EXPOSURE PROBLEM IN MULTI-UNIT AUCTIONS

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Abstract

We characterize the optimal bidding strategies of local and global bidders for two heterogenous licenses in a multi-unit simultaneous ascending auction. The global bidder wants to win both licenses to enjoy synergies; therefore, she bids more than her stand-alone valuation of a license. This exposes her to the risk of losing money even when she wins all licenses. We determine the optimal bidding strategies in the presence of an exposure problem. By using simulation methods, first, we show the frequency of inefficient allocation in the simultaneous ascending auction. Then, we show that the Vickrey-Clarke-Groves (VCG) mechanism may generate more revenue than the simultaneous ascending auction.

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1 Introduction

In a typical American or Canadian spectrum license auction, hundreds of (heterogenous) licenses are sold simultaneously. Each of these licences gives the spectrum usage right of a geographical area to the winning firm. Some *'local'* firms are interested in winning only specific licenses in order to serve local markets while other *'global'* firms are interested in winning all the licenses in order to serve nationwide.[1] The global firms enjoy synergies if they win all the licenses which gives them an incentive to bid over their stand-alone valuations for some licenses. As a result, there is a risk of incurring losses. Therefore, global bidders lower their bids. This is known as the exposure problem.[2]

In a model simplifying the American and the recent Canadian spectrum license auctions, we derive the optimal bidding strategies of local and global firms in a simultaneous ascending auction of two licenses. We mainly focus on the optimal bidding strategies when there is the possibility of an exposure problem, and through simulations, we determine how frequently the exposure problem (i.e., ex-post loss) occurs. In addition, we decompose the frequency into two cases; the case in which the exposure problem occurs when the global bidder wins only one license, and the case in which the exposure problem occurs when the global bidder wins all licenses.

Exposure problem indicates that the allocation may not be efficient. We compare the efficiency and revenue properties of the simultaneous ascending auction with those of the Vickrey-Clarke-Groves (VCG) mechanism when bidders are allowed to bid on packages. VCG is an efficient auction that gives the highest revenue among all incentive compatible, individually rational, efficient auctions. In the literature (e.g., Ausubel and Milgrom (2006)), there are examples which show that VCG mechanism may give extreme low revenue in complete information settings. We show that VCG mechanism may give higher revenue to

[1] In the recent Canadian Advanced Wireless Spectrum auction, firms such as Globalive and Rogers were interested in all licenses whereas firms such as Bragg Communication and Manitoba Telecom Services (MTS) were interested in East Coast and Manitoba licenses, respectively.

[2] We will interchangeably use exposure problem as follows. We say that an exposure problem occurred whenever the global bidder incurs a loss ex-post.

the seller for many parameter spaces and various distributions in incomplete information setting. We also show the frequency of inefficient allocation when simultaneous ascending auction is used.

The multi-unit auction literature generally assumes that global bidders have either equal valuations (Englmaier et. al (2009), Albano et. al. (2001), Kagel and Levin (2005), Katok and Roth (2004), Rosenthal and Wang (1996), and Krishna and Rosenthal (1996)) or very large synergies (Albano et al. (2006)). The spectrum licenses for different geographic areas are not homogenous objects; hence, the equal valuation assumption does not fit the Canadian or the American spectrum license auction. Moreover, in a heterogeneous license environment, bidders may not drop out of both auctions simultaneously. This enables us to analyze bidding behavior in the remaining auction, and hence, the exposure problem in detail.

We allow for moderate synergies, and our focus is on the exposure problem and the comparison of revenue and efficiency properties of the simultaneous ascending auction with those of the VCG auction, unlike Albano et.al (2006).[3] In our paper, the global bidder will lower his bid because of the exposure problem; however, their optimal strategy still requires him to bid over his stand alone valuation for at least one license. If he wins this license by receiving a potential loss, then he may need to stay in the other license auction to minimize his loss. Therefore, there are cases in which the exposure problem may arise even when the bidder wins all the licenses.

Two additional papers related to this paper are Goerre and Lien (2010) and Zheng (2008). Goerre and Lien assume that the valuation of winning a given number of licenses is the same regardless of the composition. Hence, they find that the optimal drop out price is the same for both licenses. In our paper, marginal valuations are different. Global bidder's valuation of license A or license B is different.[4] Hence, our paper shows that the optimal drop out price

[3]They assume large synergies so no exposure problem exists in equilibrium. Our results coincide with theirs when we also assume large synergies.

[4]We model situations in which winning the spectrum license for, say, Iowa City is different than winning the license for New York city. This flexibility comes at the expense of allowing only one global bidder. Goerre and Lin (2010)'s assumptions allow them to write a tractable model in which they can allow multiple global bidders.

is not the same, and the exposure problem of this case will lead to different results such as winning all licenses and still incurring a loss. In addition, through simulations, we show the frequency of inefficient allocation for the simultaneous ascending auction. Our common point with Goerre and Lien (2010) is that we also find VCG mechanism may give higher revenue than the simultaneous ascending auction. Zheng (2008) is mainly interested in showing that jump-bidding will alleviate the exposure problem. We do not allow jump-bidding as the Canadian spectrum auction has not allowed this.

