrey, 1961), the efficient allocation rule with a reserve price (Hartline and Roughgarden, 2009; Dhangwatnotai et al., 2010) for approximate optimal auction design, and Myerson's virtual utility maximizer allocation rule (Myerson, 1981) for optimal auction design.

1.1 Relationship with Literature

A central result in mechanism design is that the efficient allocation rule in the single object auction private values model is implementable using the Vickrey auction (Vickrey, 1961; Clarke, 1971; Groves, 1973). However, the set of implementable allocation rules is very rich. As shown by Myerson (1981), implementability is equivalent to a monotonicity property of the allocation rules [2]. The monotonicity property is equivalent to requiring that for every agent i and for every valuation profile of other agents, there is a cutoff valuation of agent i below which he does not get the object and above which he gets the object. Myerson (1981) uses this characterization to show that the expected revenue maximizing allocation rule is a particular type of virtual utility maximizer [3].

While the description of the efficient allocation rule and Myerson's virtual utility maximizer allocation rule is an explicit prescription for designing a mechanism, the description of implementable allocation rules using the monotonicity property is indirect. To understand this better, consider a setting with two agents. Suppose both the agents have valuations in $[0, 1]$. According to the monotonicity property, an implementable allocation rule must specify cutoff valuations. How does one go about designing such an allocation rule? For instance, if we fix the valuation of agent 2 at 0.5 and then fix the cutoff valuation of agent 1 at 0.5, this means that for any valuation profile $(v_1, 0.5)$, the allocation rule must allocate the object to agent 1 if $v_1 > 0.5$ and must not allocate the object to agent 1 if $v_1 < 0.5$. But this already puts restrictions on what can be done at many other valuation profiles. For instance, fix the valuation of agent 1 at 0.8, then the cutoff of agent 2 must be greater than 0.5 - otherwise we violate monotonicity since we allocate the object to agent 1 at $(0.8, 0.5)$.

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[2] See also extensions of this characterization to the multidimensional private values models in Bikhchandani et al. (2006); Saks and Yu (2005); Ashlagi et al. (2010); Cuff et al. (2012); Mishra and Roy (2012).

[3]The results in Myerson (1981) are more general. In particular, he considers implementation in Bayes-Nash equilibrium and allows for randomization. But the expected revenue maximizing allocation rule is a deterministic and dominant strategy implementable allocation rule.

It is in this sense that the monotonicity (or cutoff based) characterization of implementable allocation rule is an implicit characterization.

The relationship between our results and the monotonicity characterization can best be illustrated by reference to parallel results in strategic voting literature. Muller and Satterthwaite (1977) show that Maskin monotonicity [4] is necessary for dominant strategy implementation, and if the domain is unrestricted then it is also sufficient. However, the seminal results of Gibbard (1973) and Satterthwaite (1975) show that dictatorship is the only dominant strategy implementable voting rule.

In a very general quasi-linear private values (and multidimensional type space) set up, Roberts (1979) shows that if there are at least three alternatives and the type space of every agent is unrestricted, then every onto implementable allocation rule is an affine maximizer. An affine maximizer is a (linear) generalization of the efficient allocation rule. With some mild restriction on affine maximizers, it can be shown that every affine maximizer is implementable [5].

Roberts' theorem can be thought of as the counterpart of the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) in quasi-linear private values environments. After the seminal result of Gibbard (1973) and Satterthwaite (1975), a vast literature in social choice theory has pursued the characterization of implementable allocation rules in restricted "voting" domains, e.g., the median voting rule and its generalizations characterize implementable allocation rules in single-peaked domains (Moulin, 1980; Barbera et al., 1993). Indeed, these characterizations of implementable allocation rules are all in the spirit of Roberts' theorem - they describe the precise *parameters* that are required to design an implementable allocation rule.

A single object auction domain is a restricted domain - every agent gets positive utility from only one alternative, the alternative where he gets the object. Consequently, the result in Roberts (1979) does not apply in this domain. There have been extensions of Roberts' theorem to certain environments. For instance, Mishra and Sen (2012) show that Roberts' theorem holds in certain bounded but full dimensional type spaces under an additional condition of *neutrality*. Their neutrality condition is vacuous in the single object auction model. Moreover, the type space in the single object auction model is not full dimensional. Carbajal et al. (2012) extend Roberts' theorem to certain restricted type spaces which satisfy some technical conditions. Though it covers many interesting models, including those with

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[4]Maskin monotonicity can be thought of as the counterpart of monotonicity in strategic voting models.

[5] Carbajal et al. (2012) show that if there are at least three alternatives and the type space of every agent is unrestricted, then an onto allocation rule is implementable if and only if it is a *lexicographic* affine maximizer. Lexicographic affine maximizers contain a particular class of affine maximizers where ties are broken carefully.

infinite set of alternatives, the single object auction model does not satisfy their technical conditions. Marchant and Mishra (2012) extend Roberts' theorem to the case of two alternatives. Since the number of alternatives in the single object auction model is more than two, their results do not hold in our model. One particular characterization of Lavi et al. (2003) in a restricted domain stands out. They focus on a particular restricted domain, which they call order-based domains (this includes some auction domains). Under various additional restrictions on the allocation rule (which includes an independence condition), they show that every implementable allocation rule must be an "almost" affine maximizer - roughly, almost affine maximizers are affine maximizers for large enough values of types of agents. Finally, there have been many simplifications of the original proof of Roberts (Jehiel et al., 2008; Lavi, 2007; Dobzinski and Nisan, 2009; Vohra, 2011; Mishra and Sen, 2012). But none of these proofs show how Roberts' theorem can be extended to a restricted domain like the single object auction model. Unlike most of the literature, our goal is not to characterize "affine maximizers" - indeed, all our characterizations capture a larger class of implementable allocation rules than affine maximizers.

A feature of our virtual utility maximizer characterization is the use of the non-bossy axiom. Non-bossiness requires that if agent i is not getting the object at a valuation profile, and he changes his valuation such that he continues to not get the object at the new valuation profile, then the allocation of no agent must change between these two profiles. The use of non-bossiness axiom in social choice theory with private good allocations, specially matching problems, is extensive - it was first used by Satterthwaite and Sonnenschein (1981). For instance, Svensson (1999) characterizes the serial dictatorship allocation rules in the context of matching problems (without monetary transfers) using *strategy-proofness, non-bossiness,* and *neutrality.* Similarly, Papai (2000) characterizes the set of *hierarchical exchange rules* in the context of matching problems using *strategy-proofness, non-bossiness, Pareto optimality,* and *reallocation-proofness* - see also Ehlers (2002); Hatfield (2009) [6]. Non-bossiness has also been used in quasi-linear environments. In the context of cost sharing of a binary public good, Mutuswami (2005) shows that any mechanism with a non-bossy allocation rule and satisfying other extra conditions must be a *weak group strategy-proof* mechanism.

Though, we characterize implementable allocation rules, by virtue of revenue equivalence [7], this also characterizes the set of dominant strategy mechanisms. An alternate approach is to characterize the set of dominant strategy mechanisms directly by imposing conditions on mechanisms rather than just on allocation rules. A contribution along this line is Ashlagi and Serizawa (2011). They show that any mechanism which always allocates

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[6]Strategy-proofness and non-bossiness in these models is equivalent to a form of groupstrategy-proofness - see for instance Papai (2000).

[7]Revenue equivalence holds if the set of possible valuations of every agent is an interval (Myerson, 1981).

the object, satisfies individual rationality, non-negativity of payments, *anonymity in net utility*, and dominant strategy incentive compatiblity must be the Vickrey auction. This result is further strengthened by Mukherjee (2012), who shows that any strategy-proof and anonymous (in net utility) mechanism which always allocates the object must use the efficient allocation rule. Further, Sakai (2012) characterizes the Vickrey auction with a reserve price using various axioms on the *mechanism* (this includes an axiom on the allocation rule which requires a weak version of efficiency). By placing minimal axioms on *allocation rules*, we are able to characterize a broader class of mechanisms (using revenue equivalence) than these papers.

1.2   Discussions of the Main Results

Our characterization of implementability shows that implementability is equivalent to maximizing generalized utilities. Generalized utilities transform the original valuation of an agent to a new utility, which depends on the valuations of *all* the agents. This is similar to implementing the *efficient* allocation rule in an interdependent values model with the qualification that we allow generalized utilities to be negative, which is precluded in the standard interdependent value model. It is well known that the efficient allocation rule is not generally implementable in the interdependent values single object auction. However, Maskin (1992) shows that single crossing of value functions is a sufficient condition for implementing the efficient allocation rule in this model [8]. The single crossing condition that we require for generalized utility functions is weaker than the usual single crossing condition used in the interdependent value models. Nevertheless, our result reveals a surprising and interesting connection between these seemingly unrelated models.

Our virtual utility maximization result is obtained by exploiting the connection between implementability and rationalizability of allocation rules. The rationalizability approach views the mechanism designer as a *decision maker*, who is choosing among various utility vectors (associated with each alternative) at every valuation profile. Observe that at any valuation profile an allocation rule chooses a vector of utilities that are the payoffs of the agents. For instance, if agent i is assigned the object at a valuation profile (v_i, v_{-i}), then each agent realizes a utility of zero except agent i who realizes a utility of v_i [9]. If the alternative where the seller keeps the object is chosen, then every agent realizes a utility of zero. Let D be the set of all such vectors of utilities - these are vectors in $\mathbb{R}^n$ and lie on one of the n axes

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[8]See sharper negative results about implementation in multidimensional interdependent values models in Jehiel et al. (2006) and their resolution in Bikhchandani (2006).

[9]To clarify, these are the utilities agents derive from the allocation alone, and do not include the net utility from payments.

of $\mathbb{R}^n$, where n is the number of agents [10]. An allocation rule is strongly rationalizable if there exists an ordering $\succ$ on D with the following property: at every valuation profile, the allocation rule chooses the utility vector that is $\succ$-maximal at that profile. We show that an allocation rule is implementable and non-bossy if and only if it is strongly rationalizable. This result is of independent interest. Once we establish this result, we use some classical results concerning utility representation of orderings (a la Debreu (1954)) to complete our characterization of virtual utility maximizers.

This rationalizability approach was first used in Mishra and Sen (2012). They consider general quasi-linear environments with private values. They show that if the type space is a *multidimensional open interval*, then every implementable and *neutral* allocation rule is rationalizable. Rationalizability here is weaker than strong rationalizability in the sense that it does not require the underlying ordering to be a linear ordering. Our results depart from those in Mishra and Sen (2012) in many ways. First, as discussed earlier, their domain condition is not satisfied in our model, and neutrality is vacuous in the single object auction models. Second, we show that implementability and non-bossiness is *equivalent to strong rationalizability*. Mishra and Sen (2012) do not provide any such equivalence. Indeed, the non-bossiness that we use, is a condition that is specific to private good allocation problems, and cannot be used in general mechanism design problems.

2   THE SINGLE OBJECT AUCTION MODEL

A seller is selling an indivisible object to n potential agents (buyers). The set of agents is denoted by $N := \{1, \ldots, n\}$. The private value of agent i for the object is denoted by $v_i \in \mathbb{R}_{++}$. The set of all possible private values of agent i is $V_i \subseteq \mathbb{R}_{++}$ - note that we do not allow zero valuations. We will use the usual notations v_{-i} and V_{-i} denote a profile of valuations without agent i and the set of all profiles of valuations without agent i respectively. Let $V := V_1 \times V_2 \times \ldots \times V_n$.

