An Axiomatization of the Serial Cost-Sharing Method

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Abstract

We offer an axiomatization of the serial cost-sharing method of Friedman and Moulin (1999). The key property in our axiom system is Group Demand Monotonicity, asking that when a group of agents raise their demands, not all of them should pay less.

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1. Introduction

Serial cost sharing was proposed by Moulin and Shenker (1992) as a method for allocating the cost of production of a single good among n agents. Friedman and Moulin (1999) generalized it to the context where each agent consumes a possibly different good: total cost varies with the consumption profile but need no longer be a function of the sum of the agents' consumptions. The problem is to allocate the cost $C(x)$ generated by the demand profile $x = (x_1, ..., x_n)$ on the basis of the knowledge of x and the information contained in the cost function C defined on $\mathbb{R}^n_+$, which is assumed to be nondecreasing, continuously differentiable, and to display no fixed costs. This is the standard cost-sharing model developed by Billera and Heath (1982), Mirman and Tauman (1982), and Samet and Tauman (1982). Assuming without loss of generality that $x_1 \leq x_2 \leq ... \leq x_n$, Friedman and Moulin's serial method charges agent i the integral of her marginal cost along the "constrained egalitarian path" made up of the line segments linking 0 to $(x_1, ..., x_1)$ to $(x_1, x_2, ..., x_2)$, and so on to x. This is an alternative to the better known method derived from the Aumann-Shapley (1974) value for nonatomic games, which integrates marginal costs along the diagonal from 0 to x.

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Friedman and Moulin (1999) proposed an axiomatization of their method. A key axiom in their work states that if all goods are perfect substitutes –that is, $C(z) = c(\sum_{i\in N} z_i)$– then an agent's cost share should not exceed the cost of producing n times her own demand. This condition offers a protection against the risk of paying an exceedingly high cost share because of the much higher demands of others. It is certainly in the original spirit of the serial method but remains perhaps too reminiscent of the very definition of the method to provide an independent justification for it. In fact, the axiom rules out virtually all the popular cost-sharing methods. The only noticeable exception we are aware of is the so-called "cross-subsidizing serial method" of Moulin and Sprumont (2006) which differs from the Friedman-Moulin method but retains its serial structure.

We offer an alternative axiomatization of the serial method which does not suffer from the above criticism. The general normative principle motivating our choice of axioms is the one that underlies most of the theory of cost sharing: an agent should pay –fully but only– the fraction of the cost generated by her own demand[1]. Of course, unless the cost function is additively separable, this general principle is ambiguous. The challenge is to formulate unambiguous statements that capture the essential aspects of it.

In order to do that, we find it useful to break down an agent's influence on total cost into two components: the marginal cost function associated with the good she consumes and the size of her demand. If agents must be charged "the cost of their demand", then cost shares should somehow

(a) be positively associated with marginal cost functions,

(b) be positively associated with demand sizes,

(c) be independent of any cost-irrelevant information.

With one exception –Additivity–, our axioms are meant to be unambiguous statements interpreting these three desiderata. Of course, desideratum (b) is compelling only when each good is consumed by a clearly identifiable agent who can be held responsible for the entire demand of that good. That is the interpretation of the cost-sharing model we have in mind[2].

The first component of our axiom system is nothing more than the extension to the cost-sharing model of the system used by Shapley (1953) to characterize the value: Additivity (cost shares depend additively on the cost function), Dummy (an agent pays nothing if total cost never increases with her consumption), and Anonymity (the identity of an agent does not affect what she pays). If Additivity is used for tractability –the world of nonadditive methods is virtually uncharted territory that we do not want to venture into–, the other two axioms follow naturally from desiderata (a) and (c) above. Dummy is a minimal expression of the view that cost shares should be positively related to marginal cost functions and Anonymity follows from the principle forbidding the use of cost-irrelevant information. As

[1] There are contexts where this "full responsibility" principle is not warranted: see Moulin and Sprumont (2006) for a discussion and an alternative view.

[2] A good example is the problem of allocating overhead costs among the various divisions of a large firm (Shubik (1962)). Desideratum (b) is not compelling when the demand for a given good results from the agregation of many small individual demands, as in the telephone pricing problem studied by Billera, Heath and Raanan (1978) and other applications of Aumann-Shapley pricing.

a matter of fact, we do employ a strengthened version of the Dummy axiom requiring also that a change in the demand of a dummy agent should have no effect on cost shares. This requirement too follows naturally from (c).

All these properties are well known and the serial method shares them with the other two central methods discussed in the literature, the Aumann-Shapley and Shapley-Shubik methods. Within the class delimited by Additivity, Anonymity and our strengthened version of the Dummy axiom, the Aumann-Shapley method is known to be the only one satisfying Scale Invariance (cost shares do not depend on the units in which consumptions are measured) and the property that cost shares are proportional to demands when goods are perfect substitutes. See Billera and Heath (1982) and Mirman and Tauman (1982) for details[3]. In the same class, the Shapley-Shubik method proposed by Shubik (1962), is the only method satisfying Scale Invariance and Demand Monotonicity (an agent's cost share does not decrease when her demand goes up). See Friedman and Moulin (1999).[4]

The second component of our axiom system is Group Demand Monotonicity. Introduced by Moulin and Sprumont (2005), this axiom says that when a group of agents raise their demands, not all of them should end up paying less. This is stronger than Demand Monotonicity but still follows naturally from desideratum (b). Group Demand Monotonicity is satisfied by a number of well known methods that have nothing in common with the serial method, such as equal or proportional cost sharing.

We show that in conjunction with the properties forming the first component of our axiom system, Group Demand Monotonicity characterizes the serial method. The model, the axioms, and our theorem are presented in Section 2. That section also contains a brief sketch of the key ideas underlying the proof. The proof itself is given in Section 3. A discussion of our result and further comparison with related work is offered in Section 4.

2. The model and the result

Let $N = \{1, ..., n\}$ be a finite set of agents, $n \geq 3$. A *cost function* is a mapping $C : \mathbb{R}_+^N \to \mathbb{R}_+$ that is nondecreasing, continuously differentiable, and satisfies $C(0) = 0$. The set of cost functions is denoted $\mathcal{C}$. A *demand profile* is a point $x \in \mathbb{R}_+^N$. A (*cost-sharing*) *method* is a mapping φ which assigns to each (*cost-sharing*) *problem* $(C, x) \in \mathcal{C} \times \mathbb{R}_+^N$ a vector of nonnegative cost shares $\varphi(C, x) \in \mathbb{R}_+^N$ satisfying the *budget balance* condition $\sum_{i \in N} \varphi_i(C, x) = C(x)$.

As is well known, this model can be reinterpreted as a surplus-sharing model: C is then viewed as a production function, x_i is agent i's input contribution and $\varphi_i(C, x)$ is her share of the total output produced. All our axioms remain meaningful under this alternative

[3] Additivity, Scale Invariance and the property that cost shares are proportional to demands when goods are perfect substitutes actually suffice to characterize the Aumann-Shapley method.

[4] Additivity, Dummy, Anonymity and Demand Monotonicity suffice to pin down the Shapley-Shubik method. As observed in Moulin and Sprumont (2007), the axiom of Continuity at Zero is redundant in the corollary to Theorem 1 in Friedman and Moulin (1999).

interpretation. We maintain the cost-sharing interpretation throughout the rest of the paper to avoid confusion.

If $C \in \mathcal{C}$ and $i \in N$, we denote by $\partial_i C(z)$ the ith partial derivative of C at z if $z_i > 0$ and its ith right partial derivative at z if $z_i = 0$. For all $z, z' \in \mathbb{R}_+^N$, we let $z \wedge z' = (\min(z_1, z_1'), ..., \min(z_n, z_n'))$. The *(Friedman-Moulin) serial method* is the cost-sharing method φ^* defined by

$$\varphi_i^*(C, x) = \int_0^{x_i} \partial_i C((\alpha, \alpha, ..., \alpha) \wedge x) d\alpha \tag{2.1}$$

for all $C \in \mathcal{C}$, $x \in \mathbb{R}_+^N$, and $i \in N$. This method reduces to the well known serial formula proposed by Moulin and Shenker (1992) in the particular case of perfectly substitutable goods. If there exists a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that $C(z) = c(\sum_{i \in N} z_i)$ for all $z \in \mathbb{R}_+^N$, then, assuming without loss of generality that $x_1 \leq x_2 \leq ... \leq x_n$, the cost shares in (2.1) become $\varphi_1^*(C, x) = \frac{c(nx_1)}{n}$, $\varphi_2^*(C, x) = \frac{c(nx_1)}{n} + \frac{c(x_1+(n-1)x_2)-c(nx_1)}{n-1}$, ... $\varphi_n^*(C, x) = \frac{c(nx_1)}{n} + \frac{c(x_1+(n-1)x_2)-c(nx_1)}{n-1} + ... + \frac{c(x_1+...+x_n)-c(x_1+...+x_{n-2}+2x_{n-1})}{1}$.

Just like the Aumann-Shapley method, the serial method belongs to the class of "path methods" (Friedman (2004)). A *path* to $x \in \mathbb{R}_+^N$ is a continuous nondecreasing function $r_x : [0, 1] \to [0, x]$ such that $r_x(0) = 0$ and $r_x(1) = x$. A cost-sharing method φ is a *path method* if to each $x \in \mathbb{R}_+^N$ there is a path r_x such that

$$\varphi_i(C, x) = \int_0^1 \partial_i C(r_x(t)) \frac{dr_x}{dt}(t) dt \tag{2.2}$$

for all $C \in \mathcal{C}$ and $i \in N$. Note that this expression is well defined because r_x is differentiable almost everywhere. The serial method is generated by the collection of "constrained egalitarian" paths $r_x(t) = (tx_n, tx_n, ..., tx_n) \wedge x$ whereas the Aumann-Shapley method is generated by the collection of "diagonal" paths $r_x(t) = tx$.

We now present our axioms. The first four are adapted from the properties used by Shapley (1953) to characterize the value in the model of cooperative games.

Additivity. For all $C, C' \in \mathcal{C}$ and $x \in \mathbb{R}_+^N$, $\varphi(C + C', x) = \varphi(C, x) + \varphi(C', x)$.

As mentioned in the Introduction, our primary motivation for this axiom is tractability. Additive methods can be described fairly explicitly: Friedman and Moulin (1999) and Friedman (2004) offer characterizations of the class of additive methods satisfying the Dummy axiom and Moulin and Vohra (2003) propose a description of the entire class in the discrete version of the cost-sharing model. By comparison, only a few specific nonadditive rules were studied in the literature –see for instance Sprumont (1998) and Koster (2007)– and no general characterization result is available. Beyond tractability, however, the practical advantages of Additivity should not be underestimated. As many authors have noted, an additive method is easily implementable. When total cost arises from independent production processes, applying the method to the cost function corresponding to each process and adding the resulting cost shares is equivalent to applying it to the aggregated cost function. This guarantees that the proper level of application of the method is not a matter of dispute.

Following standard terminology, we call agent i a dummy (agent) if $\partial_i C(z) = 0$ for all $z \in \mathbb{R}_+^N$. The familiar Dummy axiom states that a dummy agent pays nothing: if $\partial_i C(z) = 0$ for all $z \in \mathbb{R}_+^N$, then $\varphi_i(C, x) = 0$ for all $x \in \mathbb{R}_+^N$. We replace this axiom by the two properties of Weak Dummy and Dummy Independence. The conjunction of these properties is stronger than Dummy.

Weak Dummy. For all $C \in \mathcal{C}$, $x \in \mathbb{R}_+^N$, and $i \in N$, if $x_i = 0$ and $\partial_i C(z) = 0$ for all $z \in \mathbb{R}_+^N$, then $\varphi_i(C, x) = 0$.

Weak Dummy requires that a dummy agent who demands nothing pay nothing. This is an extremely weak axiom. If the statement that an agent should pay only the fraction of the cost generated by her own demand entails any well defined restriction on φ, this must be one.

Dummy Independence. For all $C \in \mathcal{C}$, $x, x' \in \mathbb{R}_+^N$, and $i \in N$, if $\partial_i C(z) = 0$ for all $z \in \mathbb{R}_+^N$ and $x_j = x'_j$ for all $j \in N \setminus \{i\}$, then $\varphi(C, x) = \varphi(C, x')$.

This axiom says that if total cost is independent of an agent's demand, then cost shares should also be. Together with Weak Dummy, Dummy Independence allows one to essentially ignore all dummy agents. This seems to be a very natural separability condition for a theory aiming at charging agents according to their own impact on total cost. Dummy Independence is satisfied by all the popular cost-sharing methods proposed in the literature, including the Aumann-Shapley and Shapley-Shubik methods.

Within the class of methods satisfying Dummy, the axiom of Dummy Independence may also be defended from a strategic viewpoint. Indeed, a method satisfying Dummy and violating Dummy Independence would be vulnerable to manipulations by pairs consisting of a dummy and a non-dummy agent: an increase in the dummy agent's demand could reduce her partner's cost share without increasing her own.