Our paper can be contrasted with Kagel and Levin (2005) and Krishna and Rosenthal (1996). Krishna and Rosenthal (1996) study a second price auction (simultaneous and sequential auctions), and do not specifically analyze exposure problem. Kagel and Levin (2005) use a single global bidder that competes with several local bidders. Their ascending bid version of the uniform price auction is different than ours since their two goods are sold in a single auction. In the Canadian spectrum auction, the licenses are sold in separate auctions so we use different auctions in our model. We also use simulations to calculate the probability of exposure problem occurrence, and the comparison of the revenue and efficiency properties of this auction with the VCG auction.

One of the contributions of this paper is to use the simulation methods to show the probability of the exposure problem occurring in the simultaneous ascending auction. We also decomposed this probability into two cases; when the global bidder wins only one license, and when he wins both licenses. We call the first one "exposure problem I" and the latter one "exposure problem II." We use four different distributions to draw the valuations; uniform, beta distribution with alpha and beta equal to two, beta distribution with alpha equal to one and beta equal to four, and beta distribution with alpha equal to four and beta equal to one.

We compare our the revenue and the efficiency properties of this auction with those of the VCG auction. When we use uniform distribution and one local bidder on each licences, we show that the revenue is 8 per cent higher but the allocation is inefficient 4 per cent of the time.

Almost all proofs are included in the Appendix.

2 The Model

There are 2 licenses, license A and B for sale.[5] There are one global bidder who demands both licenses and $m_j = m - 1$ local bidders who demand only license $j = A, B$. Specifically, m_j will denote the number of active local bidders on the auction.[6] Both local bidders and the global bidder have a private stand alone valuation for a single license, v_{ij}, where i and j represent the bidder and the license, respectively. The valuations v_{ij} are drawn from the continuous distribution function $F(v_{ij})$ with support on $[0, 1]$ and probability density function $f(v_{ij})$ which is positive everywhere with the only exception that $f(0) \geq 0$ is allowed. The bidders' type, global or local, is publicly known.

We consider a setting where the licenses are auctioned off simultaneously through an ascending multi-unit auction. Each license is auctioned off at a different auction (like Krishna and Rosenthal (1996) but unlike Kagel and Levin (2005)) but at the same time. Prices start from zero for both licenses and increase simultaneously and continuously at the same rate. Bidders choose when to drop out. When only one bidder is left on a given license, the clock stops for that license, and the sole remaining bidder wins the license at the price at which the last bidder dropped out. If there are more than one bidder remaining on the other license, its price will continue to increase. If n bidders drop out at the same price and nobody is left in the auction, then each one of them will win the license with probability $\frac{1}{n}$.

The drop-out decision is irreversible. Once a bidder drops out of bidding for a given license, he cannot bid for this license later.[7] The number of active bidders and the drop-out prices are publicly known. We also assume that there is no budget constraints for the bidders.

[5] We use two licenses like Albano et. al. (2001 and 2006), Brusco and Lopomo (2002), Chow and Yavas (2009), and Menucicci (2003).

[6] Allowing different number of local bidders per license will not change our qualitative results.

[7] In the real-world auctions, there is activity rule. If the bidders do not have enough highest standing bids, then the number of licenses they may bid on is decreased (in the next rounds). Hence, when there are two licenses, this translates into an irreversible drop-out.

We assume that there is a homogeneous positive synergy for the global bidder. Specifically, letting bidder 1 be the global bidder, the global bidder's total valuation, given that it wins two licenses is, $V_1 = v_{1A} + v_{1B} + \alpha$, where the synergy term α is assumed to be strictly positive and public knowledge.[8] His stand-alone valuation of license A or B is given by v_{1A} or v_{1B}. Bidder iA, $i = 2, 3, ...m$ is only interested in license A, and her private valuation is v_{iA}. Bidder iB is only interested in license B, and her private valuation is v_{iB}. [9] A local bidder who is interested in license j participates only in license j auction.

We derive a symmetric perfect Bayesian equilibrium through a series of lemmas that follow. First, we describe the equilibrium strategy of the local bidder.

Lemma 1 *Each local bidder has a weakly dominant strategy to stay in the auction until the price reaches his stand alone valuation.*

This is a well-known result so we skip the proof.

Now, suppose all the local bidder drops out of license B auction, and hence, the global bidder wins license B at the price p_B, which in equilibrium is equal to $p_B = max\{v_{2B}, ..., v_{(m)B}\}$ by lemma 1. Then, as the price for license A increases, the global bidder will compare the payoff from dropping out from license A auction at the clock price p (which is $v_{1B} - p_B$) and the payoff from winning license A at price p (which is $v_{1A} + v_{1B} + \alpha - p_B - p$). The updated optimal drop out price, p_A, is found by equating these two equations: $v_{1A}+v_{1B}+\alpha-p_B-p_A = v_{1B} - p_B \Rightarrow p_A = v_{1A} + \alpha$. If global bidder wins license A first, the updated optimal drop out price, p_B, can be found symmetrically. We state this as lemma 2.