The set of alternatives is denoted by $A := \{a_0, a_1, \ldots, a_n\}$, where a_0 is the alternative where the seller keeps the object and for every $i \in N$, a_i is the alternative where agent i gets the object. Notice that our model focuses on deterministic alternatives. Every agent $i \in N$ gets zero value from any alternative where he does not get the object. An allocation rule is a mapping $f : V \to A$. For convenience, for every $v \in V$ and for every $i \in N$, we use the notation $f_i(v) \in \{0, 1\}$ to denote if agent i gets the object ($f_i(v) = 1$) or not ($f_i(v) = 0$) at valuation profile v in allocation rule f.

Payments are allowed and agents have quasi-linear utility functions over payments. A payment rule of agent $i \in N$ is a mapping $p_i : V \to \mathbb{R}$.

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[10] The structure of D depends on how the set of valuations (type space) of each agent looks like.

DEFINITION 1 *An allocation rule f is* **implementable** *(in dominant strategies) if there exists payment rules $(p_1, \ldots, p_n)$ such that for every agent $i \in N$ and for every $v_{-i} \in V_{-i}$*

$$v_i f_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i f_i(v_i', v_{-i}) - p_i(v_i', v_{-i}) \qquad \forall \ v_i, v_i' \in V_i.$$

In this case, we say $(p_1, \ldots, p_n)$ implement f and the mechanism $(f, p_1, \ldots, p_n)$ is **incentive compatible**.

Notice that we focus on deterministic dominant strategy implementation.

Myerson (1981) showed that the following notion of monotonicity is equivalent to implementability.

DEFINITION 2 *An allocation rule f is* **monotone** *if for every $i \in N$, for every $v_{-i} \in V_{-i}$, and for every $v_i, v_i' \in V_i$ with $v_i < v_i'$ and $f_i(v_i, v_{-i}) = 1$, we have $f_i(v_i', v_{-i}) = 1$.*

Myerson (1981) shows that an allocation rule is implementable if and only if it is monotone - this result does not require any restriction on the space of valuations (see Vohra (2011), for instance).

3   THE COMPLETE CHARACTERIZATION

In this section, we provide a complete characterization of implementable allocation rules. In particular, we show that an implementable allocation rule is equivalent to a *generalized utility maximizer* allocation rule.

A **generalized utility function (GUF)** of agent $i \in N$ is a function $u_i : V \to \mathbb{R}$. Notice that the generalized utility of an agent may be negative also. We will need the following version of single crossing property.

DEFINITION 3 *The GUFs $(u_1, \ldots, u_n)$ satisfy* **top single crossing** *if for every $i \in N$, for every $v_{-i} \in V_{-i}$, and for every $v_i, v_i' \in V_i$ with $v_i > v_i'$ and $u_i(v_i', v_{-i}) \geq \max(0, \max_{k \in N} u_k(v_i', v_{-i}))$, we have $u_i(v_i, v_{-i}) > \max(0, \max_{k \in N \setminus \{i\}} u_k(v_i, v_{-i}))$.*

The top single crossing condition is a very general monotonicity condition. We give below a standard definition of a "single crossing" property, which implies top single crossing.

DEFINITION 4 *GUFs $(u_1, \ldots, u_n)$ satisfy* **single crossing** *if for every $i, j \in N$, for every $v_{-i} \in V_{-i}$, for every $v_i', v_i \in V_i$ with $v_i > v_i'$, we have $u_i(v_i, v_{-i}) - u_i(v_i', v_{-i}) > u_j(v_i, v_{-i}) - u_j(v_i', v_{-i})$.*

A GUF u_i is **increasing** if for every $v_{-i} \in V_{-i}$ and for every $v_i, v_i' \in V_i$ with $v_i > v_i'$ we have $u_i(v_i, v_{-i}) > u_i(v_i', v_{-i})$.

LEMMA 1 *If GUFs $(u_1, \ldots, u_n)$ satisfy single crossing and u_i is increasing for every $i \in N$, then they satisfy top single crossing.*

Proof: Consider $i \in N$ and $v_{-i} \in V_{-i}$. Let $v_i, v'_i \in V_i$ such that $v_i > v'_i$ and $u_i(v'_i, v_{-i}) \geq \max(0, \max_{k \in N} u_k(v'_i, v_{-i}))$. Since u_i is increasing, $u_i(v_i, v_{-i}) > u_i(v'_i, v_{-i}) \geq 0$. Further, by single crossing, $u_i(v_i, v_{-i}) - u_i(v'_i, v_{-i}) > u_j(v_i, v_{-i}) - u_j(v'_i, v_{-i})$ for all $j \neq i$. Using the fact that $u_i(v'_i, v_{-i}) \geq u_j(v'_i, v_{-i})$ for all $j \neq i$, we get that $u_i(v_i, v_{-i}) > u_j(v_i, v_{-i})$ for all $j \neq i$. Hence, $u_i(v_i, v_{-i}) > \max(0, \max_{k \in N \setminus \{i\}} u_k(v_i, v_{-i}))$. ∎

We are now ready to introduce a new class of implementable allocation rules.

DEFINITION 5 *An allocation rule f is a **generalized utility maximizer** if there exist GUFs $(u_1, \ldots, u_n)$ satisfying top single crossing such that for every $v \in V$, if $f(v) = a_i$ then $i \in \arg\max_{i \in N \cup \{0\}} u_i(v)$, where $u_0(v) = 0$.*

Generalized utility maximizers are implementable.

LEMMA 2 *If f is a generalized utility maximizer, then it is implementable.*

Proof: Fix a generalized utility maximizer f, and let $(u_1, \ldots, u_n)$ be the corresponding increasing GUFs satisfying top single crossing. Consider agent i and $v_{-i} \in V_{-i}$. Also, consider any $v_i, v'_i \in V_i$ with $v_i > v'_i$ and $f(v'_i, v_{-i}) = a_i$. By definition, $u_i(v'_i, v_{-i}) \geq \max(0, \max_{k \in N} u_k(v'_i, v_{-i}))$. By top single crossing, $u_i(v_i, v_{-i}) > \max(0, \max_{k \in N \setminus \{i\}} u_k(v_i, v_{-i}))$. Hence, $f(v_i, v_{-i}) = a_i$. So, f is monotone, and hence, implementable. ∎

This leads to the main result of this section.

THEOREM 1 *Suppose $V_i = (0, \beta_i)$, where $\beta_i \in \mathbb{R}_{++}$, for all $i \in N$. Then, f is implementable if and only if it is a GUF maximizer allocation rule.*

Proof: Lemma 2 showed that every GUF maximizer is implementable. Now, for the converse, suppose f is implementable. Fix an agent $i \in N$ and $v_{-i} \in V_{-i}$. If $f(v_i, v_{-i}) \neq a_i$ for all $v_i \in V_i$, then define $\kappa_i^f(v_{-i}) = \beta_i$. Else, define $\kappa_i^f(v_{-i}) = \inf\{\alpha \in V_i : f(v_i, v_{-i})\}$. Notice that $\kappa_i^f(v_{-i})$ is well defined. Further, since f is monotone, for every agent $i \in N$, for every v_{-i}, and for every $v_i \in V_i$, if $v_i > \kappa_i^f(v_{-i})$, we have $f(v_i, v_{-i}) = a_i$ and for every $v_i < \kappa_i^f(v_{-i})$ we have $f(v_i, v_{-i}) \neq a_i$. Define for every $i \in N$ and for every (v_i, v_{-i}), $u_i(v_i, v_{-i}) := v_i - \kappa_i^f(v_{-i})$. By definition, if $f(v) = a_i$, then $v_i - \kappa_i^f(v_{-i}) \geq 0$ and $v_j - \kappa_j^f(v_{-j}) \leq 0$ for all $j \neq i$. Hence, $i \in \arg\max_{k \in N \cup \{0\}} u_k(v)$, where $u_0(v) = 0$.

To show that $(u_1, \ldots, u_n)$ satisfy top single crossing, consider $i \in N$ and $v_{-i} \in V_{-i}$. Let $v_i, v'_i \in V_i$ such that $v_i > v'_i$ and $u_i(v'_i, v_{-i}) \geq \max(0, \max_{k \in N} u_k(v'_i, v_{-i}))$. Notice that

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$u_i(v_i, v_{-i}) > u_i(v_i', v_{-i}) \geq 0$. By definition of $u_1, \ldots, u_n$, if $u_i(v_i, v_{-i}) > 0$, then $v_i > \kappa_i^f(v_{-i})$, and hence, $f(v_i, v_{-i}) = a_i$. But, this implies that $u_k(v_i, v_{-i}) = v_k - \kappa_i^f(v_{-k}) \leq 0$ for all $k \neq i$. Hence, $u_i(v_i, v_{-i}) > \max(0, \max_{k \in N \setminus \{i\}} u_k(v_i, v_{-i}))$. $\blacksquare$

Remark. Consider a general mechanism design set up with private values and quasi-linear utility. Let A be a finite set of alternatives. Suppose $|A| \geq 3$. The type of agent i is denoted as $v_i \in \mathbb{R}^{|A|}$ and $v_i(a)$ denotes the valuation of agent i for alternative a. Roberts (1979) shows that if type space of every agent is $\mathbb{R}^{|A|}$, then for every onto and implementable allocation rule f, there exists $\lambda_1, \ldots, \lambda_n \geq 0$, not all of them equal to zero, and $\kappa : A \to \mathbb{R}$ such that at every valuation profile v, $f(v) \in \arg\max_{a \in A}[\sum_{i \in N} \lambda_i v_i(a) + \kappa(a)]$. Such allocation rules are called affine maximizer allocation rules. Theorem 1 can be thought of as the analogue of Roberts' affine maximizer theorem in the single object auction model. Since the generalized utility functions are quite general than the affine maximizers, it reflects the richness of the set of implementable allocation rules in this model.

4   IMPLEMENTATION, RATIONALIZABILITY, AND NON-BOSSINESS

Though the generalized utility maximizers provide a complete characterization of implementable allocation rules, GUFs satisfying top single crossing are not easy to construct. The objective of this section is to characterize implementability in the presence of additional conditions. This allows us to characterize a simpler class of implementable allocation rules than generalized utility maximizers. For this, we extend idea of virtual utility functions in Myerson (1981). Virtual utility function of an agent only depends on the valuation of that agent. Clearly, a virtual utility function is also a GUF. Using additional conditions besides implementability, we characterize the virtual utility maximizer allocation rules.

The backbone of this characterization is a notion of rationalizability in our model. We introduce this idea of rationalizability in the single object auctions next.

4.1   Rationalizability

To define rationalizability, we view the mechanism designer as a decision maker who is making choices using his allocation rule. Notice that at every profile of valuations, by choosing an alternative $a \in A$, the mechanism designer assigns values to each agent - zero to all agents who do not get the object but positive value to the agent who gets the object. Denote by $\mathbf{1}_{v_i}$ the vector of valuations in $\mathbb{R}_+^n$, where all the components except agent i has zero and the component corresponding to agent i has v_i. Further, denote by $\mathbf{1}_0$ the n-dimensional zero vector. For convenience, we will write $\mathbf{1}_0$ as $\mathbf{1}_{v_0}$ at any valuation profile.

Using this notation, at a valuation profile $(v_1, \ldots, v_n)$, a mechanism designer's choice of an alternative in A can lead to the selection of one of the following $(n+1)$ vectors in $\mathbb{R}_+^n$ to be chosen - $\mathbf{1}_{v_0}, \mathbf{1}_{v_1}, \ldots, \mathbf{1}_{v_n}$. We will refer to these vectors as *utility vectors*. Any allocation rule f can alternatively thought of choosing utility vectors at every valuation profile. The domain of valuations V_i of agent i gives rise to a set of feasible utility vectors where only agent i gets positive value. In particular define for every $i \in N$, $D_i := \{\mathbf{1}_{v_i} : v_i \in V_i\}$. Further, let $D_0 := \{\mathbf{1}_{v_0}\}$ and $V_0 = \{0\}$. Denote by $D := D_0 \cup D_1 \cup D_2 \cup \ldots \cup D_n$ the set of all utility vectors consistent with the domain of profile of valuations V.