Our fourth axiom uses the following notation. If i, j are two distinct agents, we denote by π^{ij} the permutation on N which exchanges i and j : $\pi^{ij}(i) = j$, $\pi^{ij}(j) = i$ and $\pi^{ij}(k) = k$ if $k \in N \setminus \{i, j\}$. If $x \in \mathbb{R}_+^N$ and $C \in \mathcal{C}$, we define $\pi^{ij}x$ by $(\pi^{ij}x)_{\pi^{ij}(k)} = x_k$ for all $k \in N$ and we define $\pi^{ij}C$ by $\pi^{ij}C(\pi^{ij}z) = C(z)$ for all $z \in \mathbb{R}_+^N$. Note that $\pi^{ij}C \in \mathcal{C}$.

Anonymity. For all $C \in \mathcal{C}$, $x \in \mathbb{R}_+^N$, and distinct $i, j \in N$, if $x_i = x_j$, then $\varphi_i(C, x) = \varphi_j(\pi^{ij}C, x)$.

This requirement expresses the familiar idea that the names of the agents should play no role in the computation of the cost shares. This is widely accepted as a very basic notion of fairness and is consistent with condition (c) in the Introduction: characteristics unrelated to the cost function or the demand profile should be ignored. Our formulation is rather weak insofar as it does not impose restrictions on the cost shares across demand profiles, in contrast with the condition used in Sprumont (2008) for instance. On the other hand, our axiom does impose restrictions across cost functions; it is stronger than the requirement that agents with equal demands pay the same cost share when the cost function is symmetric in the goods they demand.

Our fifth axiom has no counterpart in Shapley's characterization of the value.

Group Demand Monotonicity. For all $C \in \mathcal{C}$, all $x, x' \in \mathbb{R}_+^N$, and all nonempty $S \subseteq N$, if $x_i < x_i'$ for all $i \in S$ and $x_i = x_i'$ for all $i \in N \backslash S$, then there exists $i \in S$ such that $\varphi_i(C, x) \leq \varphi_i(C, x')$.

This axiom simply requires that when a group of agents jointly increase their demands, not all of them pay less. It strengthens Moulin's (1995) Demand Monotonicity axiom which only requires that if $x_i < x_i'$ and $x_j = x_j'$ for all $j \in N \backslash \{i\}$, then $\varphi_i(C, x) \leq \varphi_i(C, x')$. As already discussed, Group Demand Monotonicity is in line with the normative principle that cost shares should be positively related to demand sizes. We note that the axiom is also compelling from the strategic viewpoint: in an environment where agents can easily communicate, Group Demand Monotonicity is necessary to prevent manipulations by coordinated artificial inflation of demands.

An interesting weak form of the axiom consists of restricting Group Demand Monotonicity to the groups containing no more than two agents. Combined with our first four axioms, this weaker requirement turns out to imply Group Demand Monotonicity: see Section 4 for a discussion.

Theorem. *The following statement are equivalent:*
(i) φ *is a cost-sharing method satisfying Additivity, Weak Dummy, Dummy Independence, Anonymity and Group Demand Monotonicity;*
(ii) φ *is the serial cost-sharing method.*

Observe that contrary to the classic axiomatizations of the Aumann-Shapley and Shapley-Shubik methods, our theorem does not use any measurement invariance axiom. As is well known, the Friedman-Moulin serial method is not scale invariant. It is important to keep in mind that the method was proposed as an extension of the Moulin-Shenker formula for perfect substitutes and is meant to be used in problems where the goods, although no longer necessarily perfectly substitutable, remain genuinely comparable. In such environments, scale invariance is not compelling. See Friedman and Moulin (1999) for a list of examples including queueing and scheduling cost-sharing problems, as well as the output-sharing problem in a cooperative enterprise. Of course, the serial method is invariant to a change in the common unit in which the goods are measured[5].

It is not difficult to check that statement (ii) implies statement (i) but the proof of the converse implication is long. The remainder of this section offers an informal overview of it. For simplicity, let us consider the three-agent case, $N = \{1, 2, 3\}$. Fix a method φ satisfying our axioms. Because φ satisfies Additivity and Dummy, a fundamental lemma in Friedman and Moulin (1999) guarantees that the cost shares are obtained by integrating marginal costs: for every demand profile x and every agent i, there exists a measure μ_i^x

[5] In fact, unlike the Aumann-Shapley method but like the Shapley-Shubik method, the serial method is invariant to common *ordinal* rescaling of demands.

on $[0,x]$ such that $\varphi_i(C,x) = \int_{[0,x]} \partial_i C d\mu_i^x$ for every cost function C. The proof of our theorem consists in showing that Dummy Independence, Anonymity, and Group Demand Monotonicity force the support of each of the measures $\mu_1^x, \mu_2^x, \mu_3^x$ to be included in the (range of the) constrained egalitarian path to x. The bulk of the proof is devoted to establishing this fact in the particular case when all coordinates of the demand profile x coincide: the argument is developed in the first six steps of the proof and does not rely on Dummy Independence. The last two steps of the proof exploit Dummy Independence to establish the fact for an arbitrary demand profile x.

The heart of the proof is Step 2. Let $\mu^a = (\mu_1^a, \mu_2^a, \mu_3^a)$, $\mu^b = (\mu_1^b, \mu_2^b, \mu_3^b)$ denote the systems of measures which, according to the Friedman-Moulin lemma just described, generate $\varphi(.,a)$ and $\varphi(.,b)$. Consider two demand profiles $a = (\alpha,\alpha,\beta)$ and $b = (\beta,\beta,\beta)$, where $0 < \alpha < \beta$. Suppose for a moment that φ is the Shapley-Shubik method: given a problem (C,x), each agent pays her Shapley value in the cooperative game $\gamma_{(C,x)}(S) = C(x_S, 0_{N\backslash S})$ for all $S \subseteq N$. Thus,

$$\begin{aligned}\varphi_3(C,a) &= \frac{1}{3}(C(0,0,\beta) - C(0,0,0)) + \frac{1}{6}(C(\alpha,0,\beta) - C(\alpha,0,0)) + \\ &\quad \frac{1}{6}(C(0,\alpha,\beta) - C(0,\alpha,0)) + \frac{1}{3}(C(\alpha,\alpha,\beta) - C(\alpha,\alpha,0))\end{aligned}$$

and

$$\begin{aligned}\varphi_3(C,b) &= \frac{1}{3}(C(0,0,\beta) - C(0,0,0)) + \frac{1}{6}(C(\beta,0,\beta) - C(\beta,0,0)) + \\ &\quad \frac{1}{6}(C(0,\beta,\beta) - C(0,\beta,0)) + \frac{1}{3}(C(\beta,\beta,\beta) - C(\beta,\beta,0)).\end{aligned}$$

Suppose now that the cost function C is symmetric in z_1, z_2 and that its restriction to the finite grid $\{0,\alpha,\beta\}^N$ is

$$C(z) = \begin{cases} 1 \text{ if } z \geq (\beta,0,\beta) \text{ or } z \geq (\alpha,\alpha,\beta) \text{ or } z \geq (0,\beta,\beta), \\ \qquad\qquad 0 \text{ otherwise.}\end{cases}$$

Then $\varphi_3(C,a) = 0+0+0+\frac{1}{3} = \frac{1}{3}$ and $\varphi_3(C,b) = 0+\frac{1}{6}+\frac{1}{6}+\frac{1}{3} = \frac{2}{3}$. But $C(a) = C(b) = 1$. Since C is symmetric in z_1, z_2 and φ is anonymous, we must have $\varphi_i(C,a) = \frac{1}{3}$ and $\varphi_i(C,b) = \frac{1}{6}$ for $i = 1,2$, violating Group Demand Monotonicity.

Observe that the measure μ_3^b attaches a positive measure (namely, $\frac{1}{6}$) to each of the intervals $[(\beta,0,0),(\beta,0,\beta)]$, $[(0,\beta,0),(0,\beta,\beta)]$. The violation of Group Demand Monotonicity arises because these intervals lie outside the region of $[0,b]$ where $z_i \leq \alpha$ for both $i = 1,2$ or $z_i \geq \alpha$ for both $i = 1,2$. It is for this reason, and because μ_3^a is simply the projection of μ_3^b on the interval $[0,a]$ (in the sense that $\mu_3^a(Z) = \mu_3^b(\{z \in [0,b] \mid z \wedge a \in Z\})$ for every measurable $Z \subseteq [0,a]$), that we were able to find a cost function C, symmetric in z_1, z_2 and such that $C(a) = C(b)$, for which $\varphi_3(C,a) < \varphi_3(C,b)$. The anonymity of φ then precipitated the violation.

It should be clear that this argument extends beyond the particular case of the Shapley-Shubik method. Step 2 shows that whenever the measure μ_3^a is the projection of μ_3^b on the interval $[0, a]$, Anonymity and Group Demand Monotonicity force the support of μ_3^b to be included in the region of $[0, b]$ where $z_i \leq \alpha$ for $i = 1, 2$ or $z_i \geq \alpha$ for $i = 1, 2$. Proving this only requires a careful specification of the above cost function C outside the grid $\{0, \alpha, \beta\}^N$.

As it turns out, the assumption that μ_3^a is the projection of μ_3^b on the interval $[0, a]$ is superfluous. Step 1 shows that it follows from Anonymity and Group Demand Monotonicity. The proof is similar to that of Step 2: assuming that φ is anonymous and that the projection property does not hold, we again exhibit a cost function $\widetilde{C}$ such that $\varphi_i(\widetilde{C}, a) > \varphi_i(\widetilde{C}, b)$ for $i = 1, 2$.

The rest is tedious but rather straightforward. Step 3 lets α vary between 0 and β. It follows immediately from Step 2 that the support of μ_3^b is included in the region of $[0, b]$ where $z_1 = z_2$. Likewise, the support of μ_1^b is included in the plane $z_2 = z_3$ and the support of μ_2^b is included in the plane $z_1 = z_3$.

Combining these three conditions with budget balance, it is intuitive that the support of each of the measures $\mu_1^b, \mu_2^b, \mu_3^b$ must in fact be included in the ray $z_1 = z_2 = z_3$. The formal proof in Steps 4 to 6 relies on the rather cumbersome restrictions (identified in Friedman and Moulin (1999)) that budget balance entails on those measures.

Step 7 identifies the key restriction that Dummy Independence imposes on the entire system of measures characterizing φ. A simple argument shows that for any demand profile $x = (x_1, x_2, x_3)$, the measures $\mu_1^{(x_1,x_2,0)}, \mu_2^{(x_1,x_2,0)}$ must be the projections of μ_1^x, μ_2^x on $[0, (x_1, x_2, 0)]$.

Step 8 concludes the proof. Since the supports of μ_1^b, μ_2^b are included in the egalitarian ray $z_1 = z_2 = z_3$, the projection property established in Step 7 implies that the supports of $\mu_1^{(\beta,\beta,0)}, \mu_2^{(\beta,\beta,0)}$ are included in the ray $z_1 = z_2$. As is well known, Demand Monotonicity (which follows from Group Demand Monotonicity) then implies that the supports of $\mu_1^{(\alpha,\beta,0)}, \mu_2^{(\alpha,\beta,0)}$ are included in the (range of the) constrained egalitarian path to $(\alpha, \beta, 0)$ whenever $0 \leq \alpha \leq \beta$. A corresponding statement also holds for every permutation of $(\alpha, \beta, 0)$. Invoking the projection property proved in Step 7 again, we conclude that the supports of $\mu_1^x, \mu_2^x, \mu_3^x$ are included in the (range of the) constrained egalitarian path to x whenever $0 \leq x \leq b$. This implies that φ must be the serial method.

3. The proof

Proof that (ii) implies (i). It is well known and easy to check that the serial method φ^* satisfies our first four axioms. To check Group Demand Monotonicity, fix a cost function C, a group of agents $S \subseteq N$, and two demand profiles x, x' such that $x_i < x'_i$ for all $i \in S$ and $x_i = x'_i$ for all $i \in N \backslash S$. We claim that the cost share of any agent with minimal demand in S at x cannot decrease when the demand profile changes from x to x'. Indeed, let $i \in S$ be such that $x_i \leq x_j$ for all $j \in S$. Then $\varphi_i^*(C, x) = \int_0^{x_i} \partial_i C((\alpha, \alpha, ..., \alpha) \wedge x) d\alpha = \int_0^{x_i} \partial_i C((\alpha, \alpha, ..., \alpha) \wedge x') d\alpha \leq \int_0^{x'_i} \partial_i C((\alpha, \alpha, ..., \alpha) \wedge x') d\alpha = \varphi_i^*(C, x')$.■

As noted in Moulin and Sprumont (2005), this argument can be generalized to show that all "fixed-path methods" satisfy Group Demand Monotonicity. In our continuous framework, a *fixed-path method* is a path method generated by a collection of paths r_x having the property that if $x \leq x'$, then $r_x([0,1])$ is the projection of $r_{x'}([0,1])$ on $[0,x]$, namely $r_x([0,1]) = \{y \wedge x \mid y \in r_{x'}([0,1])\}$. One can think of such a method as being generated by a single fixed "unbounded path" r. The serial method is a fixed-path method whereas the Aumann-Shapley method is not. To see why a fixed-path method φ is group demand monotonic, consider again a group S and two demand profiles x, x' such that $x_i < x'_i$ for all $i \in S$ and $x_i = x'_i$ for all $i \in N \backslash S$. If the demand of agent i is among those in S that are reached first along the path r_x (in the sense that $t_x(i) := \inf\{t \mid (r_x)_i(t) \geq x_i\}$ $\leq t_x(j) := \inf\{t \mid (r_x)_j(t) \geq x_j\}$ for all $j \in S$), then agent i's cost share cannot decrease from x to x' since $\varphi_i(C,x) = \int_0^{t_x(i)} \partial_i C(r_x(t)) \frac{d(r_x)_i}{dt}(t)dt = \int_0^{t_x(i)} \partial_i C(r_{x'}(t)) \frac{d(r_{x'})_i}{dt}(t)dt \leq \int_0^1 \partial_i C(r_{x'}(t)) \frac{d(r_{x'})_i}{dt}(t)dt = \varphi_i(C,x')$.

