Sample Exam for 2012

Booklet No. TEST CODE: REI

Forenoon

Questions: 9	**Time: 2 hours**

- On the answer booklet write your Name, Registration number, Test Code, Number of the booklet etc. in the appropriate places.
- The test has 9 questions. **Answer any five**. All questions carry equal (20) marks.

1. (a) Prove by induction or otherwise

$$\frac{\left(\frac{5}{4}\right)^n}{n!} < \frac{1}{2^{n-2}} \text{ for } n = 2, 3, \ldots \textbf{ (10 marks)}$$

(b) Use this to show that $\log_e 5 > \frac{5}{4}$. **(10 marks)**

2. (a) For any finite set X, let $|X|$ denote the number of elements in X. Suppose A, B, C are three finite sets with $|A| = 8$, $|B| = 10$, $|C| = 12$, $|A \cup B| = 14$, $|A \cup C| = 14$, $|B \cup C| = 17$, $|A \cup B \cup C| = 17$. Find $|A \cap B \cap C|$. **(12 marks)**

(b) Define for any sets X, Y, $X \setminus Y = \{x \in X : x \notin Y\}$ (i.e., $X \setminus Y$ consists of all elements of X which are not in Y) and $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. In 3(a), find $|A \triangle B|$. **(8 marks)**

3. Convex Sets

(a) Consider the following system of linear equations with three variables x_1, x_2, x_3.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

Here, a_{ij} with $i, j \in \{1, 2, 3\}$ and b_1, b_2, and b_3 are real numbers. Suppose $(1, 1, 0)$ and $(0, 0, 1)$ are solutions to this system of equations. Is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ a solution to this system of linear equations? Explain your answer. **(5 marks)**

(b) Let $S_1, S_2 \subseteq \mathbb{R}^n$ be two convex sets. For each of the following sets, either prove that it is convex or give a counterexample.

(i) $S_1 + S_2 = \{z \in \mathbb{R}^n : z = x + y, x \in S_1, y \in S_2\}$. **(5 marks)**

(ii) $S_1 \cup S_2$. **(5 marks)**

(iii) $S_1 \cap S_2$. **(5 marks)**

4. (a) There are three cards. The first is green on both sides, the second is red on both sides and the third is green on one side and red on the other. We choose a card at random and we see one side (also chosen at random). If the side we see is green, what is the probability that the other side is green? **(12 marks)**

   (b) Let x be a random variable with cumulative distribution function F. Suppose F is strictly increasing. Consider $y = F(x)$. For any $a \in [0, 1]$, find Probability$(y < a)$. Based on this, what distribution does y follow? **(8 marks)**

5. (a) Find the maximum of the function $f(x) = x^{\frac{1}{x}}$ for all $x > 0$. **(15 marks)**

   (b) Find the maxima and the minima, if any, of the function $f(x) = x^3 - 6x^2 + 24x$ for all real values of x. **(5 marks)**

6. (a) Prove that $e^x = x + 2$ has a solution. **(10 marks)**

   (b) Three cards, say I, II, III, are lying on a table (see Figure 1). A number or a letter is printed on each side of every card.

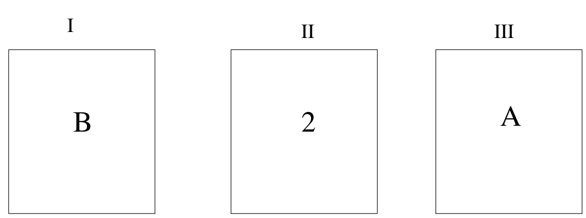

Figure 1: Three cards

Consider the statement

> If an even number is printed on one side of a card, then the letter "A" is printed on the other side.

Which cards would one have to turn over to check the truth of the statement? **(10 marks)**

7. (a) Consider the matrix given below.

$$\mathbf{A} = \begin{bmatrix} x & 0 & k \\ 1 & x & k-3 \\ 0 & 1 & 1 \end{bmatrix}$$

Suppose determinant of $\mathbf{A}$ is zero. What is the least positive integral value of k for which x can take real and more than one distinct values? **(10 marks)**

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave function. Let A be an $n \times m$ matrix, and let $b \in \mathbb{R}^n$. Consider the function $h : \mathbb{R}^m \to \mathbb{R}$ defined by

$$h(x) = f[Ax + b], \; x \in \mathbb{R}^m.$$

Is h concave? Why or why not? Explain clearly. **(10 marks)**

8. Consider the following joint distribution function of random variables x and y

$$F_{xy} = P(x \leq t, y \leq s) = \begin{cases} 0 & t < 0 \text{ or } s < 0, \\ \frac{ts}{2} + \frac{\alpha t(1-t)s(2-s)}{4} & 0 \leq t \leq 1, 0 \leq s \leq 2, \\ t & 0 \leq t \leq 1, s \geq 2, \\ \frac{s}{2} & t \geq 1, 0 \leq s \leq 2, \\ 1 & t \geq 1 \text{ and } s \geq 2 \end{cases}$$

where α is some real number between -1 and 1.

Determine the marginal distribution functions F_x and F_y. For what value(s) of α are the random variables x and y independent. **(6+6+8 marks)**

9. (a) Consider the following function

$$f(x) = \begin{cases} -ae^x & x < 0 \\ a^2 - 1 & x \geq 0. \end{cases}$$

where a is a constant. Find the real values of a for which f is continuous at $x = 0$. Is f differentiable at $x = 0$ for these values of a? Explain your answer. **(8+4 marks)**

(b) Find the solutions to the following optimization problem. **(8 marks)**

$$\max_{x_1, x_2 \in \mathbb{R}} \; x_1 x_2$$
$$\text{s.t.}$$
$$x_1^2 + x_2^2 \leq 16.$$