An example with two agents will clarify some of the concepts. With two agents, there are three alternatives $A = \{a_0, a_1, a_2\}$. Let $V_1 = V_2 = \mathbb{R}_{++}$. In that case, D is a subset of $\mathbb{R}^2$. In particular, D_0 is the origin, D_1 and D_2 are the axes in $\mathbb{R}^2$. At any valuation profile, there will be three points from D, one being D_0, one chosen from D_1, and the other chosen from D_2.

For every allocation rule f, let $G^f : V \to D$ be a *social welfare function* induced by f, i.e., for all $v \in V$, $G^f(v) = \mathbf{1}_{v_j}$ if $f(v) = a_j$ for any $j \in \{0, 1, \ldots, n\}$. Further, for every allocation rule f, let $D^f := \{x \in D : G^f(v) = x \text{ for some } v \in V\}$. We impose some natural requirements on the allocation rule of a mechanism designer - the selection of an alternative at every valuation profile must be *rational* in the following sense.

To define the notion of a rational allocation rule, we will use orderings (reflexive, complete, and transitive binary relation) on the set of utility vectors D. For any ordering $\succeq$ on D, let $\succ$ be the asymmetric component of $\succeq$ and $\sim$ be the symmetric component of $\succeq$. A linear ordering has no symmetric component. An ordering $\succeq$ on D is monotone if for every $i \in N$, for every $v_i, v_i' \in V_i$ with $v_i > v_i'$, we have $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_i'}$. Our notion of rational allocation requires that at every profile of valuations it must choose a maximal element among the utility vectors at that valuation profile, where the maximal element is defined using a monotone ordering on D.

DEFINITION **6** *An allocation rule f is **rationalizable** if there exists a monotone ordering $\succeq$ on D such that for all $v \in V$, $G^f(v) \succeq \mathbf{1}_{v_j}$ for all $j \in \{0, 1, \ldots, n\}$. In this case, we say $\succeq$ rationalizes f.*

*An allocation rule f is **strongly rationalizable** if there exists a monotone linear ordering $\succ$ on D such that for all $v \in V$, $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$ for all $j \in \{0, 1, \ldots, n\} \setminus \{i\}$, where $G^f(v) = \mathbf{1}_{v_i}$. In this case, we say $\succ$ strongly rationalizes f.*

We will investigate the relationship between (strongly) rationalizable allocation rules and implementable allocation rules. The following lemma establishes that a rational allocation rule is implementable.

LEMMA **3** *Every rationalizable allocation rule is implementable.*

Proof: Consider a rationalizable allocation rule f and let $\succeq$ be the corresponding ordering on D. Fix an agent i and valuation profile v_{-i}. Consider two valuations of agent i: v_i and v_i' with $v_i < v_i'$ with $f(v_i, v_{-i}) = a_i$. By definition of $\succeq$, $\mathbf{1}_{v_i} \succeq \mathbf{1}_{v_j}$ for all $j \in (N \cup \{0\}) \setminus \{i\}$. Since $\succeq$ is monotone, $\mathbf{1}_{v_i'} \succ \mathbf{1}_{v_i}$. By transitivity, $\mathbf{1}_{v_i'} \succ \mathbf{1}_{v_j}$ for all $j \in (N \cup \{0\}) \setminus \{i\}$. Then, by the definition of $\succeq$, $f(v_i', v_{-i}) = a_i$. Hence, f is monotone. ∎

The converse of Lemma 3 is not true. The following example establishes that.

EXAMPLE **1**

Suppose there are two agents: $N = \{1, 2\}$. Suppose $V_1 = V_2 = \mathbb{R}_{++}$. Consider an allocation rule f defined as follows. At any valuation profile (v_1, v_2), if $\max(v_1 - 2v_2, v_2 - v_1) < 0$, then $f(v_1, v_2) = a_0$. Else, if $v_1 - 2v_2 < v_2 - v_1$, then $f(v_1, v_2) = a_2$ and if $v_1 - 2v_2 \geq v_2 - v_1$, then $f(v_1, v_2) = a_1$. It is easy to verify that f is monotone, and hence, implementable.

We argue that f is not a rationalizable allocation rule. Assume for contradiction that f is a rationalizable allocation rule and $\succeq$ is the corresponding monotone ordering. Consider the profile of valuation (v_1, v_2), where $v_1 = 1$ and $v_2 = 2$. For $\epsilon > 0$ but arbitrarily close to zero, $f(v_1, v_2 - \epsilon) = a_2$. Hence, $\mathbf{1}_{v_2 - \epsilon} \succeq \mathbf{1}_{v_0}$. By monotonicity, $\mathbf{1}_{v_2} \succ \mathbf{1}_{v_0}$. Now, consider the profile of valuations (v_1', v_2), where $v_1' = 2 + \epsilon$ and $v_2 = 2$. Note that $f(v_1', v_2) = a_0$. Hence, $\mathbf{1}_{v_0} \succeq \mathbf{1}_{v_2}$. This is a contradiction.

A feature of this example is that at valuation profile (v_1, v_2), the allocation rule was choosing a_2. But when valuation of agent 1 changed to v_1', it chose a_0 at valuation profile (v_1', v_2). Hence, agent 1 could change the outcome without changing his own outcome. As we show next, such allocation rules are incompatible with rationalizability.

4.2 Non-bossy Single Object Auctions

In this section, we will characterize the set of implementable allocation rules which are rationalizable. Besides implementability, we will impose an additional condition on an allocation rule.

DEFINITION **7** *An allocation rule f is* **non-bossy** *if for every $i \in N$, for every $v_{-i} \in V_{-i}$ and for every $v_i, v_i' \in V_i$ with $f_i(v_i, v_{-i}) = f_i(v_i', v_{-i}) = 0$, we have $f(v_i, v_{-i}) = f(v_i', v_{-i})$.*

Non-bossiness requires that if an agent does not change his own allocation (i.e., whether he is getting the object or not) by changing his valuation, then he should not be able to change the allocation of anyone. It was first proposed by Satterthwaite and Sonnenschein (1981). As discussed in the introduction, it is a plausible condition to impose in private good allocation problems and has been extensively used in the strategic social choice theory literature.

We give an example of a bossy and a non-bossy allocation rule in Figure 1(a) and Figure 1(b) respectively. These figures indicate a scenario with two agents. The possible outcomes of the allocation rules at different valuation profiles are depicted in the Figures. In Figure 1(a), the allocation rule is bossy since if we start from a region where alternative a_2 is chosen and agent 1 increases his value, then we can come to a region where alternative a_0 is chosen (i.e., agent 1 can change the outcome without changing his own outcome). However, such a problem is absent for the allocation rule in Figure 1(b).
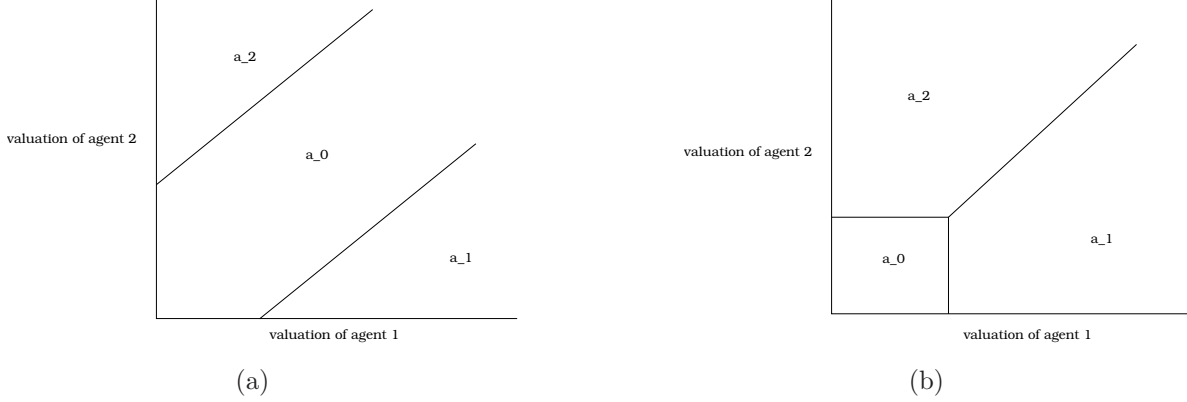

Figure 1: Bossy and non-bossy allocation rules

LEMMA **4** *A strongly rationalizable allocation rule is non-bossy.*

Proof: Let f be a strongly rationalizable allocation rule with $\succ$ being the corresponding ordering on D. Fix an agent i and $v_{-i} \in V_{-i}$. Consider $v_i, v'_i \in V_i$ such that $f(v_i, v_{-i}) = a_j \neq a_i$ and $f(v'_i, v_{-i}) = a_l \neq a_i$. By definition, $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_k}$ for all $k \in (N \cup \{0\}) \setminus \{j\}$. Suppose $f(v'_i, v_{-i}) = a_l \neq a_i$. By definition, $\mathbf{1}_{v_l} \succ \mathbf{1}_{v_k}$ for all $k \in (N \cup \{0\}) \setminus \{l\}$. Assume for contradiction $a_l \neq a_j$. Then, we get that $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_l}$ and $\mathbf{1}_{v_l} \succ \mathbf{1}_{v_j}$, which is a contradiction. ∎

This leads to the formal connection between implementability and rationalizability.

THEOREM **2** *An allocation rule is implementable and non-bossy if and only if it is strongly rationalizable.*

Proof: By virtue of Lemmas 3 and 4, we only need to show that if an allocation rule f is implementable and non-bossy then it is strongly rationalizable. We do the proof in several steps.

STEP 1. For any $i, j \in N \cup \{0\}$ with $i \neq j$, consider $\mathbf{1}_{v_i}$ and $\mathbf{1}_{v_j}$ for some $v_i \in V_i$ and $v_j \in V_j$. Suppose for some v_{-ij}, we have $f(v_i, v_j, v_{-ij}) = a_i$. We will show that

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if f is implementable and non-bossy, then $f(v_i, v_j, v'_{-ij}) \neq a_j$ for all v'_{-ij}. Assume for contradiction $f(v_i, v_j, v'_{-ij}) = a_j$ for some v'_{-ij}. Now, consider a valuation profile v''_{-ij} where $v''_k = \min(v_k, v'_k)$ for all $k \in N \setminus \{i, j\}$. Since f is implementable (monotone) and non-bossy $f(v_i, v_j, v''_{-ij}) = a_i$ if (v_i, v_j, v_{-ij}) is changed to (v_i, v_j, v''_{-ij}) agent-by-agent and $f(v_i, v_j, v''_{-ij}) = a_j$ if (v_i, v_j, v'_{-ij}) is changed to (v_i, v_j, v''_{-ij}) agent-by-agent. This is a contradiction.

STEP 2. We will first define an ordering $\succeq$, and then convert it into an anitsymmetric ordering. The ordering $\succeq$ is defined using f as follows. Let the symmetric and asymmetric components of $\succeq$ be $\sim$ and $\succ$ respectively. Define $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$ if there is some v_{-ij} such that $f(v_i, v_j, v_{-ij}) = a_i$ and $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_i}$ if there is some v_{-ij} such that $f(v_i, v_j, v_{-ij}) = a_j$. Further, for every $i \in N$ and every $v_i \in V_i$, define $\mathbf{1}_{v_i + \epsilon} \succ \mathbf{1}_{v_i}$ for all $\epsilon > 0$ such that $(v_i + \epsilon) \in V_i$. If $f(v_i, v_j, v_{-ij}) \notin \{a_i, a_j\}$ for all v_{-ij} then $\mathbf{1}_{v_i} \sim \mathbf{1}_{v_j}$. Further define, $\mathbf{1}_{v_i} \sim \mathbf{1}_{v_i}$. By Step 1, this is a well defined binary relation.