We now turn to the proof that only the serial method satisfies our axioms. The following notation will be used throughout. Vector inequalities are written $\leq, <, \ll$. For all $S \subseteq N$ and $z \in \mathbb{R}^N$ we denote by $z_S \in \mathbb{R}^S$ the restriction of z to S. If $z, z' \in \mathbb{R}^N$, we denote by $(z_S, z'_{N\backslash S})$ the point in $\mathbb{R}^N$ whose restrictions to S and $N \setminus S$ are z_S and $z'_{N\backslash S}$, respectively. If $Z \subseteq \mathbb{R}^N$, we let $Z_S = \left\{z_S \in \mathbb{R}^S \mid \exists z_{N\backslash S} \in \mathbb{R}^{N\backslash S} : (z_S, z_{N\backslash S}) \in Z\right\}$.

Our proof relies on Friedman and Moulin's (1999) characterization of the cost-sharing methods satisfying Additivity and Dummy. For any $x \in \mathbb{R}^N_+$, denote by $\mathcal{B}([0,x])$ the set of Borel subsets of $[0,x]$. If $i \in N$, $a \in [0,x]$, and μ_i^x is a Radon measure on $\mathcal{B}([0,x])$, define $m_i^x(a) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mu_i^x(\{z \in [0,x] \mid a_i \leq z_i \leq a_i + \varepsilon \text{ and } z_j \geq a_j \text{ for all } j \in N \setminus i\})$. A *measure system* is a mapping μ on $\mathbb{R}^N_+$, $x \mapsto \mu^x = (\mu_1^x, ..., \mu_n^x)$, where each μ_i^x is a nonnegative Radon measure on $\mathcal{B}([0,x])$ such that

$$\sum_{i \in S} m_i^x(a) = 1 \text{ for all } S \subseteq N \text{ and almost all } a \in [0,x] \text{ such that } a_{N\backslash S} = 0, \tag{3.1}$$

where the term "almost all" is understood with respect to the $|S|$-dimensional Lebesgue measure on $\left[0, (x_S, 0_{N\backslash S})\right]$. A useful implication of (3.1) (derived by taking $S = \{i\}$) is that the projection of μ_i^x on the one-dimensional interval $[0, x_i]$ is the Lebesgue measure:

$$\mu_i^x\left(\{z \in [0,x] \mid a_i \leq z_i \leq b_i\}\right) = b_i - a_i \text{ whenever } 0 \leq a_i \leq b_i \leq x_i. \tag{3.2}$$

Lemma (Friedman and Moulin, 1999). *For any function $\varphi : \mathcal{C} \times \mathbb{R}^N_+ \to \mathbb{R}^N_+$, the following statements are equivalent:*
(i) *φ is a cost-sharing method satisfying Additivity and Dummy;*
(ii) *there exists a measure system μ such that*

$$\varphi_i(C,x) = \int_{[0,x]} \partial_i C d\mu_i^x \text{ for all } i \in N,\ C \in \mathcal{C}, \text{ and } x \in \mathbb{R}^N_+. \tag{3.3}$$

The role of (3.1) is to guarantee that the cost shares defined by (3.3) satisfy budget balance. It may be useful to outline the argument in the case of a cost-sharing problem (C,x) where C is sufficiently differentiable. For any set $S = \{i_1, ..., i_S\} \subseteq N$, denote by $\partial_S C$ the partial derivative $\partial_{i_1} ... \partial_{i_S} C$. Compute the cost shares in (3.3) using repeated integration by parts: for any $i \in N$, $\varphi_i(C,x) = \int_{[0,x]} \partial_i C(z) d\mu_i^x(z) = \sum_{S \subseteq N} \int_{[0,(x_S, 0_{N \setminus S})]} \partial_{S \setminus i} \partial_i C(z) m_i^x(z) dz = \sum_{S \subseteq N} \int_{[0,(x_S, 0_{N \setminus S})]} \partial_S C(z) m_i^x(z) dz$, where dz refers to the $|S|$-dimensional Lebesgue measure. Then $\sum_{i \in N} \varphi_i(C,x) = \sum_{i \in N} \sum_{S \subseteq N} \int_{[0,(x_S, 0_{N \setminus S})]} \partial_S C(z) m_i^x(z) dz = \sum_{S \subseteq N} \int_{[0,(x_S, 0_{N \setminus S})]} \partial_S C(z) (\sum_{i \in S} m_i^x(z)) dz = \sum_{S \subseteq N} \int_{[0,(x_S, 0_{N \setminus S})]} \partial_S C(z) dz$ $= C(x)$. Friedman and Moulin (1999) invoke an approximation to extend the proof to all cost-sharing problems.

The measure system μ in the Friedman-Moulin lemma is unique; we say that it *generates* φ. We refer to μ^x as a *measure system at* x. By the *support of* μ^x we mean the union of the supports of the measures $\mu_1^x, ..., \mu_n^x$. We denote by μ^* the measure system generating the serial method φ^* and call μ^* the *serial measure system.* This system is an example of a *fixed measure system*. In a fixed measure system μ, when $x \leq x'$, each measure μ_i^x is the *projection of* $\mu_i^{x'}$ onto $[0,x]$, namely, the measure $p_x \mu_i^{x'}$ defined on $\mathcal{B}([0,x])$ by

$$p_x \mu_i^{x'}(Z) = \mu_i^{x'}(\{z \in [0,x'] \mid z \wedge x \in Z \text{ and } z_i \leq x_i\}). \tag{3.4}$$

For any $b = (\beta, \beta, ..., \beta) \in \mathbb{R}_+^N$, the support of the serial measure system μ^{*b} at b is the set $\{(\alpha, \alpha, ..., \alpha) \mid 0 \leq \alpha \leq \beta\}$, the diagonal of $[0,b]$. Using (3.2), it is easy to see that this property determines μ^{*b} uniquely, as noted in the proof of Theorem 2 in Friedman and Moulin (1999). For any $x \leq b$, the serial measure system at x is defined by the projection property $\mu_i^{*x} = p_x \mu_i^{*b}$ for all $i \in N$.

The support of μ^{*x} is the constrained egalitarian path to x. Suppose, without loss of generality, that $x_1 \leq x_2 \leq ... \leq x_n$. For all $i \in N$, define the demand profile $x^i \in \mathbb{R}_+^N$ by $x_j^i = \min(x_i, x_j)$ for all $j \in N$. The support of μ^{*x} is

$$S^{*x} = \cup_{i=1}^n co\left\{x^{i-1}, x^i\right\}, \tag{3.5}$$

where $x^0 = 0$ and $co\{x^{i-1}, x^i\}$ is the line segment joining x^{i-1} to x^i. See Figure 1.

Proof that (i) implies (ii). Let φ be a cost-sharing method satisfying Additivity, Weak Dummy, Dummy Independence, Anonymity and Group Demand Monotonicity. Since Weak Dummy and Dummy Independence imply Dummy, it follows from the Friedman-Moulin lemma that there exists a measure system μ generating φ. Let β be a positive real number, fix the demand profile $b = (\beta, \beta, \beta, ..., \beta)$, and let $B = [0,b]$.

Step 1. *We claim that if* $0 < \alpha < \beta$, *and* $a = (\alpha, \alpha, \beta, ..., \beta)$, *then* $\mu_i^a = p_a \mu_i^b$ *for* $i = 3, ..., n$.

Fix α such that $0 < \alpha < \beta$, let $a = (\alpha, \alpha, \beta, ..., \beta)$ and write $A = [0,a]$. We use the following terminology and notation. A set $E \subseteq \mathbb{R}^N$ is an *interval* (in $\mathbb{R}^N$) if $E = \times_{i \in N} E_i$, where each E_i is an (open, half-open, or closed) interval in $\mathbb{R}$. If E is nonempty, we denote its endpoints by $e^-(E)$, $e^+(E)$ or simply e^-, e^+. An *open interval* in $\mathbb{R}^N$ is an interval which is also an open set: if nonempty, it takes the form $E = \{z \in \mathbb{R}^N \mid e^- \ll z \ll e^+\}$

where $e^- \ll e^+$, and we write $E =]e^-, e^+[$. Let $\mathcal{E}$ and $\mathcal{E}^o$ denote the set of intervals and the set of open intervals, respectively. The set of intervals which are below the hyperplane $z_1 = z_2$ is $\mathcal{E}_< = \{E \in \mathcal{E} \mid z_2 < z_1 \text{ for all } z \in E\}$; the set of intervals above it is $\mathcal{E}_> = \{E \in \mathcal{E} \mid z_2 > z_1 \text{ for all } z \in E\}$, and the set of intervals whose endpoints are on this hyperplane is $\mathcal{E}_= = \{E \in \mathcal{E} \mid e_1^-(E) = e_2^-(E) \text{ and } e_1^+(E) = e_2^+(E)\}$.

1.1. *We claim that* $\mu_i^a(E \cap A) \geq p_a \mu_i^b(E \cap A)$ *for* $i = 3, ..., n$ *and all* $E \in \mathcal{E}^o \cap \mathcal{E}_<$.

We only give a sketch of the argument and refer the reader to the Appendix for details. Fix $E \in \mathcal{E}^o \cap \mathcal{E}_<$ and let $e^-, e^+ \in \mathbb{R}^N$, $e^- \ll e^+$, be the endpoints of E. Assume that

$$e_1^+ \leq \alpha \text{ or } e_1^+ > \beta. \tag{3.6}$$

This assumption entails no loss of generality. (If $\alpha < e_1^+ \leq \beta$, choose $e_1^{++} > \beta$ and consider the open interval $E^+ =]e^-, (e_1^{++}, e_{N\setminus 1}^+)[$. Apply the argument below to E^+ rather than E to obtain $\mu_i^a(E^+ \cap A) \geq p_a \mu_i^b(E^+ \cap A)$ for $i = 3, ..., n$. Since $E^+ \cap A = E \cap A$, our claim follows.) Assumption (3.6) guarantees that

$$p_a \mu_i^b(E \cap A) = \mu_i^b(E \cap B) \text{ for } i = 3, ..., n. \tag{3.7}$$

Suppose now, by way of contradiction, that, say,

$$\mu_3^a(E \cap A) < p_a \mu_3^b(E \cap A). \tag{3.8}$$

Assume also that $0 \leq e_3^-$ and $e_3^+ \leq \beta$: this too is without loss of generality because $\mu_3^a(\{z \in A \mid z_3 = 0\}) = \mu_3^a(\{z \in A \mid z_3 = \beta\}) = p_a \mu_3^b(\{z \in A \mid z_3 = 0\}) = p_a \mu_3^b(\{z \in A \mid z_3 = \beta\}) = 0$ by (3.2). For any set $Z \subseteq \mathbb{R}^N$, let $\pi^{12} Z = \{\pi^{12} z \mid z \in Z\}$, where we recall that π^{12} is the permutation exchanging agents 1 and 2, and let $Z_* = Z \cup \pi^{12} Z$: this is the smallest superset of Z that is symmetric with respect to the hyperplane $z_1 = z_2$.

Suppose we could construct a cost function C such that (a) $C(a) = C(b)$, (b) $C(z)$ is independent of $z_4, ..., z_n$, and (c) $\partial_3 C$ is a positive constant k on $E \cap \mathbb{R}_+^N$ and zero elsewhere. Define the function C_* on $\mathbb{R}_+^N$ by $C_*(z) = C(z)$ if $z_1 \geq z_2$ and $C_*(z) = \pi^{12} C(z)$ otherwise.

By Anonymity $\varphi_1(C, a) + \varphi_2(C, a) = \varphi_2(\pi^{12} C, a) + \varphi_1(\pi^{12} C, a)$ and by Dummy $\varphi_i(C, a) = 0 = \varphi_i(\pi^{12} C, a)$ for $i = 4, ..., n$. Since $C(a) = \pi^{12} C(a)$, budget balance implies $\varphi_3(C, a) = \varphi_3(\pi^{12} C, a)$. Since $\varphi_3(C, a) = \int_A \partial_3 C d\mu_3^a = k \mu_3^a(E \cap A)$ and $\varphi_3(\pi^{12} C, a) = \int_A \partial_3 \pi^{12} C d\mu_3^a = k \mu_3^a(\pi^{12}(E \cap A))$, we obtain $\mu_3^a(E \cap A) = \mu_3^a(\pi^{12}(E \cap A))$ and therefore $\mu_3^a((E \cap A)_*) = 2\mu_3^a(E \cap A)$.

Likewise, since $\varphi_3(C, b) = k \mu_3^b(E \cap B) = k p_a \mu_3^b(E \cap A)$ (by (3.7)) and $\varphi_3(\pi^{12} C, b) = k p_a \mu_3^b(\pi^{12}(E \cap A))$, a similar argument yields $p_a \mu_3^b((E \cap A)_*) = 2 p_a \mu_3^b(E \cap A)$. Therefore inequality (3.8) implies $\mu_3^a((E \cap A)_*) < p_a \mu_3^b((E \cap A)_*)$.