STEP 3. We now show that $\succeq$ satisfies the following conditions:

1. for every $x, y \in D^f$, $x \sim y$, where $D^f = \{x \in D : G^f(v) = x \text{ for some } v \in V\}$,

2. for every $x \in D^f$ and for every $y \notin D^f$, $x \succ y$,

3. for all $v \in V$, $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$ for all $j \in \{0, 1, \ldots, n\} \setminus \{i\}$, where $G^f(v) = \mathbf{1}_{v_i}$.

- PROOF OF (1). Pick $x, y \in D^f$. By definition, there is $v \in V$, such that $G^f(v) = x$. If $x = \mathbf{1}_{v_i}$, then $f(v) = a_i$. Suppose $y = \mathbf{1}_{v'_i}$. Then, by definition, either $x \succ y$ or $y \succ x$. Hence, suppose $y = \mathbf{1}_{v'_j}$ for some $j \neq i$. Then, by monotonicity and non-bossiness, $f(v_i, v'_j, v_{-ij}) \in \{a_i, a_j\}$. Hence, $x \succ y$ or $y \succ x$ but not both.

- PROOF OF (2). Pick $x \in D^f$ but $y \notin D^f$. By definition, there is $v \in V$, such that $G^f(v) = x$. If $x = \mathbf{1}_{v_i}$, then $f(v) = a_i$. Suppose $y = \mathbf{1}_{v'_i}$. Then, if $v'_i > v_i$, we have $f(v'_i, v_{-i}) = a_i$ by monotonicity, and this contradicts the fact that $y \notin D^f$. Hence, $v'_i < v_i$, and by definition, $x \succ y$.

  Suppose $y = \mathbf{1}_{v'_j}$ for some $j \neq i$. Then, by monotonicity and non-bossiness, $f(v_i, v'_j, v_{-ij}) \in \{a_i, a_j\}$. Using the fact that $y \notin D^f$, we get that $f(v_i, v'_j, v_{-ij}) = a_i$. Hence, $x \succ y$.

- PROOF OF (3). At any valuation profile $(v_1, \ldots, v_n)$, if $f(v_1, \ldots, v_n) = a_i$, then, by definition, $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$ for all $j \neq i$.

STEP 4. We show that $\succeq$ is an ordering. By definition $\succeq$ is reflexive and complete. To show transitivity of $\succeq$, we show transitivity of $\succ$ and $\sim$. Pick $v_i \in V_i, v_j \in V_j$ and $v_k \in V_k$

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such that $\mathbf{1}_{v_i} \sim \mathbf{1}_{v_j}$ and $\mathbf{1}_{v_j} \sim \mathbf{1}_{v_k}$. Note that i, j, k are distinct. Consider any profile of valuations (v_i, v_k, v'_{-ik}). By Step (3), $f(v_i, v_j, v_k, v'_{-ijk}) \notin \{a_i, a_j, a_k\}$. By monotonicity and non-bossiness, $f(v_i, v_k, v'_{-ik}) \notin \{a_i, a_k\}$. Hence, $\mathbf{1}_{v_i} \sim \mathbf{1}_{v_k}$. This shows that $\sim$ is transitive.

Now, we show that $\succ$ is transitive. Suppose for some $i \in N$, $\mathbf{1}_{v_i + \epsilon} \succ \mathbf{1}_{v_i}$ for some $\epsilon > 0$ such that $v_i + \epsilon \in V_i$. Also, for some $j \neq i$, $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$. Then, by definition, for some v_{-ij}, $f(v_i, v_j, v_{-ij}) = a_i$. By monotonicity, $f(v_i + \epsilon, v_j, v_{-ij}) = a_i$. Hence, $\mathbf{1}_{v_i + \epsilon} \succ \mathbf{1}_{v_j}$.

We also know that for some $i \in N$ and for some $\epsilon > 0, \delta > 0$, if $\mathbf{1}_{v_i + \epsilon + \delta} \succ \mathbf{1}_{v_i + \epsilon}$ and $\mathbf{1}_{v_i + \epsilon} \succ \mathbf{1}_{v_i}$, then $\mathbf{1}_{v_i + \epsilon + \delta} \succ \mathbf{1}_{v_i}$.

Finally, pick $v_i \in V_i, v_j \in V_j$ and $v_k \in V_k$ such that $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$ and $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_k}$, where i, j, k are distinct. This means, $f(v_i, v_j, v'_{-ij}) = a_i$ for some v'_{-ij}. By monotonicity and non-bossiness, $f(v_i, v_j, v_k, v'_{-ijk}) \in \{a_i, a_k\}$. But, $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_k}$ implies that $f(v_i, v_j, v_k, v'_{-ijk}) \neq a_k$. Hence, $f(v_i, v_j, v_k, v'_{-ijk}) = a_i$. Hence, $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_k}$. This shows that $\succ$ is transitive. So, $\succeq$ is an ordering.

STEP 5. We show that f is strongly rationalizable. To do show, we generate a new linear ordering $\succ'$ on D from $\succeq$. The linear ordering $\succ'$ is defined as follows. For every $x, y \in D^f$, $x \succ' y$ if and only if $x \succ y$. For every $x \in D^f$ and $y \notin D^f$, $x \succ' y$ (note that this means $x \succ' y$ if and only if $x \succ y$). Now, construct any linear ordering $\succ''$ of the elements of $D \setminus D^f$. Make $\succ'$ coincide with $\succ''$ on elements of $D \setminus D^f$. Notice that $\succ'$ is a well-defined ordering on D. Further, by definition of $\succ'$ and Step 3, at any valuation profile $(v_1, \ldots, v_n)$, if $f(v_1, \ldots, v_n) = a_i$, then, by definition, $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$ for all $j \neq i$. Hence, f is strongly rationalizable. ∎

If the linear ordering we constructed in the proof of Theorem 2 can be represented using a utility function, then the characterization will be even more direct. If for every agent $i \in N$, V_i is finite, then it is possible. But, as the next example illustrates, this is not always possible.

EXAMPLE 2

Suppose $N = \{1, 2\}$ and $V_1 = V_2 = \mathbb{R}_{++}$. Consider the allocation rule f such that for all valuation profiles (v_1, v_2), $f(v_1, v_2) = a_1$ if $v_1 \geq 1$, $f(v_1, v_2) = a_2$ if $v_1 < 1$ and $v_2 \geq 1$, and $f(v_1, v_2) = a_0$ otherwise. It can be verified that f is implementable (monotone) and non-bossy. By Theorem 2, f is strongly rationalizable. Now, consider the linear order defined in the proof of Theorem 2 that strongly rationalizes f - denote it by $\succ^f$. If $v_1 = v_2 = 1$, we have $f(v_1, v_2) = a_1$. Hence, $\mathbf{1}_{v_1} \succ^f \mathbf{1}_{v_2}$.

Now, consider the following definition.

DEFINITION 8 *An ordering $\succeq$ on the set D is separable if there exists a countable set $Z \subseteq D$ such that for every $x, y \in D$ with $x \succ y$, there exists $z \in Z$ such that $x \succeq z \succeq y$.*

It is well known that an ordering on D has a utility representation if and only if it is separable - the result goes back to at least Debreu (1954) (see also Ok (2012) and Kreps (1988) for details). We show that $\succ^f$ is not separable. Note that since $\succ^f$ is monotone, any utility vector between $\mathbf{1}_{v_1}$ and $\mathbf{1}_{v_2}$ will be of the form $\mathbf{1}_{v_2+\epsilon}$ or $\mathbf{1}_{v_1-\epsilon}$ for some $\epsilon > 0$. But, $f(v_1, v_2 + \epsilon) = a_2$ implies that $\mathbf{1}_{v_2+\epsilon} \succ^f \mathbf{1}_{v_1}$ for all $\epsilon > 0$. Also, $f(v_1 - \epsilon, v_2) = a_2$ implies that $\mathbf{1}_{v_2} \succ^f \mathbf{1}_{v_1-\epsilon}$ for all $\epsilon > 0$. Hence, there cannot exist $z \in D$ such that $\mathbf{1}_{v_1} \succ^f z \succ^f \mathbf{1}_{v_2}$.

4.3   Virtual Utility Maximization

We saw that the linear ordering that strongly rationalizes an allocation rule may not have a utility representation. The aim of this section is to explore minimal conditions that allow us to define a new ordering for any implementable and non-bossy allocation rule which has a utility representation. This allows us to characterize a broad class of allocation rules. Our extra condition is a *continuity* condition.

DEFINITION 9 *An allocation rule f is **continuous** if for every $i, j \in N$ $(i \neq j)$, for every value profile v with $f(v) = a_i$, and for every $\epsilon > 0$, there exists a $\delta_{\epsilon,v} > 0$ such that $f(v_i + \epsilon, v_j + \delta_{\epsilon,v}, v_{-ij}) = a_i$.*

*An allocation rule f is **uniformly continuous** if for every $i, j \in N$ $(i \neq j)$ and for every v_{-ij}, for every $\epsilon > 0$, there exists a $\delta_{\epsilon,v_{-ij}} > 0$ such that for every v_i, v_j with $f(v_i, v_j, v_{-ij}) = a_i$, we have $f(v_i + \epsilon, v_j + \delta_{\epsilon,v_{-ij}}, v_{-ij}) = a_i$.*

Both continuity and uniform continuity require a version of robustness of the allocation rule. Note that if f is uniformly continuous, then it is continuous. Further, in the definitions above, we can assume $\delta_{\epsilon,v_{-ij}}$ (or $\delta_{\epsilon,v}$ for continuous case) to be less than ϵ without loss of generality if f is monotone and non-bossy.

We will now introduce a new class of allocation rules.

DEFINITION 10 *An allocation rule f is a **virtual utility maximizer (VUM)** if there exists an increasing function $U_i : V_i \to \mathbb{R}$ for every $i \in N \cup \{0\}$, where $U_0(0) = 0$, such that for every valuation profile $v \in V$, $f(v) = a_j$ implies that $j \in \arg\max_{j \in N \cup \{0\}} U_j(v_j)$.*

LEMMA 5 *A VUM allocation rule is implementable.*

Proof: Suppose f is a VUM allocation rule with corresponding virtual utility functions $U_0, U_1, \ldots, U_n$. Fix an agent i and the valuation profile of other agents at v_{-i}. Consider v_i, v_i' such that $v_i < v_i'$ and $f(v_i, v_{-i}) = a_i$. Then, by VUM maximization, $U_i(v_i) \geq U_j(v_j)$ for all $j \in N \cup \{0\}$. Since U_i is increasing, $U_i(v_i') > U_j(v_j)$ for all $j \in (N \cup \{0\}) \setminus \{i\}$. This implies that $f(v_i', v_{-i}) = a_i$. So, f is monotone, and hence, implementable. ∎

The VUM allocation rule can have many allocations which maximize the sum of virtual utilities. We propose a modification of the VUM allocation rules which breaks these ties using an ordering.

DEFINITION **11** *An allocation rule f is a* **virtual utility maximizer (VUM) with order-based tie-breaking** *if there exists an increasing function $U_i : V_i \to \mathbb{R}$ for every $i \in N \cup \{0\}$, where $U_0(0) = 0$, and a monotone linear ordering $\succ$ on D such that for every valuation profile $v \in V$, $f(v) = a_j$ implies that $j \in \arg\max_{i \in N \cup \{0\}} U_i(v_i)$ and $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_k}$ for all $k \neq j$ and $k \in \arg\max_{i \in N \cup \{0\}} U_i(v_i)$, i.e., $\mathbf{1}_{v_j}$ is the unique virtual utility maximizer according to $\succ$.*

A VUM allocation rule with order-based tie-breaking is also a VUM allocation rule. Hence, by Lemma 5, it is implementable. However, unlike a VUM allocation rule, a VUM allocation rule with order-based tie-breaking is non-bossy.

LEMMA **6** *A VUM allocation rule with order-based tie-breaking is non-bossy.*

Proof: Let f be a VUM allocation rule with order-based tie-breaking and v be a valuation profile such that $f(v) \neq a_j$ for some $j \in N$. Suppose $f(v_j', v_{-j}) \neq a_j$. Then, by definition, the unique virtual utility maximizer of f remains the same in (v_i, v_{-i}) and (v_i', v_{-i}). So, $f(v_i, v_{-i}) = f(v_i', v_{-i})$, and hence, f is non-bossy. ∎

We are now ready to state the main result of this section.

THEOREM **3** *Suppose $V_i = (0, \beta_i)$, where $\beta_i \in \mathbb{R}_{++} \cup \{\infty\}$, for all $i \in N$ and f is a uniformly continuous allocation rule. Then, the following statements are equivalent.*

1. *f is an implementable and non-bossy allocation rule.*

2. *f is a virtual utility maximizer allocation rule with order-based tie-breaking.*

4.3.1 Remarks on Theorem 3

- VUM ALLOCATION RULES AND NON-BOSSINESS. It can be easily seen that not every VUM allocation rule is non-bossy. For instance, consider the efficient allocation rule that allocates the good to an agent with the highest value. Suppose there are three

agents with valuations $10, 10, 8$ respectively and suppose that the efficient allocation rule allocates the object to agent 1. Consider the valuation profile $(10, 10, 9)$ and suppose that the efficient allocation rule now allocates the object to agent 2. This violates non-bossiness. Theorem 3 shows that such violations can happen in case of ties (as was the case here with ties between agents 1 and 2), and when ties are broken carefully (using an order-based tie-breaking), a VUM allocation rule becomes non-bossy.

- SOME VIRTUAL UTILITY MAXIMIZERS. An efficient allocation rule is also a VUM allocation rule, where $U_i(v_i) = v_i$ for all $i \in N$ and for all $v_i \in V_i$. Similarly, we can define for every $i \in N$ and for every $v_i \in V_i$, $U_i(v_i) = \lambda_i v_i + \kappa_i$ for some $\lambda_i \geq 0$ and $\kappa_i \in \mathbb{R}$, and this VUM will correspond to the affine maximizer allocation rules of Roberts (1979). The virtual utility function in Myerson (1981) takes the form $U_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$, where F_i and f_i are respectively the cumulative density function and density function of the distribution of valuation of agent i.

- PAYMENTS. It is well known that revenue equivalence (Myerson, 1981) implies that for any implementable allocation rule, the payments are determined uniquely up to an additive constant. Suppose V_i is an interval for all $i \in N$. For any implementable allocation rule f, define the cutoff for agent i and valuation profile v_{-i} as $\kappa_i^f(v_{-i}) = \inf\{\alpha \in V_i : f(\alpha, v_{-i}) = a_i\}$, where $\kappa_i^f(v_{-i}) = 0$ if $f(\alpha, v_{-i}) \neq a_i$ for all $\alpha \in V_i$. It is well known that for every $i \in N$ and for every $(v_i, v_{-i}) \in V$, $p_i^f(v_i, v_{-i}) = \kappa_i^f(v_{-i})$ if $f(v_i, v_{-i}) = a_i$ and $p_i^f(v_i, v_{-i}) = 0$ if $f(v_i, v_{-i}) \neq a_i$ is a payment rule which implements f. Further, by revenue equivalence, any payment rule p which implements f must satisfy for every $i \in N$ and for every (v_i, v_{-i}), $p_i(v_i, v_{-i}) = p_i^f(v_i, v_{-i}) + h_i(v_{-i})$, where $h_i : V_{-i} \to \mathbb{R}$ is any function. Such cutoffs are easy to determine for generalized utility maximizers. Thus, by characterizing implementable allocation rules, we characterize the class of dominant strategy incentive compatible mechanisms.

- OTHER VERSIONS OF NON-BOSSIENESS. Another version of non-bossiness, which seem appealing is the *utility non-bossiness*. Utility non-bossiness is a condition on *mechanisms* rather than on allocation rules only. In particular, an incentive compatible mechanism (f, p) satisfies **utility non-bossiness** if for every $i \in N$, for every v_{-i}, and for every $v_i, v_i' \in V_i$, such that $v_i f_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) = v_i' f_i(v_i', v_{-i}) - p_i(v_i', v_{-i})$, we have $v_j f_j(v_i, v_{-i}) - p_j(v_i, v_{-i}) = v_j f_j(v_i', v_{-i}) - p_j(v_i', v_{-i})$ for all $j \in N$. In words, if an agent changes his valuation such that his net utility does not change, then the net utility of every agent must remain unchanged.

We do not impose such version of utility non-bossiness because this is a condition on

mechanisms, and we are interested in conditions on *allocation rules*. Further, utility non-bossiness is not satisfied by many canonical mechanisms. For instance, the second-price Vickrey auction is not utility non-bossy. To see this, consider an example with two agents with valuations 10 and 7 respectively. Note that the allocation rule in a second-price Vickrey auction is an efficient allocation rule. The net utilities of agents 1 and 2 in the second-price Vickrey auction are 3 and 0 respectively. Now, consider the valuation profile $(10, 8)$. At this valuation profile, agent 2 continues to get zero net utility in the second price Vickrey auction, but the net utility of agent 1 is reduced to 2. This shows that the second-price Vickrey auction is not utility non-bossy. On the other hand, the efficient allocation rule with order-based tie-breaking is a non-bossy allocation rule.

4.3.2   Proof of Theorem 3

By Lemmas 5 and 6, a VUM allocation rule with order-based tie-breaking is implementable and non-bossy. We show that every implementable, non-bossy, and uniformly continuous allocation rule is a VUM allocation rule with order-based tie-breaking. We do the proof in various steps. Throughout we assume that $V_i = (0, \beta_i)$, where $\beta_i \in \mathbb{R}_{++} \cup \{\infty\}$, for all $i \in N$.

STEP 1. In this step, we show that if f is implementable, non-bossy, and uniformly continuous allocation rule, then there is an ordering $\succeq^f$ on D which rationalizes f. We construct this specific $\succeq^f$ in this step [11].

Suppose f is an implementable, non-bossy, and uniformly continuous allocation rule. We first define the notion of a **winning set**. The winning set of allocation rule f at a valuation profile v is denoted as $W^f(v)$, and defined as follows. For any $i \in N$, we say $a_i \in W^f(v)$ if for all $\epsilon > 0$, we have $f(v_i + \epsilon, v_{-i}) = a_i$, where $(v_i + \epsilon) \in V_i$. We say that $a_0 \in W^f(v)$ if for all $\epsilon > 0$, we have $f(\{v_j - \epsilon\}_{j \in N}) = a_0$, where $(v_j - \epsilon) \in V_j$ for all $j \in N$. The first claim is that $W^f(v)$ is non-empty for all valuation profiles v.

LEMMA 7 *If f is implementable and non-bossy, then for every value profile v, $f(v) \in W^f(v)$.*

Proof: Consider an implementable and non-bossy allocation rule f and a value profile v. If $f(v) = a_j \neq a_0$, then by monotonicity $f(v_j + \epsilon, v_{-j}) = a_j$ for all $\epsilon > 0$. Hence, $f(v) \in W^f(v)$.

If $f(v) = a_0$, then consider any $\epsilon > 0$ and a valuation profile v' such that $v'_i - \epsilon > 0$ for all $i \in N$. We argue that $f(v') = a_0$, and hence, $a_0 = f(v) \in W^f(v)$. Assume for

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[11]Notice that by Theorem 2, if f is implementable and non-bossy, then it is a strongly rationalizable allocation rule, and hence, a rationalizable allocation rule. The novelty of this step of the proof is to be able to construct a *specific* ordering which rationalizes f.

contradiction that $f(v') = a_j \neq a_0$. Now, we go from v' to v by increasing the valuation of one agent at a time. By monotonicity, $f(v_j, v'_{-j}) = a_j$. Now, pick any $k \in N \setminus \{j\}$. Then, either $f(v_j, v_k, v'_{-jk}) = a_k$ or by non-bossiness $f(v_j, v_k, v'_{-jk}) = a_j$. In both cases, we see that $f(v_j, v_j, v'_{-jk}) \neq a_0$. Continuing in this manner, we will reach the valuation profile v and get that $f(v) \neq a_0$, a contradiction. ∎

STEP 1.1. In this step, we show that an implementable, non-bossy, and uniformly continuous allocation rule satisfies a form of independence property.

DEFINITION 12 *An allocation rule f satisfies* **binary independence** *if for any pair of alternatives $a_j, a_k \in A$ and any pair of valuation profiles v, v' such that $\mathbf{1}_{v_j} = \mathbf{1}_{v'_j}$ and $\mathbf{1}_{v_k} = \mathbf{1}_{v'_k}$, the following conditions hold.*

1. *if $a_k \in W^f(v)$ and $a_j \in W^f(v')$, then $a_k \in W^f(v')$,*

2. *if $a_j \in W^f(v)$ and $a_k \notin W^f(v)$, then $a_k \notin W^f(v')$.*

Intuitively, the binary independence property says that the comparison of any pair of utility vectors is independent of what the other utility vectors are.

PROPOSITION 1 *An implementable, non-bossy, and uniformly continuous allocation rule satisfies binary independence.*

Proof: The proof is in the Appendix. ∎

STEP 1.2. In this step, we define an ordering on the set of utility vectors D. We denote this ordering as $\succeq^f$. The anti-symmetric part of this ordering is denoted as $\succ^f$ and the symmetric part is denoted as $\sim^f$. For any $i \in N$ and for any $v_i, v'_i \in V_i$ with $v_i > v'_i$, we define $\mathbf{1}_{v_i} \succ^f \mathbf{1}_{v'_i}$. Further, for every $i \in N$ and every $v_i \in V_i$, we define $\mathbf{1}_{v_i} \sim^f \mathbf{1}_{v_i}$ (reflexive). For any $i, j \in N \cup \{0\}$ and any $v_i \in V_i$ and $v_j \in V_j$, we define

1. $\mathbf{1}_{v_i} \succ^f \mathbf{1}_{v_j}$, if there exists a valuation profile v' such that $\mathbf{1}_{v'_i} = \mathbf{1}_{v_i}$, $\mathbf{1}_{v'_j} = \mathbf{1}_{v_j}$, and $a_i \in W^f(v')$ but $a_j \notin W^f(v')$;

2. $\mathbf{1}_{v_i} \sim^f \mathbf{1}_{v_j}$, if (a) there exists a valuation profile v' such that $\mathbf{1}_{v'_i} = \mathbf{1}_{v_i}$, $\mathbf{1}_{v'_j} = \mathbf{1}_{v_j}$, and $a_i, a_j \in W^f(v')$ or (b) at every valuation profile v' such that $\mathbf{1}_{v'_i} = \mathbf{1}_{v_i}$, and $\mathbf{1}_{v'_j} = \mathbf{1}_{v_j}$, we have $a_i, a_j \notin W^f(v')$.

We show that the binary relation $\succeq$ is well defined.

LEMMA 8 *Suppose f is implementable, non-bossy, and uniformly continuous. Then, $\succeq^f$ is well-defined.*

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Proof: Fix some $x, y \in D$. If $x, y \in D_i$ for some $i \in N$, and $x = \mathbf{1}_{v_i}$ and $y = \mathbf{1}_{v_i'}$ with $v_i > v_i'$ then, by definition, $x \succ^f y$. Similarly, if $x \in D_i$ and $y \in D_j$ for some $i \neq j$, and for every valuation profile v with $\mathbf{1}_{v_i} = x$ and $\mathbf{1}_{v_j} = y$ we have $a_i, a_j \notin W^f(v)$, then, by definition, $x \sim^f y$.