Now, since $\varphi_3(C_*, a) = k \mu_3^a((E \cap A)_*)$ and $\varphi_3(C_*, b) = k p_a \mu_3^b((E \cap A)_*)$, it follows that $\varphi_3(C_*, a) < \varphi_3(C_*, b)$. By Dummy, $\varphi_i(C_*, a) = 0 = \varphi_i(C_*, b)$ for $i = 4, ..., n$. Since $C_*(a) = C_*(b)$, budget balance implies $\varphi_1(C_*, a) + \varphi_2(C_*, a) > \varphi_1(C_*, b) + \varphi_2(C_*, b)$. By Anonymity, $\varphi_i(C_*, a) > \varphi_i(C_*, b)$ for $i = 1, 2$, contradicting Group Demand Monotonicity.

An example of a nondecreasing function C satisfying properties (a), (b) and (c) above is the following. For all $z \in \mathbb{R}^N$ and $i \in N$, define $z_i' = \frac{z_i - e_i^-}{e_i^+ - e_i^-}$. Let $z_3'' = med(0, z_3', 1)$, the median of the three numbers 0, z_3', 1, and define $C : \mathbb{R}_+^N \to [0,1]$ by

$$C(z) = \begin{cases} z_3'' \text{ if } (z_1, z_2) \in E_{\{1,2\}}, \\ 0 \text{ if } (z_1, z_2) \notin E_{\{1,2\}} \text{ and } z'_1 + z_2' < 1, \\ \frac{1}{2} \text{ if } (z_1, z_2) \notin E_{\{1,2\}} \text{ and } z'_1 + z_2' = 1, \\ 1 \text{ if } (z_1, z_2) \notin E_{\{1,2\}} \text{ and } z'_1 + z_2' > 1, \end{cases} \tag{3.9}$$

where we recall that $E_{\{1,2\}} = \{(z_1, z_2) \mid \exists z_3, ..., z_n : (z_1, z_2, z_3, ..., z_n) \in E\}$. See Figure 2 for an illustration. The only difficulty is that C is not a cost function: it is not continuously differentiable or indeed even continuous. The formal proof in the Appendix involves approximating C by a sequence of cost functions.

1.2. *We claim that* $\mu_i^a(E \cap A) \geq p_a \mu_i^b(E \cap A)$ *for* $i = 3, ..., n$ *and all* $E \in \mathcal{E}_<$.

Let $E \in \mathcal{E}_<$ be an interval with endpoints e^-, e^+. Partition N into $N_{<,<}$, $N_{<,\leq}$, $N_{\leq,<}$, $N_{\leq,\leq}$ so that $E = \{z \in \mathbb{R}^N \mid e_i^- < z_i < e_i^+ \text{ if } i \in N_{<,<},\ e_i^- < z_i \leq e_i^+ \text{ if } i \in N_{<,\leq},\ e_i^- \leq z_i < e_i^+ \text{ if } i \in N_{\leq,<}, \text{ and } e_i^- \leq z_i \leq e_i^+ \text{ if } i \in N_{\leq,\leq}\}$. For $m = 1, 2, ...,$ define $E_m = \{z \in \mathbb{R}^N \mid e_i^- < z_i < e_i^+ \text{ if } i \in N_{<,<},\ e_i^- < z_i < e_i^+ + \frac{1}{m} \text{ if } i \in N_{<,\leq},\ e_i^- - \frac{1}{m} < z_i < e_i^+ \text{ if } i \in N_{\leq,<}, \text{ and } e_i^- - \frac{1}{m} < z_i < e_i^+ + \frac{1}{m} \text{ if } i \in N_{\leq,\leq}\}$. By definition, $E_{m+1} \subseteq E_m$ for $m = 1, 2, ...$ and $\cap_{m=1}^{\infty} E_m = E$. It follows that for all $i = 3, ..., n$, $\mu_i^a(E \cap A) = \lim_{m\to\infty} \mu_i^a(E_m \cap A)$ and $p_a \mu_i^b(E \cap A) = \lim_{m\to\infty} p_a \mu_i^b(E_m \cap A)$. By Step 1.1, $\mu_i^a(E_m \cap A) \geq p_a \mu_i^b(E_m \cap A)$ for $m = 1, 2, ...$ and $i = 3, ..., n$. The claim follows.

1.3. *We claim that* $\mu_i^a(E \cap A) \geq p_a \mu_i^b(E \cap A)$ *for* $i = 3, ..., n$ *and all* $E \in \mathcal{E}$.

Mutatis mutandis, the proof that $\mu_i^a(E \cap A) \geq p_a \mu_i^b(E \cap A)$ for $i = 3, ..., n$ and all $E \in \mathcal{E}_>$ is identical to the argument in Steps 1.1 and 1.2. The proof that $\mu_i^a(E \cap A) \geq p_a \mu_i^b(E \cap A)$ for $i = 3, ..., n$ and all $E \in \mathcal{E}_=$ is also similar. When $E \in \mathcal{E}_=$ the function C in (3.9) is symmetric with respect to z_1, z_2 and C_* coincides with C. Assumption (3.6) guarantees that $C(a) = C(b)$. The only change required in the formal proof in the Appendix is that the functions $\widetilde{C}^m$ satisfying (5.2) to (5.5) must now be symmetric with respect to z_1, z_2. This causes no difficulty since E itself is symmetric with respect to z_1, z_2. To conclude the proof of Step 1.3, it suffices to note that every interval in $\mathcal{E}$ can be written as a disjoint union of intervals in $\mathcal{E}_<$, $\mathcal{E}_>$ and $\mathcal{E}_=$.

1.4. *We claim that* $\mu_i^a(E \cap A) = p_a \mu_i^b(E \cap A)$ *for* $i = 3, ..., n$ *and all* $E \in \mathcal{E}$.

Let $E \in \mathcal{E}$ be an interval with endpoints e^-, e^+, fix $i \in \{3, ..., n\}$ and assume without loss of generality that $0 \leq e_i^-$ and $e_i^+ \leq \beta$. Let $G = \{z \in \mathbb{R}^N \mid e_i^- \leq z_i \leq e_i^+\}$. Applying (3.2) to μ_i^a and $p_a \mu_i^b$,

$$\mu_i^a(G \cap A) = e_i^+ - e_i^- = p_a \mu_i^b(G \cap A). \tag{3.10}$$

Partition $G \setminus E$ into the eight disjoint intervals $G_{<,<} = G_1$, $G_{=,<} = G_2$, $G_{>,<} = G_3$, $G_{<,=} = G_4$, $G_{>,=} = G_5$, $G_{<,>} = G_6$, $G_{=,>} = G_7$ and $G_{>,>} = G_8$, where $G_{<,<} = \{z \in G \mid$

$z_1 < z_1'$ and $z_2 < z_2'$ for all $z' \in E\}$, $G_{=,<} = \{z \in G \mid z_1 = z_1'$ for some $z' \in E$ and $z_2 < z_2'$ for all $z' \in E\}$, $G_{>,<} = \{z \in G \mid z_1 > z_1'$ and $z_2 < z_2'$ for all $z' \in E\}$, and so on.

By (3.10), $\mu_i^a(E \cap A) = (e_i^+ - e_i^-) - \sum_{k=1}^{8} \mu_i^a(G_k \cap A)$ and $p_a\mu_i^b(E \cap A) = (e_i^+ - e_i^-) - \sum_{k=1}^{8} p_a\mu_i^b(G_k \cap A)$. By Step 1.3, $\mu_i^a(G_k \cap A) \geq p_a\mu_i^b(G_k \cap A)$ for $k = 1, ..., 8$. Hence $\mu_i^a(E \cap A) \leq p_a\mu_i^b(E \cap A)$. Since the opposite weak inequality holds by Step 1.3, we are done.

1.5. *We claim that* $\mu_i^a(Z) = p_a\mu_i^b(Z)$ *for* $i = 3, ..., n$ *and all* $Z \in \mathcal{B}(A)$.

Because every open set in $\mathbb{R}^N$ is the union of a countable collection of (open) intervals, Step 1.5 follows from Step 1.4, the definition of the Borel sets, and the countable additivity of the measures μ_i^a and $p_a\mu_i^b$.

Step 2. *For any real number* α *such that* $0 < \alpha < \beta$, *partition* B *into* $B^0(\alpha) = \{z \in B \mid z_1, z_2 \leq \alpha\}$, $B^1(\alpha) = \{z \in B \mid z_2 < \alpha < z_1\}$, $B^2(\alpha) = \{z \in B \mid z_1 < \alpha < z_2\}$, *and* $B^3(\alpha) = \{z \in B \mid (\alpha, \alpha) < (z_1, z_2)\}$. *We claim that*

$$\mu_i^b(B^1(\alpha) \cup B^2(\alpha)) = 0 \text{ for } i = 3, ..., n. \tag{3.11}$$

The proof works by constructing a particular cost function C and applying Anonymity and Group Demand Monotonicity to the problems (C, a), (C, b). Although the construction of C is in essence rather simple, the requirement that it be continuously differentiable introduces unavoidable minor complications. We begin by defining C on the set $\overline{B} = \{z \in B \mid z_i = \beta$ for $i = 3, ..., n\}$. If $z = (z_1, z_2, \beta, ..., \beta) \in \overline{B}$, we abbreviate notation by writing $z = (z_1, z_2)$.

Let $s : \mathbb{R} \to [0, 1]$ be a "smoothing function", namely, a nondecreasing, continuously differentiable function such that $s(0) = 0$, $s(1) = 1$, and $s'(0) = s'(1) = 0$. Define $h : \overline{B} \to [0, 1]$ by

$$h(z) = \begin{cases} s\left(\frac{2}{1+\frac{\alpha - z_2}{\alpha - z_1}}\right) & \text{if } 0 \leq z_2 \leq z_1 < \alpha, \\ s\left(\frac{2}{1+\frac{\alpha - z_1}{\alpha - z_2}}\right) & \text{if } 0 \leq z_1 < z_2 < \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

The level sets of this function are shown in Figure 3. Observe that $h(\gamma, \gamma) = 1$ whenever $0 \leq \gamma < \alpha$. The function h is continuously differentiable everywhere but at (α, α), where it is discontinuous.

Define the functions C_1, $C_2 : \overline{B} \to [0, 1]$ by

$$\begin{aligned} C_1(z) &= s\left(1 - \left(\frac{\beta - z_1}{\beta - \alpha} - h(z)\right)\left(\frac{\alpha - z_2}{\alpha}\right)\right), \\ C_2(z) &= s\left(1 - \left(\frac{\alpha - z_1}{\alpha}\right)\left(\frac{\beta - z_2}{\beta - \alpha} - h(z)\right)\right). \end{aligned}$$

Observe that $C_1(z) = C_2(z) = 1$ if $z \geq (\alpha, \alpha)$ and $C_1(\gamma, \gamma) = C_2(\gamma, \gamma)$ whenever $0 \leq \gamma < \alpha$. The functions C_1, C_2 are continuously differentiable at every point, including (α, α) where

$\partial_i C_j(\alpha,\alpha) = 0$ for $i,j \in \{1,2\}$. Moreover, one checks that $\partial_i C_1(\gamma,\gamma) = \partial_i C_2(\gamma,\gamma)$ for $0 \leq \gamma < \alpha$ and $i \in \{1,2\}$.

Partition the set $\overline{B}$ into $\overline{B}^{01}(\alpha) = \left\{z \in \overline{B} \mid z_2 < z_1 \leq \alpha\right\}$, $\overline{B}^{02}(\alpha) = \left\{z \in \overline{B} \mid z_1 \leq z_2 \leq \alpha\right\}$, $\overline{B}^{1}(\alpha) = \left\{z \in \overline{B} \mid z_2 < \alpha < z_1\right\}$, $\overline{B}^{2}(\alpha) = \left\{z \in \overline{B} \mid z_1 < \alpha < z_2\right\}$, and $\overline{B}^{3}(\alpha) = \{z \in \overline{B} \mid (\alpha,\alpha) < (z_1,z_2)\}$. Define $C : \overline{B} \to [0,1]$ by

$$C(z) = \begin{cases} C_1(z) \text{ if } z \in \overline{B}^{01}(\alpha) \cup \overline{B}^{1}(\alpha), \\ C_2(z) \text{ if } z \in \overline{B}^{02}(\alpha) \cup \overline{B}^{2}(\alpha), \\ \quad 1 \text{ if } z \in \overline{B}^{3}(\alpha). \end{cases}$$

Thanks to the properties of C_1, C_2 discussed above, C is continuously differentiable and one checks that it is nondecreasing. The level sets of C are drawn in Figure 4. Note that $C(a) = C(b) = 1$.

Finally, with a slight abuse of notation, we extend C to $\mathbb{R}_+^N$ by letting

$$C(z) = \frac{\sum_{i=3}^{n} z_i}{(n-2)\beta} C\left(\min\left(z_1,\beta\right), \min\left(z_2,\beta\right), \beta, ..., \beta\right)$$

for all $z \in \mathbb{R}_+^N$. This function belongs to $\mathcal{C}$.

Suppose now that (3.11) is false: say, $\mu_3^b(B^1(\alpha) \cup B^2(\alpha)) > 0$. Let $a = (\alpha,\alpha,\beta,...,\beta)$, $A = [0,a]$, and define the function $\partial_3^a C : \mathbb{R}_+^N \to \mathbb{R}_+$ by $\partial_3^a C(z) = \partial_3 C(z \wedge a)$. By Step 1, the measure μ_3^a is obtained by projection of μ_3^b onto A. It then follows from the definition of the Lebesgue integral that

$$\int_A \partial_3 C d\mu_3^a = \int_B \partial_3^a C d\mu_3^b.$$

From the definition of C, we have

$$\begin{aligned} \partial_3^a C(z) &= \partial_3 C(z) \text{ for all } z \in B^0(\alpha) \cup B^3(\alpha), \\ \partial_3^a C(z) &< \partial_3 C(z) \text{ for all } z \in B^1(\alpha) \cup B^2(\alpha). \end{aligned}$$

For instance, if $z \in B^1(\alpha)$, then $\partial_3^a C(z) = \partial_3 C(\alpha, z_2, z_3, ..., z_n) = \frac{1}{(n-2)\beta} C(\alpha, z_2, \beta, ..., \beta) < \frac{1}{(n-2)\beta} C(z_1, z_2, \beta, ..., \beta) = \partial_3 C(z)$.

Therefore

$$\begin{aligned} \varphi_3(C,b) - \varphi_3(C,a) &= \int_B \partial_3 C d\mu_3^b - \int_A \partial_3 C d\mu_3^a \\ &= \int_B \left(\partial_3 C - \partial_3^a C\right) d\mu_3^b \\ &= \int_{B^1(\alpha) \cup B^2(\alpha)} \left(\partial_3 C - \partial_3^a C\right) d\mu_3^b \\ &> 0, \end{aligned}$$

that is, $\varphi_3(C,a) < \varphi_3(C,b)$. Since C is symmetric in $z_3,...,z_n$ and $a_3 = ... = a_n = \beta$ and $b_3 = ... = b_n = \beta$, Anonymity implies $\varphi_3(C,a) = ... = \varphi_n(C,a)$ and $\varphi_3(C,b) = ... = \varphi_n(C,b)$. Therefore $\varphi_i(C,a) < \varphi_i(C,b)$ for $i = 3,...,n$. Since $C(a) = C(b)$, budget balance implies $\varphi_1(C,a) + \varphi_2(C,a) > \varphi_1(C,b) + \varphi_2(C,b)$. But since C is symmetric in z_1, z_2 and $a_1 = a_2 = \alpha$ and $b_1 = b_2 = \beta$, Anonymity also forces $\varphi_1(C,a) = \varphi_2(C,a)$ and $\varphi_1(C,b) = \varphi_2(C,b)$. Hence, $\varphi_1(C,a) > \varphi_1(C,b)$ and $\varphi_2(C,a) > \varphi_2(C,b)$, contradicting Group Demand Monotonicity.

Step 3. *Define $D^{12} = \{z \in B \mid z_1 = z_2\}$. We claim that*

$$\mu_i^b(D^{12}) = \mu_i^b(B) \text{ for } i = 3,...,n. \tag{3.12}$$

For any real number α such that $0 < \alpha < \beta$, let $B^{12}(\alpha) = B^1(\alpha) \cup B^2(\alpha)$ and $D^{12}(\alpha) = B \setminus B^{12}(\alpha)$. For $r = 1,2,...$, let $B_r^{12} = \cup_{k=1}^{r-1} B^{12}(\frac{k\beta}{r})$ and $D_r^{12} = B \setminus B_r^{12}$. See Figure 5 for an illustration when $n = 3$. We get

$$\begin{aligned} \mu_i^b(D_r^{12}) &= \mu_i^b(B) - \mu_i^b(B_r^{12}) \\ &= \mu_i^b(B) - \mu_i^b(\cup_{k=1}^{r-1} B^{12}(\frac{k\beta}{r})) \\ &= \mu_i^b(B) \end{aligned}$$

for $i = 3,...,n$, where the last equality holds because (3.11) guarantees that $\mu_i^b(B^{12}(\frac{k\beta}{r})) = 0$ for $k = 1,...,r-1$ and $i = 3,...,n$. Since $D_r^{12} \supseteq D_{r+1}^{12}$ for $r = 1,2,...$ and $D^{12} = \cap_{r=1}^{\infty} D_r^{12}$, we obtain

$$\begin{aligned} \mu_i^b(D^{12}) &= \mu_i^b(\cap_{r=1}^{\infty} D_r^{12}) \\ &= \lim_{r\to\infty} \mu_i^b(D_r^{12}) \\ &= \mu_i^b(B) \end{aligned}$$

for $i = 3,...,n$.

Step 4. *For all $S \subseteq N \setminus 3$ such that $|S| \geq 2$, let $D^S = \{z \in B \mid z_i = z_j \text{ for all } i,j \in S\}$. We claim that*

$$\mu_3^b(D^{N\setminus 3}) = \mu_3^b(B). \tag{3.13}$$

From Step 3, $\mu_3^b(D^{\{1,2\}}) = \mu_3^b(B)$. Since the choice of agents 1 and 2 in Steps 1, 2 and 3 was arbitrary, this conclusion generalizes to

$$\mu_3^b(D^S) = \mu_3^b(B) \text{ for all } S \subseteq N \setminus 3 \text{ such that } |S| = 2. \tag{3.14}$$

For all $S \subseteq N \setminus 3$ such that $|S| \geq 2$, define $\widehat{D}^S = \{z \in D^S \mid z_i \neq z_k \text{ for all } i \in S \text{ and all } k \in (N \setminus 3) \setminus S\}$. Statement (3.14) implies

$$\mu_3^b(\widehat{D}^S) = 0 \text{ for all } S \subseteq N \setminus 3 \text{ such that } 2 \leq |S| \leq n-2. \tag{3.15}$$

To see why, suppose there exists $S \subseteq N \setminus 3$ such that $2 \leq |S| \leq n-2$ and $\mu_3^b(\widehat{D}^S) > 0$. Because $1 \leq |S| \leq n-2$, there exist $i \in S$ and $k \in (N \setminus 3) \setminus S$ such that $\widehat{D}^S \subseteq \{z \in B \mid z_i \neq z_k\} = B \setminus D^{\{i,k\}}$. But then $\mu_3^b(B \setminus D^{\{i,k\}}) \geq \mu_3^b(\widehat{D}^S) > 0$, hence $\mu_3^b(D^{\{i,k\}}) < \mu_3^b(B)$, contradicting (3.14).

Notice that $\cup_{S \subseteq N \setminus 3 : |S| \geq 2} D^S = \{z \in B \mid \exists i, j \in N \setminus 3 : i \neq j \text{ and } z_i = z_j\}$. Since for all $S \subseteq N \setminus 3$ such that $|S| \geq 2$, $D^S = \bigcup_{T \subseteq N \setminus 3 : T \supseteq S} \widehat{D}^T$, we have

$$\bigcup_{S \subseteq N \setminus 3 : |S| \geq 2} D^S = \bigcup_{S \subseteq N \setminus 3 : |S| \geq 2} \widehat{D}^S.$$

Since $D^{N \setminus 3} = \widehat{D}^{N \setminus 3} = \{z \in B \mid z_i = z_j \text{ for all } i, j \in N \setminus 3\}$, it follows that

$$D^{N \setminus 3} = \left(\bigcup_{S \subseteq N \setminus 3 : |S| \geq 2} D^S \right) \setminus \left(\bigcup_{\substack{S \subseteq N \setminus 3: \\ 2 \leq |S| \leq n-2}} \widehat{D}^S \right).$$

Using (3.14) and (3.15), it follows that $\mu_3^b(D^{N \setminus 3}) \geq \mu_3^b(B)$. This inequality must be an equality since $D^{N \setminus 3} \subseteq B$. This completes Step 4.

Next, we establish a general property of the measure system μ that will be used in Step 6. This property does not depend on the assumption that φ satisfies Anonymity and Group Demand Monotonicity; it is implied by (3.3) and (3.1).

Step 5. *For any real number α such that $0 < \alpha < \beta$, define $E_{3+}(\alpha) = \{z \in B \mid z_3 \geq \alpha > z_j \text{ for all } j \in N \setminus 3\}$ and $E_{3-}(\alpha) = \{z \in B \mid z_3 \leq \alpha < z_j \text{ for all } j \in N \setminus 3\}$. We claim that*

$$\text{if } \mu_i^b(E_{3+}(\alpha)) = 0 \text{ for all } i \in N \setminus 3, \text{ then } \mu_3^b(E_{3+}(\alpha)) = 0, \tag{3.16}$$

and

$$\text{if } \mu_i^b(E_{3-}(\alpha)) = 0 \text{ for all } i \in N \setminus 3, \text{ then } \mu_3^b(E_{3-}(\alpha)) = 0. \tag{3.17}$$

This is illustrated in Figure 6. We prove (3.16) and leave the similar proof of (3.17) to the reader. If $i \in N$ and P is a property that points of B may have, we abbreviate notation by writing $\mu_i^b(P)$ instead of $\mu_i^b(\{z \in B \mid z \text{ satisfies property } P\})$. For all $t \in B$, $i \in N$, $S \subseteq N \setminus i$, and $\varepsilon > 0$ small enough, we define

$$m_i^S(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mu_i^b \left(t_i \leq z_i \leq t_i + \varepsilon,\ z_j < t_j \text{ if } j \in S, \text{ and } z_j \geq t_j \text{ if } j \in (N \setminus i) \setminus S\right)$$

In particular, $m_i^{\emptyset}(t) = m_i^b(t)$, as defined just before condition (3.2).

5.1. *We claim that if $0 < \alpha' < \alpha$, then*

$$\mu_3^b \left(z_i < \alpha' \text{ for } i \in N \backslash 3 \text{ and } z_3 \geq \alpha\right) = \int_\alpha^\beta m_3^{N \backslash 3}(\alpha', \alpha', z_3, \alpha', ..., \alpha') dz_3. \tag{3.18}$$

Defining $M(t_3) = \mu_3^b\,(z_i < \alpha' \text{ for } i \in N\backslash 3 \text{ and } z_3 \geq t_3)$, we have $M'(t_3) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(M(t_3{+}\varepsilon) - M(t_3)) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\mu_3^b(z_i < \alpha' \text{ for } i \in N\backslash 3 \text{ and } z_3 \geq t_3{+}\varepsilon) - \mu_3^b(z_i < \alpha' \text{ for } i \in N\backslash 3 \text{ and } z_3 \geq t_3)) = -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mu_3^b\,(z_i < \alpha' \text{ for } i \in N\backslash 3 \text{ and } t_3 \leq z_3 < t_3 + \varepsilon) = -m_3^{N\backslash 3}(\alpha', \alpha', t_3, \alpha', ..., \alpha')$, where the last equality uses property (3.2).

Hence $\int_\alpha^\beta m_3^{N\backslash 3}(\alpha', \alpha', t_3, \alpha', ..., \alpha')dt_3 = -\int_\alpha^\beta M'(t_3)dt_3 = M(\alpha) - M(\beta) = \mu_3^b(z_i < \alpha'$ for $i \in N\backslash 3$ and $z_3 \geq \alpha) - \mu_3^b(z_i < \alpha'$ for $i \in N\backslash 3$ and $z_3 \geq \beta) = \mu_3^b(z_i < \alpha'$ for $i \in N\backslash 3$ and $z_3 \geq \alpha)$.

5.2. *We claim that if* $0 < \alpha' < \alpha$, *then*

$$m_3^{N\backslash 3}(\alpha', \alpha', z_3, \alpha', ..., \alpha') = \sum_{i\in N\backslash 3} m_i^{(N\backslash 3)\backslash i}(\alpha', \alpha', z_3, \alpha', ..., \alpha') \textit{ for almost all } z_3 \in [\alpha, \beta]\,. \tag{3.19}$$

To see why this is true, fix $z_3 \in [\alpha, \beta]$ and write $a' = (\alpha', \alpha', z_3, \alpha', ..., \alpha')$. For any set S such that $3 \in S \subseteq N$, applying property (3.1) gives $\sum_{i\in S} m_i^b(a'_S, 0_{N\backslash S}) = 1$ almost surely whenever $3 \in S \subseteq N$. By definition of $m_i^T(a')$, we have $m_i^b(a'_S, 0_{N\backslash S}) = m_i^\emptyset(a'_S, 0_{N\backslash S}) = \sum_{T:\emptyset\subseteq T\subseteq N\backslash S} m_i^T(a')$ for all $i \in S$. Therefore,

$$\sum_{i\in S}\ \sum_{T:\emptyset\subseteq T\subseteq N\backslash S} m_i^T(a') = 1$$

almost surely whenever $3 \in S \subseteq N$. Adding up these conditions pre-multiplied by alternating positive and negative unit coefficients,

$$\sum_{S:3\in S\subseteq N} (-1)^{|S|-1} \left(\sum_{i\in S}\ \sum_{T:\emptyset\subseteq T\subseteq N\backslash S} m_i^T(a') \right) = 0$$

almost surely. Cancelling terms in the left-hand side of this equation, we obtain

$$m_3^{N\backslash 3}(a') - \sum_{i\in N\backslash 3} m_i^{(N\backslash 3)\backslash i}(a') = 0$$

almost surely, as claimed.