So, we just need to consider the case where $x \in D_i$ and $y \in D_j$ for some $i \neq j$, and there is a valuation profile v with $\mathbf{1}_{v_i} = x$ and $\mathbf{1}_{v_j} = y$ with either a_i or a_j or both are in $W^f(v)$. We consider two cases.

CASE 1. Suppose $a_i, a_j \in W^f(v)$. Now, consider any other valuation profile v' such that $\mathbf{1}_{v_i} = \mathbf{1}_{v_i'} = x$ and $\mathbf{1}_{v_j} = \mathbf{1}_{v_j'} = y$. By Proposition 1, $a_i \in W^f(v')$ if and only if $a_j \in W^f(v')$. This means that the relation $x \sim^f y$ is well-defined.

CASE 2. Suppose $a_i \in W^f(v)$ but $a_j \notin W^f(v)$. Now, consider any other valuation profile v' such that $\mathbf{1}_{v_i} = \mathbf{1}_{v_i'} = x$ and $\mathbf{1}_{v_j} = \mathbf{1}_{v_j'} = y$. By Proposition 1, $a_j \notin W^f(v')$. This means that the relation $x \succ^f y$ is well-defined. ∎

STEP 1.3. In this step, we show that $\succeq^f$ is an ordering, i.e., the binary relation is reflexive, complete, and transitive. The fact that $\succeq^f$ is reflexive and complete is clear. We show that $\succeq^f$ is transitive.

PROPOSITION 2 *If f is an implementable, non-bossy, and uniformly continuous allocation rule, then $\succeq^f$ is transitive.*

Proof: The proof is in the Appendix. ∎

STEP 1.4. We conclude Step 1 by showing that f is a rationalizable allocation rule and $\succeq^f$ rationalizes f. Note that the ordering $\succeq^f$, defined in Steps 1.2 and 1.3, is a monotone ordering. By Lemma 7, for every valuation profile v, $f(v) \in W^f(v)$. Hence, by definition of $\succeq^f$, $G^f(v) \succeq^f \mathbf{1}_{v_i}$ for all $i \in N \cup \{0\}$. This shows that f is a rationalizable allocation rule and $\succeq^f$ rationalizes f.

STEP 2. In this step, we show that if f is a non-bossy and uniformly continuous allocation rule, then it is implementable if and only if it is a VUM allocation rule. By Lemma 5, a VUM allocation rule is implementable. Suppose f is an implementable, non-bossy, and uniformly continuous allocation rule. By Step 1, f can be rationalized by the monotone ordering $\succeq^f$, defined as in Step 1.2. We say that $\succeq^f$ has a **utility representation** if there exists a utility function $U : D \to \mathbb{R}$ such that for all $x, y \in D$ we have $U(x) > U(y)$ if and only if $x \succ^f y$.

STEP 2.1. In this step, we will show that $\succeq^f$ is separable in the sense of Definition 8. Let $Z := \{x \in D : x = \mathbf{1}_{v_i} \text{ for some } i \in N \cup \{0\} \text{ and } v_i \text{ is rational}\}$. Note that since the set of rational numbers is countable, Z is a countable subset of D. Now, pick $x, y \in D$ such that $x \succ^f y$. If $x, y \in D_i$ for some $i \in N$, then let $x = \mathbf{1}_{v_i}$ and $y = \mathbf{1}_{v_i'}$. By definition, $v_i > v_i'$. Then, we can find a rational v_i'' such that $v_i > v_i'' > v_i'$ (this is because the set of rational numbers is a dense set). Let $z = \mathbf{1}_{v_i''}$. By definition, $z \in Z$ and $x \succ^f z \succ^f y$. Now, assume that $x = \mathbf{1}_{v_i}$ and $y = \mathbf{1}_{v_j}$ for some $i, j \in N \cup \{0\}$ with $i \neq j$. We consider various cases.

CASE A. Suppose $i \neq 0$ and $j \neq 0$. Since $x \succ^f y$, there is a valuation profile $v \equiv (v_i, v_j, v_{-ij})$ such that $a_i \in W^f(v)$ but $a_j \notin W^f(v)$. Since $a_j \notin W^f(v)$, there is some $\epsilon > 0$ such that $f(v_i, v_j + \epsilon, v_{-ij}) \neq a_j$. This means that $a_j \notin W^f(v_i, v_j + \frac{\epsilon}{2}, v_{-ij})$. Consider any $\delta > 0$. Since $f(v_i, v_j + \frac{\epsilon}{2}, v_{-ij}) \neq a_j$, by monotonicity and non-bossiness, $f(v_i + \delta, v_j + \frac{\epsilon}{2}, v_{-ij}) \neq a_j$. Since $a_i \in W^f(v)$, $f(v_i + \delta, v_j, v_{-ij}) = a_i$. By monotonicity and non-bossiness, $f(v_i + \delta, v_j + \frac{\epsilon}{2}, v_{-ij}) \in \{a_i, a_j\}$. This implies that $f(v_i + \delta, v_j + \frac{\epsilon}{2}, v_{-ij}) = a_i$. Hence, $a_i \in W^f(v_i, v_j + \frac{\epsilon}{2}, v_{-ij})$. Then, $x = \mathbf{1}_{v_i} \succ \mathbf{1}_{v_j + \frac{\epsilon}{2}} \succ \mathbf{1}_{v_j} = y$. Since the set of rational numbers is dense, we can find a $z \in Z$ arbitrarily close to $\mathbf{1}_{v_j + \frac{\epsilon}{2}}$ such that $x \succ^f z \succ^f y$.

CASE B. Suppose $i \neq 0$ and $j = 0$. Since $x \succ^f y$, there is a valuation profile (v_i, v_{-i}) such that $a_i \in W^f(v_i, v_{-i})$ but $a_0 \notin W^f(v_i, v_{-i})$. This means for some $\delta > 0$, we have $f(\{v_j - \delta\}_{j \in N}) \neq a_0$. Suppose $f(\{v_j - \delta\}_{j \in N}) = a_k$ for some $k \neq 0$. Then, $\mathbf{1}_{v_k - \delta} \succeq^f y$. Since $a_i \in W^f(v_i, v_{-i})$, we get that $x = \mathbf{1}_{v_i} \succeq^f \mathbf{1}_{v_k} \succ^f \mathbf{1}_{v_k - \delta}$. Hence, $x \succ^f \mathbf{1}_{v_k - \delta} \succeq^f y$. Since the set of rational numbers is dense, we can choose a $z \in Z$ arbitrarily close to $\mathbf{1}_{v_k - \delta}$ such that $x \succ^f z \succeq^f y$.

CASE C. Suppose $i = 0$ and $j \neq 0$. Since $x \succ^f y$, there is a valuation profile (v_j, v_{-j}) such that $a_j \notin W^f(v_j, v_{-j})$ but $a_0 \in W^f(v_j, v_{-j})$. Then, for some $\epsilon > 0$, we have $f(v_j + \epsilon, v_{-j}) = a_k$, where $k \neq j$. This implies that $\mathbf{1}_{v_k} \succeq^f \mathbf{1}_{v_j + \epsilon} \succ^f \mathbf{1}_{v_j} = y$. But $a_0 \in W^f(v_j, v_{-j})$ implies that $x \succeq^f \mathbf{1}_{v_k}$. Hence, $x \succeq^f \mathbf{1}_{v_j + \epsilon} \succ^f y$. Since the set of rational numbers is dense, we can find $z \in Z$ arbitrarily close to $\mathbf{1}_{v_j + \epsilon}$ such that $x \succeq^f z \succ^f y$.

This shows that $\succeq^f$ is separable. Using Debreu (1954), $\succeq^f$ has a utility representation. Let $U : D \to \mathbb{R}$ be a utility function representing $\succeq^f$. Without loss of generality, we can assume $U(\mathbf{1}_{v_0}) = 0$. Now, for every $i \in N \cup \{0\}$, define $U_i : V_i \to \mathbb{R}$ as follows: $U_i(v_i) = U(\mathbf{1}_{v_i})$ for all $v_i \in V_i$. Note that by the definition of $\succeq^f$, each U_i is well-defined and increasing.

Since U represents $\succeq^f$ and f is a rationalizable allocation rule with $\succeq^f$ being the corresponding ordering, we get that for all valuation profiles v, $f(v) \in \arg\max_{i \in N \cup \{0\}} U_i(v_i)$. Hence, f is a VUM allocation rule.

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By Theorem 2, f is a strongly rationalizable allocation rule. Let $\succ$ be the linear ordering that strongly rationalizes f. By definition, for all $x \in D^f$ and for all $y \notin D^f$, $x \succ y$. Further, for all $v \in V$ if $f(v) = a_j$, then $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_i}$ for all $i \neq j$. In that case, $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_k}$ for all $k \neq j$ and $k \in \arg\max_{i \in N \cup \{0\}} U_i(v_i)$. Hence, f is a VUM allocation rule with order-based tie-breaking.

5 CONCLUSIONS

We conclude by pointing some future research directions.

- Although we focus on deterministic dominant strategy implementation, randomization is a natural extension of our model. Indeed, the monotonicity characterization of Myerson (1981) extends to single object auctions with randomization. However, the generalization of Roberts' theorem with randomization is not known in any domain, let alone in the single object auction model.

- Another extension of our results is to consider the weaker notion of Bayesian implementation. Myerson (1981) shows that an appropriate extension of monotonicity is necessary and sufficient for Bayesian implementation. The extension of our parametric characterizations to Bayesian implementation is a direction for future research. Recently, Manelli and Vincent (2010) show an equivalence between Bayesian and dominant strategy implementation in single object auction models with randomization - see its generalizations in Gershkov et al. (2012). Hence, characterizing dominant strategy implementation with randomization also characterizes all equivalent (in the sense of Manelli and Vincent (2010)) Bayesian implementable allocation rules.

- It will be interesting to extend our results and Roberts' theorem to other mechanism design problems - for instance to multi-object auction models. However, as we saw in Theorem 1, the set of implementable allocation rules in such restricted environments can be quite complex to capture using extensions of Roberts' theorem. Are there reasonable assumptions that can be imposed on implementable allocation rules such that a simpler class of allocation rules can be characterized in the spirit of Roberts' theorem in these models?

APPENDIX: OMITTED PROOFS

PROOF OF PROPOSITION 1.

Proof: We will use the following lemma to prove the proposition.

LEMMA **9** *Suppose v and v' are two distinct valuation profiles such that $v_i \geq v'_i$ for all $i \in N$. Let $B(v, v') = \{a_i \in A : v_i > v'_i\}$. If f is an implementable and non-bossy allocation rule, then $W^f(v) \setminus B(v, v') \subseteq W^f(v')$.*

Proof: Let f be an implementable and non-bossy allocation rule and v and v' be two distinct valuation profiles with $v_i \geq v'_i$ for all $i \in N$. We will go from v to v' by lowering one agent's value at a time. Pick any $a_j \in B(v, v')$. Consider a new type profile v'' such that the value of every agent $i \neq j$ remains v_i and the value of agent j is v'_j, which is strictly less than v_j. Pick any $a_k \in W^f(v)$ such that $a_k \neq a_j$. Then, we consider two cases.

CASE 1: $a_k \neq a_0$. We argue that $a_k \in W^f(v'')$. Assume for contradiction that $a_k \notin W^f(v'')$. Then, for some $\epsilon > 0$, we have $f(v_k + \epsilon, v'_j, v_{-kj}) \neq a_k$. If $f(v_k + \epsilon, v'_j, v_{-kj}) = a_j$, then by monotonicity, we have $f(v_k + \epsilon, v_j, v_{-kj}) = a_j$. This is a contradiction since $a_k \in W^f(v)$. If $f(v_k + \epsilon, v'_j, v_{-kj}) = a_l \notin \{a_j, a_k\}$. But monotonicity and non-bossiness implies that $f(v_k + \epsilon, v_j, v_{-kj}) \in \{a_l, a_j\}$. But this contradicts $a_k \in W^f(v)$.

CASE 2: $a_k = a_0$. Since $a_0 \in W^f(v)$, for any $\epsilon > 0$ such that $\bar{v}_i := v_i - \epsilon > 0$ for all $i \in N$, we have $f(\bar{v}_1, \ldots, \bar{v}_n) = a_0$. Note that $v'_i - \epsilon = v_i - \epsilon = \bar{v}_i$ for all $i \neq j$ for any ϵ. Now, fix any $\epsilon > 0$ such that $v'_j - \epsilon > 0$. Consider the valuation profile $(\bar{v}_{-j}, v'_j - \epsilon)$. Since $f(\bar{v}_1, \ldots, \bar{v}_n) = a_0$ and $\bar{v}_j = v_j - \epsilon > v'_j - \epsilon$, by monotonicity and non-bossiness, we have $f(\bar{v}_{-j}, v'_j - \epsilon) = a_0$. Hence, $a_0 \in W^f(v'')$.

This establishes that $a_k \in W^f(v'')$ for any $a_k \neq a_j$. Hence, $W^f(v) \setminus \{a_j\} \subseteq W^f(v'')$. Repeating this argument for other elements of $B(v, v')$ one by one, we conclude that $W^f(v) \setminus B(v, v') \subseteq W^f(v')$. ∎

Now, let f be an implementable, non-bossy, and uniformly continuous allocation rule. Pick any pair of alternatives $a_j, a_k \in A$ and any pair of valuation profiles v, v' such that $\mathbf{1}_{v_j} = \mathbf{1}_{v'_j}$ and $\mathbf{1}_{v_k} = \mathbf{1}_{v'_k}$. We will show that f satisfies both (1) and (2) of Definition 12.

1. Suppose $a_k \in W^f(v)$ and $a_j \in W^f(v')$. We will show that $a_k \in W^f(v')$. Construct a new type profile v'' such that $v''_i = \min(v_i, v'_i)$ for all $i \in N$. Note that $\mathbf{1}_{v''_j} = \mathbf{1}_{v_j} = \mathbf{1}_{v'_j}$ and $\mathbf{1}_{v''_k} = \mathbf{1}_{v_k} = \mathbf{1}_{v'_k}$. By Lemma 9, $a_j, a_k \in W^f(v'')$. Now, assume for contradiction that $a_k \notin W^f(v')$. We now consider various cases.

   CASE 1: $a_j, a_k \in A \setminus \{a_0\}$. Since $a_k \notin W^f(v')$, there exists $\epsilon > 0$ such that $f(v'_k + \epsilon, v'_{-k}) \neq a_k$. By monotonicity and non-bossiness, for all $\epsilon' > 0$ we have $f(v'_j + \epsilon', v'_k + \epsilon, v'_{-jk}) \neq a_k$. Further, we show that $f(v'_j + \epsilon', v'_k + \epsilon, v'_{-jk}) = a_j$ for all $\epsilon' > 0$. To see

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this, suppose $f(v'_j + \epsilon', v'_k + \epsilon, v'_{-jk}) = a_l$ for some $a_l \notin \{a_j, a_k\}$. Then, by monotonicity and non-bossiness, we get $f(v'_j + \epsilon', v'_k, v'_{-jk}) = a_l$, and this contradicts $a_j \in W^f(v')$. Hence, $f(v'_j + \epsilon', v'_k + \epsilon, v'_{-jk}) = a_j$ for all $\epsilon' > 0$. Now, applying monotonicity and non-bossiness again, for all $\epsilon' > 0$, we have

$$f(v'_j + \epsilon', v'_k + \epsilon, v''_{-jk}) = a_j. \tag{1}$$

Since $a_k \in W^f(v'')$, we have $f(v'_j, v'_k + \frac{\epsilon}{2}, v''_{-jk}) = a_k$. By continuity, there is an $\epsilon' > 0$ such that $f(v'_j + \epsilon', v'_k + \epsilon, v''_{-jk}) = a_k$. This contradicts Equation 1.

CASE 2: $a_j = a_0$. We have to show that $a_0 \in W^f(v')$ implies $a_k \in W^f(v')$. Assume for contradiction that $a_k \notin W^f(v')$ but $a_0 \in W^f(v')$. For this, we first show there is some $\epsilon_i > 0$ for every $i \in N$ such that $f(v'_k + \epsilon_k, \{v'_i - \epsilon_i\}_{i \neq k}) = a_0$.

To see this, suppose $f(v'_k + \epsilon_k, \{v'_i - \epsilon_i\}_{i \neq k}) = a_k$. Fix any $l \neq k$. Then, by uniform continuity, for every ϵ there is a δ such that, $f(v'_k + \epsilon_k + \epsilon, (v'_l - \epsilon_l + \delta), \{v'_i - \epsilon_i\}_{i \neq k,l}) = a_k$. By, uniform continuity, we can choose $\epsilon_l = \delta$. Also, let $\epsilon_k = \epsilon$. Hence, we get $f(v'_k + 2\epsilon_k, v'_l, \{v'_i - \epsilon_i\}_{i \neq k,l}) = a_k$. Repeating this, we reach $f(v'_k + (n-1)\epsilon_k, v'_{-k}) = a_k$. But this contradicts the fact that $a_k \notin W^f(t)$.

Similarly, suppose $f(v'_k + \epsilon_k, \{v'_i - \epsilon_i\}_{i \neq k}) = a_l$ for some $l \neq 0, k$. Then, by monotonicity and non-bossiness, we get that $f(\{v'_i - \epsilon_i\}_{i \in N}) = a_l$. This means $f(\{v'_i - \epsilon_i\}_{i \in N}) \neq a_0$. Now, choose $\epsilon' < \min_{i \in N} \epsilon_i$. Then, consider the profile $\{v'_i - \epsilon'\}_{i \in N}$. By repeated application of monotonicity and non-bossiness, $f(\{v'_i - \epsilon'\}_{i \in N}) \neq a_0$. This contradicts $a_0 \in W^f(v')$.

This shows that there is some $\epsilon_i > 0$ for all $i \in N$ such that $f(v'_k + \epsilon_k, \{v'_i - \epsilon_i\}_{i \neq k}) = a_0$. By monotonicity and non-bossiness, $f(v'_k + \epsilon_k, \{v''_i - \epsilon_i\}_{i \neq k}) = a_0$. But $a_k \in W^f(v'')$ implies that $f(v''_k + \epsilon_k, v''_{-k}) = a_k$ (to remind, $v'_k = v''_k$). But monotonicity and non-bossiness implies that $f(v'_k + \epsilon_k, \{v''_i - \epsilon_i\}_{i \neq k}) = a_k$. This gives us a contradiction.

CASE 3: $a_k = a_0$. We have to show that if $a_j \in W^f(v')$ then $a_0 \in W^f(v')$. Assume for contradiction $a_0 \notin W^f(v')$. We first show that for some $\epsilon > 0$ and $\epsilon' > 0$, $f(v'_j - \epsilon, \{v'_i - \epsilon'\}_{i \neq j}) = a_j$.

To see this, suppose that $f(v'_j - \epsilon, \{v'_i - \epsilon'\}_{i \neq j}) = a_0$ for all ϵ, ϵ'. Then, by monotonicity and non-bossiness, we see that $f(\{v'_i - \min(\epsilon, \epsilon')\}_{i \in N}) = a_0$ for all ϵ, ϵ'. But this contradicts $a_0 \notin W^f(v')$.

Similarly, suppose that $f(v'_j - \epsilon, \{v'_i - \epsilon'\}_{i \neq j}) = a_l$ for some $l \in N \setminus \{j\}$ and for all ϵ, ϵ'. By uniform continuity, there is some $\delta := \delta_{\epsilon', v'_{-lj}} < \epsilon'$ such that $f(v'_j - \epsilon + \delta, v'_l, \{v'_i - \epsilon'\}_{i \neq j,l}) =$

a_l for all ϵ, ϵ'. Since δ is independent of ϵ, we can choose $\epsilon = \frac{\delta}{2}$ for every ϵ'. Hence, we have $f(v'_j + \frac{\delta}{2}, v'_l, \{v'_i - \epsilon'\}_{i \neq j,l}) = a_l$ for every ϵ'. Further, since $a_j \in W^f(v')$, we know that $f(v'_j + \frac{\delta}{2}, v'_{-j}) = a_j$ for all ϵ'. By repeatedly applying monotonicity and non-bossiness, we get that $f(v'_j + \frac{\delta}{2}, v'_l, \{v'_i - \epsilon'\}_{i \neq j,l}) = a_j$ for every ϵ'. This gives us a contradiction.

This shows that $f(v'_j - \epsilon, \{v'_i - \epsilon'\}_{i \neq j}) = a_j$ for some $\epsilon > 0$ and $\epsilon' > 0$. By repeatedly applying monotonicity and non-bossiness, we get that $f(v'_j - \epsilon, \{v''_i - \epsilon'\}_{i \neq j}) = a_j$ for some $\epsilon > 0$ and $\epsilon' > 0$. Since $a_0 \in W^f(v'')$, we know that $f(\{v'_i - \min(\epsilon, \epsilon')\}_{i \in N}) = a_0$. By repeatedly applying monotonicity and non-bossiness, we get that $f(v'_j - \epsilon, \{v''_i - \epsilon'\}_{i \neq j}) = a_0$. This is a contradiction.

This concludes the proof of Property (1) in Definition 12.

2. Property (2) in Definition 12 follows by applying Property (1). To see this, pick any $a_j, a_k \in A$ and v, v' as in Definition 12. Suppose $a_j \in W^f(v)$ but $a_k \notin W^f(v')$. We need to show that $a_k \notin W^f(v')$. Assume for contradiction $a_k \in W^f(v')$. Then, by changing the role of v and v' in (1), we get that $a_k \in W^f(v)$, which is a contradiction.

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PROOF OF PROPOSITION 2.

Proof: For this, we will show that $\succ^f$ and $\sim^f$ are transitive, and this in turn will imply that $\succeq^f$ is transitive. Pick any $x, y, z \in D$ such that $x \neq y \neq z$. We consider three cases.

CASE 1. Suppose $x, y, z \in D_i$ for some $i \in N$ and $x = \mathbf{1}_{v_i}, y = \mathbf{1}_{v'_i}, z = \mathbf{1}_{v''_i}$. Suppose $x \succ^f y$ and $y \succ^f z$. Then, it must be $v_i > v'_i > v''_i$. By definition, we have $x \succ^f z$.

CASE 2. $x, y \in D_i$ but $z \in D_j$ for some i, j where $i \neq j$. Suppose $x = \mathbf{1}_{v_i}, y = \mathbf{1}_{v'_i}$, and $z = \mathbf{1}_{v_j}$. Suppose $x \succ^f y$ and $y \succ^f z$. Note that $x \succ^f y$ implies $v_i > v'_i$. We consider two subcases.