5.3. Assume now that $\mu_i^b(E_{3+}(\alpha)) = 0$ for all $i \in N \setminus 3$. Combining (3.18) and (3.19),

$$\mu_3^b\,(z_i < \alpha' \text{ for } i \in N\backslash 3 \text{ and } z_3 \geq \alpha) = \int_\alpha^\beta \sum_{i\in N\backslash 3} m_i^{(N\backslash 3)\backslash i}(\alpha', \alpha', z_3, \alpha', ..., \alpha')dz_3$$

whenever $0 < \alpha' < \alpha$. But if $i \in N \setminus 3$ and $\alpha \leq t_3 \leq \beta$, then $m_i^{(N\backslash 3)\backslash i}(\alpha', \alpha', t_3, \alpha', ..., \alpha') = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mu_i^b\,(\alpha' \leq z_i \leq \alpha' + \varepsilon,\ z_j < \alpha' \text{ for } j \in (N\backslash 3)\backslash i, \text{ and } z_3 \geq t_3) = 0$ because $\mu_i^b(E_{3+}(\alpha))$

$= 0$ and $\{z \in B \mid \alpha' \leq z_i \leq \alpha' + \varepsilon,\ z_j < \alpha'$ for $j \in (N\backslash 3)\backslash i$, and $z_3 \geq t_3\} \subseteq E_{3+}(\alpha)$ when ε is sufficiently small. Therefore

$$\mu_3^b\left(z_i < \alpha' \text{ for } i \in N\backslash 3 \text{ and } z_3 \geq \alpha\right) = 0 \tag{3.20}$$

whenever $0 < \alpha' < \alpha$.

A standard limit argument completes Step 5. Writing $E_{3+}^k(\alpha) = \{z \in B \mid z_i < \alpha - \frac{1}{k}$ for $i \in N \setminus 3$ and $z_3 \geq \alpha\}$ for $k = 1, 2, ...$, we have $\mu_3^b(E_{3+}^k(\alpha)) = 0$ from (3.20). Since $E_{3+}^k(\alpha) \subseteq E_{3+}^{k+1}(\alpha)$ for all k and $\bigcup_{k=1}^{\infty} E_{3+}^k = E_{3+}(\alpha)$, we get $\mu_3^b(E_{3+}(\alpha)) = \lim_{k\to\infty} \mu_3^b(E_{3+}^k(\alpha)) = 0$.

Step 6. *We claim that*

$$\mu^b = \mu^{*b}. \tag{3.21}$$

Let $D = \{z \in B \mid z_i = z_j$ for all $i, j \in N\}$. We first show that

$$\mu_3^b(D) = \mu_3^b(B), \tag{3.22}$$

that is, the support of μ_3^b is included in the diagonal of B.

Partition $D^{N\backslash 3}$ into $D = \{z \in D^{N\backslash 3} \mid z_3 = z_i$ for all $i \in N\backslash 3\}$, $D_+^{N\backslash 3} = \{z \in D^{N\backslash 3} \mid z_3 > z_i$ for all $i \in N \setminus 3\}$, and $D_-^{N\backslash 3} = \{z \in D^{N\backslash 3} \mid z_3 < z_i$ for all $i \in N \setminus 3\}$. Suppose, contrary to our claim, that $\mu_3^b(D) < \mu_3^b(B)$. Then $\mu_3^b(D_+^{N\backslash 3}) > 0$ or $\mu_3^b(D_-^{N\backslash 3}) > 0$. We consider the case where $\mu_3^b(D_+^{N\backslash 3}) > 0$ and derive a contradiction. If $\mu_3^b(D_-^{N\backslash 3}) > 0$, a completely similar argument (using (3.17) instead of (3.16)) leads to a similar contradiction.

For $k = 1, 2, ...,$ define $D_+^{N\backslash 3}(k) = \{z \in B \mid z_3 - \frac{1}{k} \geq z_i = z_j$ for all $i, j \in N \setminus 3\}$. See Figure 7. Since $D_+^{N\backslash 3}(k) \subseteq D_+^{N\backslash 3}(k+1)$ for all k and $\bigcup_{k=1}^{\infty} D_+^{N\backslash 3}(k) = D_+^{N\backslash 3}$, we have $\mu_3^b(D_+^{N\backslash 3}) = \lim_{k\to\infty} \mu_3^b(D_+^{N\backslash 3}(k))$. Therefore there is some k such that

$$\mu_3^b(D_+^{N\backslash 3}(k)) > 0. \tag{3.23}$$

Let Γ be a finite subset of $[0, \beta]$ such that

$$D_+^{N\backslash 3}(k) \subseteq \bigcup_{\alpha \in \Gamma} E_{3+}(\alpha) \tag{3.24}$$

where, as in Step 5, $E_{3+}(\alpha) = \{z \in B \mid z_3 \geq \alpha > z_j \text{ for all } j \in N \setminus 3\}$. For instance, we may choose $\Gamma = \left\{\frac{1}{2k}, \frac{2}{2k}, ..., \frac{2k-1}{2k}\right\}$. From (3.23) and (3.24) follows that there exists $\alpha \in \Gamma$ such that $\mu_3^b(E_{3+}(\alpha)) > 0$. By (3.16) in Step 5, there must exist some $i \in N \setminus 3$ such that

$$\mu_i^b(E_{3+}(\alpha)) > 0. \tag{3.25}$$

Since the choice of agents 1 and 2 in Step 1 and the choice of agent 3 in Steps 4 and 5 was arbitrary, an equation analogous to (3.13) holds for agent i as well, namely,

$$\mu_i^b(D^{N\backslash i}) = \mu_i^b(B).$$

But $E_{3+}(\alpha) \subseteq B \setminus D^{N\setminus i}$ (since $z \in E_{3+}(\alpha) \Rightarrow z_j \neq z_3$ for all $j \in N \setminus 3 \Rightarrow z_j \neq z_3$ for all $j \in N \setminus \{3, i\} \Rightarrow z_j \neq z_k$ for some $j, k \in N \setminus i \Rightarrow z \in B \setminus D^{N\setminus i}$). Therefore $\mu_i^b(E_{3+}(\alpha)) \leq \mu_i^b(B \setminus D^{N\setminus i}) = 0$, contradicting (3.25). This proves (3.22).

Since the support of μ_3^b is included in the diagonal of B, it follows from (3.2) that μ_3^b is uniquely determined and, by definition of the serial measure system, $\mu_3^b = \mu_3^{*b}$. Since the choice of agents 1 and 2 in Step 1 and the choice of agent 3 in Steps 4 and 5 was arbitrary, $\mu_i^b = \mu_i^{*b}$ for all $i \in N$, completing Step 6.

Next we identify the key restriction imposed on μ by the Dummy Independence axiom. Let $x \in \mathbb{R}_+^N$ and $X = [0, x]$. Define the demand profile $x(12) = (x_1, x_2, 0, ..., 0)$ and let $X(12) = [0, x(12)]$. Define the demand profile $\overline{x}(12) = (0, 0, x_3, ..., x_n)$ and let $\overline{X}(12) = [\overline{x}(12), x]$. Define $\mathcal{E}_x^o(12) = \left\{E \in \mathcal{E}^o \mid E \cap X(12) \neq \emptyset \text{ and } E \cap \overline{X}(12) \neq \emptyset\right\}$.

Step 7. *We claim that for all $E \in \mathcal{E}_x^o(12)$,*

$$\mu_i^{x(12)}(E \cap X(12)) = \mu_i^x(E \cap X) \text{ for } i = 1, 2. \tag{3.26}$$

Let $E \in \mathcal{E}_x^o(12)$. Let $e^- \ll e^+$ be such that $E =]e^-, e^+[$. Since $E \cap X(12) \neq \emptyset$, we have $e_i^+ > 0$ for $i = 1, 2$. Define $e_{i+}^- = \max(e_i^-, 0)$. For $m = 3, 4, ...$, let

$$E^m = \{z \in E \mid e_{i+}^- + \frac{1}{m}(e_i^+ - e_{i+}^-) \leq z_i \leq e_i^+ - \frac{1}{m}(e_i^+ - e_{i+}^-) \text{ for } i = 1, 2\}.$$

Fix a real number $k > 0$ and let $(C^m)_{m=3,4,...}$ be a sequence of cost functions such that, for all m, (a) $C^m(z)$ is independent of $z_3, ..., z_n$, (b) $\partial_1 C^m(z) = k$ if $z \in E^m \cap \mathbb{R}_+^N$, (c) $\partial_1 C^m(z) \leq k$ if $z \in E \cap \mathbb{R}_+^N$, and (d) $\partial_1 C^m(z) = 0$ if $z \in \mathbb{R}_+^N \setminus E$. See Figure 8 for an illustration. Then,

$$\begin{aligned} k\mu_1^{x(12)}(E^m \cap X(12)) &\leq \varphi_1(C^m, x(12)) \leq k\mu_1^{x(12)}(E \cap X(12)), \\ k\mu_1^x(E^m \cap X) &\leq \varphi_1(C^m, x) \leq k\mu_1^x(E \cap X). \end{aligned}$$

Since $\lim_{m\to\infty} \mu_1^{x(12)}(E^m \cap X(12)) = \mu_1^{x(12)}(E \cap X(12))$ and $\lim_{m\to\infty} \mu_1^x(E^m \cap X) = \mu_1^x(E \cap X)$, we have $\lim_{m\to\infty} \varphi_1(C^m, x(12)) = k\mu_1^{x(12)}(E \cap X(12))$ and $\lim_{m\to\infty} \varphi_1(C^m, x) = k\mu_1^x(E \cap X)$. By Dummy Independence, $\varphi_1(C^m, x(12)) = \varphi_1(C^m, x)$ for all m. Since $\varphi_1(., x(12))$ and $\varphi_1(., x)$ are continuous (because they are of the form given in (3.3)), it follows that $\lim_{m\to\infty} \varphi_1(C^m, x(12)) = \lim_{m\to\infty} \varphi_1(C^m, x)$, hence $\mu_1^{x(12)}(E \cap X(12)) = \mu_1^x(E \cap X)$. A completely similar argument shows that $\mu_2^{x(12)}(E \cap X(12)) = \mu_2^x(E \cap X)$.

Step 8. We conclude the proof.

8.1. *Let $b(12) = (\beta, \beta, 0, ..., 0)$ and $B(12) = [0, b(12)]$. We claim that*

$$\mu^{b(12)} = \mu^{*b(12)}. \tag{3.27}$$

From Step 7, $\mu_i^{b(12)}(E \cap B(12)) = \mu_i^b(E \cap B)$ for $i = 1, 2$ and all $E \in \mathcal{E}_b^o(12)$. Using Step 6, it follows that $\mu_i^{b(12)}(E \cap B(12)) = \mu_i^{*b}(E \cap B) = 0$ for $i = 1, 2$ and all $E \in \mathcal{E}_b^o(12) \cap (\mathcal{E}_< \cup \mathcal{E}_>)$.

This means that for $i = 1, 2$ the support of $\mu_i^{b(12)}$ is included in $\{z \in B(12) \mid z_1 = z_2\}$, the diagonal of $B(12)$. Then (3.27) follows because of (3.2) and because $b_i(12) = 0$ for $i = 3, ..., n$.

8.2. *Fix a real number* α *such that* $0 \leq \alpha < \beta$ *and let* $x = (\alpha, \beta, 0, ..., 0)$. *We claim that*

$$\mu^x = \mu^{*x}. \tag{3.28}$$

Because Group Demand Monotonicity implies Demand Monotonicity, $\varphi_1(C, x) \leq \varphi_1(C, b(12))$ for all $C \in \mathcal{C}$. As Friedman and Moulin (1999) show (see Step 3 of the proof of their Theorem 1), this implies that μ_1^x and $\mu_1^{b(12)}$ coincide on $[0, x]$. Because of (3.2) and (3.1), it follows that $\mu_i^x = p_x \mu_i^{b(12)}$ for $i = 1, 2$. Hence, by Step 8.1 and the definition of the serial measure system μ^*, $\mu_i^x = p_x \mu_i^{b(12)} = p_x \mu_i^{*b(12)} = \mu_i^{*x}$ for $i = 1, 2$ and (3.28) follows because $x_i = 0$ for $i = 3, ..., n$.

8.3. Let x be an arbitrary demand profile such that $0 \leq x \leq b$. For any two distinct $i, j \in N$, let $x(ij) = (x_{\{i,j\}}, 0_{N \setminus \{i,j\}})$. Since the choice of agents $1, 2$ in Steps 7, 8.1, and 8.2 was arbitrary, (3.26) and (3.28) generalize:

$$\mu_k^{x(ij)}(E \cap X(ij)) = \mu_k^x(E \cap X) \text{ for } k = i, j \text{ and all } E \in \mathcal{E}_x^o(ij)$$

and

$$\mu^{x(ij)} = \mu^{*x(ij)}.$$

Since these two facts hold for all distinct $i, j \in N$, the support of μ^x must equal S^{*x}, the support of μ^{*x} defined in (3.5). Because of (3.2), any measure system at x whose support equals S^{*x} coincides with μ^{*x}. Thus $\mu^x = \mu^{*x}$. Since β is arbitrary, we conclude that $\mu = \mu^*$, hence $\varphi = \varphi^*$.■

4. Discussion

(1) The only other existing axiomatization of the Friedman-Moulin serial method in the continuous cost-sharing model is Theorem 2 in Friedman and Moulin (1999), which states that the serial method is characterized by Additivity, Dummy, Demand Monotonicity, and Upper Bound for Homogenous Goods. This last axiom says that if $C(z) = c(\sum_{i \in N} z_i)$, then $\varphi_i(C, x) \leq C(x_i, ..., x_i)$ for all $x \in \mathbb{R}_+^N$ and $i \in N$. We already discussed the limitations of this axiomatization in the Introduction.