CASE 2A. Suppose $j \neq 0$. Then, there is a valuation profile v'' such that $v''_i = v'_i$, $v''_j = v_j$, and $a_i \in W^f(v'')$ but $a_j \notin W^f(v'')$. Now consider the type profile $\bar{v}$, where $\bar{v}_k = v''_k$ if $k \neq i$ and $\bar{v}_i = v_i$. We show that $a_i \in W^f(\bar{v})$ and $a_j \notin W^f(\bar{v})$, and this will show that $x P^f z$. Since $a_i \in W^f(v'')$, we know that $f(v'_i + \epsilon, v_j, v''_{-ij}) = a_i$ for all $\epsilon > 0$. By monotonicity, $f(v_i + \epsilon, v_j, v''_{-ij}) = a_i$ for all $\epsilon > 0$. Hence, $a_i \in W^f(\bar{v})$. Since $a_j \notin W^f(v'')$, there is some $\epsilon > 0$ such that $f(v'_i, v_j + \epsilon, v''_{-ij}) \neq a_j$. By monotonicity and non-bossiness,

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$f(v_i, v_j + \epsilon, v''_{-ij}) \neq a_j$. Hence, $a_j \notin W^f(\bar{v})$.

CASE 2B. Suppose $j = 0$. So, z is the n-dimensional zero vector. Since $yP^f z$, there is a valuation profile $\bar{v}$ with $\mathbf{1}_{\bar{v}_i} = \mathbf{1}_{v'_i} = y$ and $a_i \in W^f(\bar{v})$ but $a_0 \notin W^f(\bar{v})$. Now, consider the valuation profile $v'' \equiv (v_i, \bar{v}_{-i})$. Since $v_i > v'_i$, by monotonicity, we have $a_i \in W^f(v'')$.

Since $a_0 \notin W^f(\bar{v})$, there is some $\epsilon > 0$ such that $f(\{\bar{v}_k - \epsilon\}_{k \in N}) \neq a_0$. Now, since $v_i > v'_i$, by monotonicity and non-bossiness, $f(v_i - \epsilon, \{\bar{v}_k - \epsilon\}_{k \neq i}) \neq a_0$. Hence, $a_0 \notin W^f(v'')$.

This completes the proof of Case 2.

CASE 3. $x \in D_i$, $y \in D_j$, $z \in D_k$, where i, j, k are distinct. Suppose $x = \mathbf{1}_{v_i}, y = \mathbf{1}_{v_j}$, and $z = \mathbf{1}_{v_k}$. Here, we will consider transitivity of both $\succ^f$ and $\sim^f$.

CASE 3A - TRANSITIVITY OF $\succ^f$. Suppose $x \succ^f y$ and $y \succ^f z$. Since $x \succ^f y$, there is some valuation profile v'' where $\mathbf{1}_{v''_i} = x, \mathbf{1}_{v''_j} = y$, and $a_i \in W^f(v'')$ but $a_j \notin W^f(v'')$.

First, note that $i \neq 0$. To see this, since $y \succ^f z$ there is a valuation profile v' where $\mathbf{1}_{v'_j} = y, \mathbf{1}_{v'_k} = z$, and $a_j \in W^f(v')$ but $a_k \notin W^f(v')$. But $\mathbf{1}_{v'_i} = x$ implies that $y \succeq^f x$, which contradicts $x \succ^f y$. Hence, $i \neq 0$.

Suppose $v''_k < v_k$. Since $a_i \in W^f(v'')$, for every $\epsilon > 0$, $f(v''_i + \epsilon, v''_j, v''_k, v''_{-ijk}) = a_i$. By monotonicity and non-bossiness, $f(v''_i + \epsilon, v''_j, v_k, v''_{-ijk}) \in \{a_i, a_k\}$ for every $\epsilon > 0$. But $f(v''_i + \epsilon, v''_j, v_k, v''_{-ijk}) = a_k$ for any $\epsilon > 0$ will imply that $z \succeq^f y$, and this will contradict $y \succ^f z$. Hence, $f(v''_i + \epsilon, v''_j, v_k, v''_{-ijk}) = a_i$ for every $\epsilon > 0$. So, $a_i \in W^f(v''_i, v''_j, v_k, v''_{-ijk})$. Since $y \succ^f z$, $a_k \notin W^f(v''_i, v''_j, v_k, v''_{-ijk})$. Hence, $x \succ^f z$.

Suppose $v''_k \geq v_k$. As before, since $a_i \in W^f(v'')$, for every $\epsilon > 0$, $f(v''_i + \epsilon, v''_j, v''_k, v''_{-ijk}) = a_i$. By monotonicity and non-bossiness, $f(v''_i + \epsilon, v''_j, v_k, v''_{-ijk}) = a_i$ for every $\epsilon > 0$. Hence, $a_i \in W^f(v''_i, v''_j, v_k, v''_{-ijk})$. Since $y \succ^f z$, $a_k \notin W^f(v''_i, v''_j, v_k, v''_{-ijk})$. Hence, $x \succ^f z$.

CASE 3B - TRANSITIVITY OF $\sim^f$. Suppose $x \sim^f y$ and $y \sim^f z$. Suppose for every valuation profile v' such that $\mathbf{1}_{v'_i} = x$ and $\mathbf{1}_{v'_j} = y$, we have $a_i, a_j \notin W^f(v')$. Further, suppose for every valuation profile $\bar{v}$ with $\mathbf{1}_{\bar{v}_j} = y$ and $\mathbf{1}_{\bar{v}_k} = z$, we have $a_j, a_k \notin W^f(\bar{v})$. Consider any valuation profile v'' such that $\mathbf{1}_{v''_i} = x$ and $\mathbf{1}_{v''_k} = z$. Assume for contradiction $a_i \in W^f(v'')$. Consider the valuation profile $\hat{v}$ such that $\mathbf{1}_{\hat{v}_j} = y$ and $\hat{v}_l = v''_l$ for all $l \neq j$. Since $\mathbf{1}_{\hat{v}_k} = z$, by definition $a_j, a_k \notin W^f(\hat{v})$. By monotonicity and non-bossiness, $a_i \in W^f(\hat{v})$. But, this is not possible since $\mathbf{1}_{\hat{v}_i} = x$ implies that $a_i, a_j \notin W^f(\hat{v})$. This means that at every valuation profile v'' with $\mathbf{1}_{v''_i} = x$ and $\mathbf{1}_{v''_k} = z$ we must have $a_i, a_k \notin W^f(v'')$. Hence, $x \sim^f z$.

Now, consider the case where $y \sim^f z$ and there is some valuation profile v' such that $\mathbf{1}_{v'_j} = y$, $\mathbf{1}_{v'_k} = z$, and $a_j, a_k \in W^f(v')$. Also, since $x \sim^f y$, there is some valuation profile

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v'' such that $\mathbf{1}_{v_i''} = x$, $\mathbf{1}_{v_j''} = y$, and $a_i, a_j \in W^f(v'')$. If $k = 0$, then $y \sim^f z$ implies that $a_k \in W^f(v'')$, and this in turn implies that $x \sim^f z$.

Suppose $k \neq 0$ and construct a new valuation profile $\bar{v}$ such that $\mathbf{1}_{\bar{v}_k} = z$ and $\bar{v}_l = v_l''$ for all $l \neq k$. We consider two possibilities.

CASE 3B(i). Suppose $i \neq 0$. There are two further possibilities. Suppose $\bar{v}_k > v_k''$. Since $a_i \in W^f(v'')$, for every $\epsilon > 0$, we have $f(v_i'' + \epsilon, v_j'', v_k'', v_{-ijk}'') = a_i$. By monotonicity and non-bossiness, $f(v_i'' + \epsilon, v_j'', \bar{v}_k, v_{-ijk}'') \in \{a_i, a_k\}$ for all $\epsilon > 0$. Denote the valuation profile $(v_i'', v_j'', \bar{v}_k, v_{-ijk}'')$ as $\hat{v}$.

Fix $\epsilon > 0$. If $f(v_i'' + \epsilon, v_j'', \bar{v}_k, v_{-ijk}'') = a_i$, then $a_i \in W^f(\hat{v})$. Else, $f(v_i'' + \epsilon, v_j'', \bar{v}_k, v_{-ijk}'') = a_k$, and by monotonicity and non-bossiness, $f(v_i'', v_j'', \bar{v}_k + \epsilon, v_{-ijk}'') = a_k$, which means that $a_k \in W^f(\hat{v})$. So, we conclude that either $a_i \in W^f(\hat{v})$ or $a_k \in W^f(\hat{v})$.

If $a_i \in W^f(\hat{v})$, using the fact that $x \sim^f y$, we must have $a_j \in W^f(\hat{v})$. Also, since $y \sim^f z$, we must have $a_k \in W^f(\hat{v})$. Hence, $x \sim^f z$. A similar argument applies if $a_k \in W^f(\hat{v})$.

Now consider the possibility where $\bar{v}_k \leq v_k''$. Since $a_i \in W^f(v'')$, for every $\epsilon > 0$, we have $f(v_i'' + \epsilon, v_j'', v_k'', v_{-ijk}'') = a_i$. By monotonicity and non-bossiness, $f(v_i'' + \epsilon, v_j'', \bar{v}_k, v_{-ijk}'') = a_i$. This shows that $a_i \in W^f(\hat{v})$. The rest of the argument is similar to the argument in the previous paragraph, and we conclude that $x \sim^f z$.

CASE 3B(ii). Suppose $i = 0$. As before, there are two possibilities. Suppose $\bar{v}_k > v_k''$. Consider any $\epsilon > 0$. Since $a_0 \in W^f(v'')$, $f(\{v_l'' - \epsilon\}_{l \in N}) = a_0$. By monotonicity and non-bossiness, $f(\bar{v}_k - \epsilon, \{v_l'' - \epsilon\}_{l \neq k}) \in \{a_0, a_k\}$. Denote the valuation profile $(v_i'', v_j'', \bar{v}_k, v_{-ijk}'')$ as $\hat{v}$.

If $f(\bar{v}_k - \epsilon, \{v_l'' - \epsilon\}_{l \neq k}) = a_0$, then $a_0 \in W^f(\hat{v})$. If $f(\bar{v}_k - \epsilon, \{v_l'' - \epsilon\}_{l \neq k}) = a_k$, then by monotonicity and non-bossiness, we can choose $(\epsilon_1, \ldots, \epsilon_n)$, all greater than zero, such that $f(\bar{v}_k - \epsilon_k, \{v_l'' - \epsilon_l\}_{l \neq k}) = a_k$. Fix $h \neq k$. By continuity, there is a $\delta > 0$ such that $f(\bar{v}_k + \epsilon_k, v_h'' - \epsilon_h + \delta, \{v_l'' - \epsilon_l\}_{l \neq k}) = a_k$. By uniform continuity, δ is independent of ϵ_h. So, we can choose $\epsilon_h = \delta$. As a result, $f(\bar{v}_k + \epsilon_k, v_h'', \{v_l'' - \epsilon_l\}_{l \neq k}) = a_k$. Repeating this argument for all $l \neq k$, we get $f(\bar{v}_k + 2(n-1)\epsilon_k, v_{-k}'') = a_k$. Since ϵ_k can be chosen arbitrarily small, $a_k \in W^f(\hat{v})$. So, we conclude that either $a_0 \in W^f(\hat{v})$ or $a_k \in W^f(\hat{v})$.

Now, suppose $\bar{v}_k \leq v_k''$. Since $a_0 \in W^f(v'')$, for every $\epsilon > 0$, $f(\{v_l'' - \epsilon\}_{l \in N}) = a_0$. By monotonicity and non-bossiness, $f(\bar{v}_k - \epsilon, \{v_l'' - \epsilon\}_{l \neq k}) = a_0$. This implies that $a_0 \in W^f(\hat{v})$.

So, in either case, $a_0 \in W^f(\hat{v})$ or $a_k \in W^f(\hat{v})$. If $a_0 \in W^f(\hat{v})$, using the fact that $x \sim^f y$, we must have $a_j \in W^f(\hat{v})$. Also, since $y \sim^f z$, we must have $a_k \in W^f(\hat{v})$. Hence, $x \sim^f z$. A similar argument applies if $a_k \in W^f(\hat{v})$. ∎

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