In the discrete version of the cost-sharing model (that is, when demands are integers and the cost function is defined over $\mathbb{N}^N$) Moulin and Sprumont (2006) offer an axiomatization of the (proper reformulation of the) Friedman-Moulin serial method based on Distributivity. That property states that the cost-sharing method should commute with the composition of cost functions. It is a technical axiom akin to Additivity with no clear normative or strategic interpretation. By contrast, Group Demand Monotonicity is meaningful on both counts.

Still in the discrete framework, Sprumont (2008) studies a combination of axioms very closely related to the one we use. The main difference is that his axiom of Independence of Dummy Changes is strictly stronger than the combination of Weak Dummy and Dummy Independence, as the example of the Aumann-Shapley method shows. Moreover, his version of Anonymity is stronger than ours. In spite of this, Sprumont's (2008) axioms fail to uniquely characterize the serial method: they circumscribe the class of so-called "nearly serial" methods. The use of the continuous framework allows us to obtain a much crisper result.

(2) Building upon results derived by Haimanko (2000a and 2000b) in the model of nonatomic games, Friedman (2004) proposes a description of the methods satisfying Additivity and Dummy which constitutes an interesting alternative to the one given by (3.3) and (3.1). He shows that at any demand profile x, any such method can be expressed as an average of the "path methods at x". Formally, if $x \in \mathbb{R}_+^N$, denote by R_x the set of paths to x. If $i \in N$ and $(C, x) \in \mathcal{C} \times \mathbb{R}_+^N$, define $f_{i,(C,x)} : R_x \to \mathbb{R}_+$ by $f_{i,(C,x)}(r_x) = \int_0^1 \partial_i C(r_x(t)) \frac{dr_x}{dt}(t)dt$. This mapping associates with every path r_x to x the cost share paid by agent i in the problem (C, x) according to the path formula generated by r_x. Friedman (2004) proves that if a method φ satisfies Additivity and Dummy, then for every $x \in \mathbb{R}_+^N$ there is a measure ν_x on R_x such that

$$\varphi_i(C, x) = \int_{R_x} f_{i,(C,x)} d\nu_x$$

for all $i \in N$ and $(C, x) \in \mathcal{C} \times \mathbb{R}_+^N$. We refer the reader to the original paper for the measure-theoretic details.

This characterization is more compact and intuitive than the Friedman-Moulin characterization in terms of measure systems. It could lead to a simpler proof of our theorem (avoiding the tedious Steps 4 to 6) and, as we suggest in comment 4 below, could also prove useful to analyze non-anonymous methods.

(3) The axioms used in our theorem are independent.

(a) A cost-sharing method satisfying all our axioms but Additivity is equal sharing among the non-dummy agents: given a problem (C, x), let $N(C) = \{i \in N \mid \exists z \in \mathbb{R}_+^N\} : \partial_i C(z) > 0$, $\varphi_i(C, x) = C(x)/|N(C)|$ if $i \in N(C)$ and $\varphi_i(C, x) = 0$ if $i \in N \setminus N(C)$.

(b) A method violating only Weak Dummy is plain egalitarianism, $\varphi_i(C, x) = C(x)/n$.

(c) A simple example of a method violating only Dummy Independence is proportionality: $\varphi_i(C, x) = x_i C(x)/\sum_{j \in N} x_j$ if $x > 0$ and $\varphi_i(C, 0) = 0$.

This method, however, violates Dummy. For an example that also satisfies Dummy, combine the serial method with the Shapley-Shubik method as follows: let $\varphi_i(C, x) = \varphi_i^*(C, x)$ if $|\{j \in N \mid x_j > 0\}| \geq 3$ and $\varphi_i(C, x) = \varphi_i^{SS}(C, x)$ otherwise, where we recall that the Shapley-Shubik method φ^{SS} charges agent i her Shapley value in the "stand-alone game" $\gamma_{(C,x)}(S) = C(x_S, 0_{N \setminus S})$ for all $S \subseteq N$. This method satisfies Group Demand Monotonicity because the Shapley-Shubik method satisfies Demand Monotonicity.

Albeit quite reasonable, Dummy Independence is an axiom that dramatically reduces the set of admissible methods. It would be interesting to know whether it can be relaxed.

Observe that the method described in the previous paragraph is not a continuous function of the demand profile. Continuity at Zero, which states that for all $(C,x) \in \mathcal{C}\times\mathbb{R}_+^N$ and all $i \in N$, $\lim_{x_i \to 0} \varphi(C,(x_i, x_{N\setminus i})) = \varphi(C,(0_i, x_{N\setminus i}))$, allowed Friedman and Moulin (1999) to dispense with Dummy Independence in their characterization of the so-called random-order methods. It is therefore tempting to conjecture that the combination of Dummy and Continuity at Zero could replace the combination of Weak Dummy and Dummy Independence in our theorem. This is not the case. Suppose $N = \{1,2,3\}$. For any $(C,x) \in \mathcal{C}\times\mathbb{R}_+^N$, let $\underline{x} = (\min(x_1,x_2,x_3), \min(x_1,x_2,x_3), \min(x_1,x_2,x_3))$ and define $C_{\underline{x}} \in \mathcal{C}$ by $C_{\underline{x}}(z) = C(\underline{x}+z) - C(\underline{x})$. The method $\varphi_i(C,x) = \varphi_i^*(C,\underline{x}) + \varphi_i^{SS}(C_{\underline{x}}, x - \underline{x})$ satisfies Additivity, Dummy, Continuity at Zero, Anonymity and Group Demand Monotonicity. This last axiom is satisfied because the Shapley-Shubik method is demand-monotonic and because at most two agents demand a quantity higher than the smallest demand. It is unclear whether similar examples can be constructed when there are more than three agents.

(d) For a method violating only Anonymity, consider any fixed-path method (as defined in the second paragraph of Section 3) other than the serial method: the simplest example is the so-called incremental method $\varphi_i(C,x) = \gamma_{(C,x)}(\{1,...,i\}) - \gamma_{(C,x)}(\{1,...,i-1\})$.

(e) Finally, the Aumann-Shapley and Shapley-Shubik methods are examples of methods violating only Group Demand Monotonicity.

(4) We conjecture that the fixed-path methods are the only methods satisfying Additivity, Weak Dummy, Dummy Independence and Group Demand Monotonicity.

It is not difficult to see that a nondegenerate convex combination of two fixed-path methods cannot be group demand monotonic. Here is a sketch of the argument in the three-agent case. Let r, r' be two fixed increasing unbounded paths in $\mathbb{R}_+^N$ and let $\varphi^r, \varphi^{r'}$ be the methods they generate: for each demand profile x, the paths to x used to compute the cost shares are the projections of r and r' on $[0,x]$, which we write r_x and r'_x. Let $\varphi = \lambda\varphi^r + (1-\lambda)\varphi^{r'}$, where $0 < \lambda < 1$.

Because the two paths r, r' are different, there exists a demand profile x at which the order in which the individual demands $x_1, ..., x_n$ are reached along the path r_x differs from the order in which they are reached along r'_x. To be more precise, write, for each $x \in \mathbb{R}_+^N$ and $i \in N$, $t_x(i) := \inf\{t \mid (r_x)_i(t) \geq x_i\}$ and $t'_x(i) := \inf\{t \mid (r'_x)_i(t) \geq x_i\}$. There must exist some demand profile x and two agents, say, 1 and 2, such that

$$t_x(1) < t_x(2) \text{ and } t'_x(2) < t'_x(1). \tag{4.1}$$

This is shown in Figure 9. The figure illustrates the case where $t_x(1) < t_x(2) < t_x(3)$ and $t'_x(2) < t'_x(1) < t'_x(3)$ but the order in which x_3 is reached relative to x_1 and x_2 is unimportant. Let $a(i) = r_x(t_x(i))$ and $a'(i) = r'_x(t'_x(i))$ for $i = 1,2$.

Let $x' = (x'_1, x'_2, x_3)$, where $x'_i > x_i$ for $i = 1,2$. Let C be a cost function such that (i) $\partial_1 C$ is positive everywhere along the path r'_x between $a'(2)$ and $a'(1)$ except in a neighborhood of $a'(2)$ (where it is zero), (ii) $\partial_2 C$ is positive everywhere along the path r_x between $a(1)$ and $a(2)$ except in a neighborhood of $a(1)$, but (iii) both $\partial_1 C$ and $\partial_2 C$ (but not $\partial_3 C$) are zero everywhere along the path $r_{x'}$ between $a(1)$ and x' as well as along the path $r'_{x'}$ between $a'(2)$ and x'. Then

$$\begin{aligned}
& \varphi_1(C,x') - \varphi_1(C,x) \\
= \; & \lambda \left(\varphi_1^r(C,x') - \varphi_1^r(C,x)\right) + (1-\lambda)\left(\varphi_1^{r'}(C,x') - \varphi_1^{r'}(C,x)\right) \\
= \; & -(1-\lambda)\int_{t'_x(2)}^{t'_x(1)} \partial_1 C(r'_x(t))\frac{dr'_x}{dt}(t)dt \\
< \; & 0
\end{aligned}$$

and

$$\begin{aligned}
& \varphi_2(C,x') - \varphi_2(C,x) \\
= \; & \lambda \left(\varphi_2^r(C,x') - \varphi_2^r(C,x)\right) + (1-\lambda)\left(\varphi_2^{r'}(C,x') - \varphi_2^{r'}(C,x)\right) \\
= \; & -\lambda\int_{t_x(1)}^{t_x(2)} \partial_2 C(r_x(t))\frac{dr_x}{dt}(t)dt \\
< \; & 0,
\end{aligned}$$

a violation of Group Demand Monotonicity.

A possible proof of our conjecture would exploit the Friedman (2004) characterization of Additivity and Dummy discussed in comment 2. The main difficulties would be (i) to show that every method satisfying Additivity, Weak Dummy, Dummy Independence and Group Demand Monotonicity can be written as a convex combination of fixed-path methods and (ii) to extend the argument above to arbitrary convex combinations of fixed-path methods, including the uncountable ones.

(5) Group Demand Monotonicity may be replaced in the statement of our theorem with the weaker requirement that members of groups *of size one or two* cannot all lower their cost shares by jointly raising their demands. This is because our proof does not use the full power of the axiom. Group Demand Monotonicity is employed in Steps 1, 2, and 8. Steps 1 and 2 only use the restriction of the condition to groups of two agents while Step 8 only uses its restriction to single agents. Of course, Demand Monotonicity could not replace Group Demand Monotonicity: numerous demand-monotonic methods, including the Shapley-Shubik method, satisfy our first four axioms.

(6) One can think of axioms of responsiveness to marginal costs that would strengthen Dummy. One very natural requirement would stipulate that if the marginal cost function associated with an agent increases, that agent should not end up paying less: if $\partial_i C^1(z) \leq \partial_i C^2(z)$ for all $z \in \mathbb{R}_+^N$, then $\varphi_i(C^1,x) \leq \varphi_i(C^2,x)$ for all $x \in \mathbb{R}_+^N$. This property is automatically satisfied by every additive method satisfying Dummy, including the serial method, because of the Friedman-Moulin lemma (see formula (3.3)).

In the same spirit of cost responsiveness, Young (1985) proposed a powerful condition dubbed Symmetric (Cost) Monotonicity: if $\partial_i C^1(z) \leq \partial_j C^2(z)$ for all $z \in [0,x]$, then $\varphi_i(C^1,x)/x_i \leq \varphi_j(C^2,x)/x_j$. The fact, proved by Young, that only the Aumann-Shapley method possesses this property illustrates the trade-off existing between the fundamental desiderata of responsiveness to marginal costs and responsiveness to demand sizes. In our interpretation of the cost-sharing model where each good is consumed by a clearly identifiable agent, Symmetric Monotonicity is not compelling because average cost shares have no particular ethical relevance.

(7) One can also think of axioms of responsiveness to demand size that would strengthen Group Demand Monotonicity. One such property is Strong Group Demand Monotonicity: the sum of the cost shares of the agents in a group should not decrease when they jointly raise their demands. This condition is violated by the serial method[6]. In fact, one can show (by adapting the arguments in Moulin and Sprumont (2005) to our continuous model) that no additive method satisfies Strong Group Demand Monotonicity and Dummy. This fact is another illustration of the trade-off between cost responsiveness and demand responsiveness within the class of additive methods. It also shows how restrictive Additivity is. Indeed, it is easy to construct non additive methods satisfying Dummy and Strong Group Demand Monotonicity: equal sharing among the non-dummy agents is a very simple example; it actually satisfies the condition that *none* of the agents who jointly raise their demands pays less.

5. Appendix

We provide the proof of the claim made in Step 1.1. To do so, we begin by constructing a sequence of continuous functions C^m approximating the function C in (3.9). Recall that $E =]e^-, e^+[$ is an open interval below the plane $z_1 = z_2$. For $m = 3, 4, ...,$ and $\lambda \in [0, 1]$, define the set

$$E^m(\lambda) = \left\{ z \in E \mid \min\left(\frac{1}{m}, \frac{2\lambda}{m}\right) \leq z_i' \leq \min\left(1 - \frac{1}{m}, 1 - \frac{2\lambda}{m}\right) \text{ for } i = 1, 2 \right\}.$$

Figure 10 shows the set $E^m_{\{1,2\}}(\lambda)$ for some values of λ. Observe that $E^m(\frac{1}{2}) = \{z \in E \mid e_i^- + \frac{1}{m}(e_i^+ - e_i^-) \leq z_i \leq e_i^+ - \frac{1}{m}(e_i^+ - e_i^-) \text{ for } i = 1, 2\} \subseteq E^m(\lambda)$ for all $\lambda \in [0, 1]$. From now on we write E^m instead of $E^m(\frac{1}{2})$. Notice that

$$E^m \subseteq E^{m+1} \text{ for } m = 3, 4, ... \text{ and } \cup_{m=3}^{\infty} E^m = E. \tag{5.1}$$

For $m = 3, 4, ...,$ and $\lambda \in [0, 1]$, define $C_0^m(., ., \lambda)$, $C_1^m(., ., \lambda)$ on $\mathbb{R}_+^{\{1,2\}}$ by

$$\begin{aligned} C_0^m(z_1, z_2, \lambda) &= \max\left(0,\ \lambda m \min(z_1', z_2'),\ \frac{1-m}{2} + \frac{m}{2}(z_1' + z_2')\right), \\ C_1^m(z_1, z_2, \lambda) &= \min\left(1,\ 1 - (1-\lambda) m \min(1 - z_1', 1 - z_2'),\ \frac{1-m}{2} + \frac{m}{2}(z_1' + z_2')\right). \end{aligned}$$

[6] A much more modest strengthening of Group Demand Monotonicity would require that if $x_i < x_i'$ for all $i \in S$ and $x_i = x_i'$ for all $i \in N \backslash S$, then either there exists $i \in S$ such that $\varphi_i(C, x) < \varphi_i(C, x')$ or else $\varphi_i(C, x) = \varphi_i(C, x')$ for all $i \in S$. This condition too is violated by the serial method.

Figure 11 illustrates the function $C_0^m(.,.,\lambda)$ (for $\lambda = \frac{3}{4}$) and the function $C_1^m(.,.,\lambda)$ (for $\lambda = \frac{1}{2}$). Define $C^m : \mathbb{R}_+^N \to [0,1]$ by

$$C^m(z) = \begin{cases} z_3'' \text{ if } (z_1,z_2) \in E_{\{1,2\}}^m(z_3''), \\ C_0^m\left(z_1,z_2,\max(z_3'',\frac{1}{2})\right) \text{ if } (z_1,z_2) \notin E_{\{1,2\}}^m(z_3'') \text{ and } \frac{1-m}{2} + \frac{m}{2}(z_1'+z_2') \le z_3'', \\ C_1^m\left(z_1,z_2,\min(z_3'',\frac{1}{2})\right) \text{ if } (z_1,z_2) \notin E_{\{1,2\}}^m(z_3'') \text{ and } \frac{1-m}{2} + \frac{m}{2}(z_1'+z_2') > z_3''. \end{cases}$$

See Figure 12. Because of (5.1), the sequence $(C^m)_{m=3,4,...}$ converges pointwise to the function C defined in (3.9).

The functions C^m are not continuously differentiable. Our next step consists in smoothing them off. We begin by slightly modifying them to obtain functions that are continuously differentiable in z_3. Let k be a large positive real number. For $m = 3,4,...$, let $f^m : \mathbb{R}_+ \to [0,1]$ be a continuously differentiable nondecreasing function such that (a) $f^m(z_3) = z_3''$ whenever $z_3 \le e_3^-$ or $e_3^- + \frac{1}{m}(e_3^+ - e_3^-) \le z_3 \le e_3^+ - \frac{1}{m}(e_3^+ - e_3^-)$ or $e_3^+ \le z_3$ and (b) the derivative of f^m is bounded above by k. Define $C^{m,m} : \mathbb{R}_+^N \to [0,1]$ by replacing z_3'' with $f^m(z_3)$ in the definition of the function C^m above. Notice that $C^{m,m}$ coincides with C^m outside E. Because the sequence $(f^m)_{m=3,4,...}$ converges pointwise to the function $f(z_3) = z_3''$, the sequence $(C^{m,m})_{m=3,4,...}$ converges pointwise to the function C defined in (3.9).

Next, we modify the functions $C^{m,m}$ to obtain functions that are also continuously differentiable in z_1 and z_2. For $m = 3,4,...$ and $\lambda \in [0,1]$, define $E_\lambda^m = \{z \in E^m \cap \mathbb{R}_+^N \mid z_3'' = \lambda\}$ and $E_\lambda = \{z \in E \cap \mathbb{R}_+^N \mid z_3'' = \lambda\}$. Given a function $\widetilde{C} : \mathbb{R}_+^N \to [0,1]$, define

$$E(\widetilde{C},\lambda) = \left\{ z \in E \mid \widetilde{C}(z) = \lambda \text{ and } z_3'' = \lambda \right\}.$$

Note that $E(C,\lambda) = E_\lambda$. Let $(\widetilde{C}^m)_{m=3,4,...}$ be a sequence of cost functions satisfying the following conditions:

$$\forall m = 3,4,... \text{ and } z \notin E,\ \widetilde{C}^m(z) = C^m(z), \tag{5.2}$$

$$\forall m = 3,4,... \text{ and } \lambda \in \left[\frac{1}{m+1}, 1 - \frac{1}{m+1}\right],\ E_\lambda^m \subseteq E(\widetilde{C}^{m+1},\lambda) \cap E_\lambda^{m+1}, \tag{5.3}$$

$$\forall \lambda \in [0,1],\ E(\widetilde{C}^m,\lambda) \to E_\lambda \text{ in the Hausdorff metric}, \tag{5.4}$$

$$\forall m = 3,4,... \text{ and } z \in E,\ \partial_3 \widetilde{C}^m(z) \le k. \tag{5.5}$$

The construction of such a sequence causes no difficulty: see Figure 13 for an illustration.

We make two sets of claims regarding this sequence. First,

$$\widetilde{C}^m \to C \text{ pointwise}, \tag{5.6}$$

where C is given in (3.9). To see why this is true, check first, using (5.4) and the continuity of the cost functions $\widetilde{C}^m$, that for all $\lambda \in [0,1]$, $\{z \in E \mid \widetilde{C}^m(z) = \lambda\} \to E_\lambda$ in the Hausdorff metric. This in turn implies, using the continuity of the cost functions again, that for all $z \in E$, $\widetilde{C}^m(z) \to C(z)$. Combining this with (5.2) and the fact that $C^m \to C$ pointwise yields (5.6).

Second, we claim that each cost function $\widetilde{C}^m$ has properties similar to C. Specifically, (a) $\widetilde{C}^m(a) = \widetilde{C}^m(b)$, (b) $\widetilde{C}^m(z)$ is independent of $z_4, ..., z_n$, and (c) $\partial_3 \widetilde{C}^m$ is a positive constant on a set $\widetilde{E}(\widetilde{C}^m)$ which tends to $E \cap \mathbb{R}^N_+$ as m grows, and zero outside $E \cap \mathbb{R}^N_+$. Properties (a) and (b) are clear. As for (c), let $E^{m,m} = \left\{z \in E^m \mid z_3'' \in \left[\frac{1}{m}, 1 - \frac{1}{m}\right]\right\}$ for $m = 3, 4, ...$ and notice that

$$\cup_{m=3}^{\infty} E^{m,m} = E. \tag{5.7}$$

For $m = 3, 4, ...$, define the set

$$\widetilde{E}^m = \left\{z \in E^{m,m} \mid \widetilde{C}^m(z_{-3}, \lambda e_3^+ + (1-\lambda) e_3^-) = \lambda \text{ for all } \lambda \in \left[\frac{1}{m}, 1 - \frac{1}{m}\right]\right\}.$$

By construction,

$$\partial_3 \widetilde{C}^m(z) = \begin{cases} \frac{1}{e_3^+ - e_3^-} \text{ for all } z \in \widetilde{E}^m, \\ 0 \text{ for all } z \notin E, \end{cases} \tag{5.8}$$

and we claim that

$$\widetilde{E}^m \subseteq \widetilde{E}^{m+1} \text{ for } m = 3, 4, ..., \text{ and } \cup_{m=3}^{\infty} \widetilde{E}^m = E \cap \mathbb{R}^N_+. \tag{5.9}$$

To prove this claim, fix $m \geq 3$ and, for all $\lambda \in \left[\frac{1}{m}, 1 - \frac{1}{m}\right]$, let $\widetilde{E}^m(\lambda) = \{z \in \widetilde{E}^m \mid z_3'' = \lambda\}$. Using (5.3), it is straightforward to check that $E_\lambda^m \subseteq \widetilde{\widetilde{E}}^{m+1}(\lambda)$ for all $\lambda \in \left[\frac{1}{m+1}, 1 - \frac{1}{m+1}\right]$, hence $E^{m,m} \subseteq \widetilde{E}^{m+1}$, and the first statement in (5.9) follows. As for the second statement, we have $\cup_{m=3}^{\infty} \widetilde{E}^m = \cup_{m=2}^{\infty} \widetilde{E}^{m+1} \supseteq \cup_{m=2}^{\infty} E^{m,m} = E$ because of (5.7).

We are now ready to complete the proof of Step 1.1. The function $\widetilde{C}^m_*$ defined on $\mathbb{R}^N_+$ by

$$\widetilde{C}^m_*(z) = \begin{cases} \widetilde{C}^m(z) \text{ if } z_1 \geq z_2, \\ \pi^{12} \widetilde{C}^m(z) \text{ otherwise,} \end{cases}$$

need not be a cost function because it may fail to be differentiable when $z_1 = z_2$. However, it is straightforward to construct a cost function $\widetilde{C}^{m,m}_*$ which (a) coincides with $\widetilde{C}^m_*$ whenever $|z_1 - z_2| < \frac{1}{m}$, (b) is symmetric in z_1, z_2, (c) is independent of $z_4, ..., z_n$, and (d) is such that $\partial_3 \widetilde{C}^{m,m}(z) = 0$ for all $z \notin E$. For m large enough, (3.7) and (5.9) guarantee that $p_a \mu_3^b(\widetilde{E}^m \cap A) = \mu_i^b(\widetilde{E}^m \cap B)$. Using (5.8) and Anonymity,

$$\varphi_3(\widetilde{C}^{m,m}_*, a) = \frac{2\mu_3^a(\widetilde{E}^m \cap A)}{e_3^+ - e_3^-} + 2 \int_{(E \cap A) \backslash (\widetilde{E}^m \cap A)} \partial_3 \widetilde{C}^m d\mu_3^a$$

and

$$\begin{aligned} \varphi_3(\widetilde{C}^{m,m}_*, b) &= \frac{2\mu_3^b(\widetilde{E}^m \cap B)}{e_3^+ - e_3^-} + 2 \int_{(E \cap B) \backslash (\widetilde{E}^m \cap B)} \partial_3 \widetilde{C}^m d\mu_3^b \\ &= \frac{2p_a \mu_3^b(\widetilde{E}^m \cap A)}{e_3^+ - e_3^-} + 2 \int_{(E \cap B) \backslash (\widetilde{E}^m \cap B)} \partial_3 \widetilde{C}^m d\mu_3^b. \end{aligned}$$

By (5.5), $\int_{(E\cap A)\setminus(\widetilde{E}^m\cap A)} \partial_3\widetilde{C}^m \; d\mu_3^a \leq k\mu_3^a((E\cap A)\setminus(\widetilde{E}^m\cap A))$ for all m. By (5.9), $\lim_{m\to\infty}\mu_3^a((E\cap A)\setminus(\widetilde{E}^m\cap A)) = 0$, hence $\lim_{m\to\infty}\int_{(E\cap A)\setminus(\widetilde{E}^m\cap A)} \partial_3\widetilde{C}^m d\mu_3^a = 0$. Similarly, $\lim_{m\to\infty}\int_{(E\cap B)\setminus(\widetilde{E}^m\cap B)} \partial_3\widetilde{C}^m d\mu_3^b = 0$. Therefore

$$\begin{aligned} & \lim_{m\to\infty}\left(\varphi_3(\widetilde{C}_*^{m,m},a) - \varphi_3(\widetilde{C}_*^{m,m},b)\right) \\ = \; & \frac{2}{e_3^+ - e_3^-}\lim_{m\to\infty}\left(\mu_3^a(\widetilde{E}^m\cap A) - p_a\mu_3^b(\widetilde{E}^m\cap A)\right) \\ = \; & \frac{2}{e_3^+ - e_3^-}\left(\mu_3^a(E\cap A) - p_a\mu_3^b(E\cap A)\right) < 0 \end{aligned}$$

by (5.9) and (3.8).

Because of (5.6), $\widetilde{C}_*^{m,m}\to C_*$ pointwise. Hence, since $\varphi_3(.,a)$ and $\varphi_3(.,b)$ are continuous (by (3.3)), there exists m such that $\varphi_3(\widetilde{C}_*^{m,m},a) - \varphi_3(\widetilde{C}_*^{m,m},b) < 0$. As in the sketch of the argument at the beginning of Step 1.1, budget balance, Dummy, and Anonymity now imply that $\varphi_i(\widetilde{C}_*^{m,m},a) > \varphi_i(\widetilde{C}_*^{m,m},b)$ for $i = 1,2$, contradicting Group Demand Monotonicity.

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