Lemma 2 *If the global bidder wins license B (or A) first, then it will stay in license A (or B) auction until the price reaches $v_{1A} + \alpha$ (or $v_{1B} + \alpha$)*

The global bidder will not drop out before the price reaches his minimum of stand-alone valuations. Otherwise, they will lose the chance of winning both licenses and enjoying the

[8]Public knowledge assumption can be removed, and all results are still valid. We assume public knowledge not to complicate the notation.

[9]We do not assume that $v_{iA} > v_{iB}$ since local firms are different; hence, their efficiency may differ.

synergy. In addition, if the global bidder's average valuation, $\frac{V_1}{2} = \frac{v_{1A}+v_{1B}+\alpha}{2}$, exceeds 1, bidding up to his average valuation will shut out the local bidders since local bidders' stand alone valuation can be at most 1.If α is large enough, this condition will always be satisfied. In such a case, the global bidder always wins both licenses in equilibrium. We summarize these results as lemma 3.

Lemma 3 *a) The global bidder stays in both license auctions at least until the price reaches the minimum of his stand-alone valuations.*

b) If his average valuation is greater than 1, the global bidder's equilibrium strategy is to stay in until the price reaches his average valuation.

To calculate the optimal drop out price for the global bidder, consider first the case in which $v_{1A} > v_{1B}$.[10] The global bidder must compare the payoffs for two cases at each price p as the clock is running: **Case 1** is the payoff from dropping out from license B auction at price p and optimally continuing on license A auction. **Case 2** is the payoff from winning license B at price p and optimally continuing on license A auction.[11] At the beginning of the auction, that is $p = 0$, the second case payoff is higher so the global bidder will start by staying in the auction. We show that the difference between these two cases are monotonic in p ; therefore, there is a unique price that makes the global bidder indifferent between these two cases (assuming that the local bidders are still active). This is the optimal drop out price, p_1^*. We show that this price can be calculated at the beginning of the auction. Note that according to Lemma 3, $p_1^* \geq v_{1B}$, and the optimal updated drop out price for license A, after winning license B at price p, is $v_{1A} + \alpha$.

We denote the expected profit of the global bidder for Case 1 by $E\Pi_1^1$ and his expected profit for Case 2 by $E\Pi_1^2$, respectively.

Let $p_A = \max\{v_{2A}, ..., v_{(m+1)A}\}$ be the price the global bidder will pay for the license A, if he wins license A. Payoffs are as follows:

[10] The other case can be calculated symmetrically.
[11] The global bidder will drop out of license B first since $v_{1A} > v_{1B}$, if he has not won license A yet.

$$E\Pi_1^1 = Max\{0, \int_p^{v_{1A}} (v_{1A} - p_A) g(p_A|p) dp_A\} \tag{1}$$

$$E\Pi_1^2 = \int_p^{Min\{v_{1A}+\alpha,1\}} (V_1 - p - p_A) g(p_A|p) dp_A + \int_{Min\{v_{1A}+\alpha,1\}}^1 (v_{1B} - p) g(p_A|p) dp_A \tag{2}$$

The explanation of equation 1 is as follows. After the global bidder drops out of the auction for license B at p, it becomes just like a local bidder, and hence, will continue to stay in the auction for license A until v_{1A}. If he wins, he will pay p_A since the local bidder with highest valuation of license A will drop out last (by Lemma 1). In order to calculate his expected profit, global bidder will be using $G(p_A|p)$ (highest order statistic) which is the distribution function of the local bidders' highest valuation p_A for license A given p. When there are $m-1$ local bidders in license A, the distribution function $G(p_A|p)$ and its density function $g(p_A|p)$ are:

$$G(p_A|p) = (F(p_A|p))^{m-1} = (\frac{\int_p^{p_A} f(v)dv}{\int_p^1 f(v)dv})^{m-1} \tag{3}$$

$$g(p_A|p) = (m-1)(\frac{\int_p^{p_A} f(v)dv}{\int_p^1 f(v)dv})^{m-2}(\frac{f(p_A)}{\int_p^1 f(v)dv}). \tag{4}$$

The first term of $E\Pi_1^2$ is Firm 1's expected profit of winning both licenses; assuming that he wins license B at the price p. If the highest local bidder's valuation p_A is less than the global bidder's (updated) willingness to pay, $v_{1A} + \alpha$, then the global bidder wins license A and pays p_A. Since $p_A < 1$, we use the minimum function in the upper limit of the first integral. The second term of $E\Pi_1^2$ is Firm 1's expected profit of winning only license B which can happen only if $p_A > v_{1A} + \alpha$. Note that the second term is non-positive by Lemma 3 (which is the exposure problem arising from winning only one license).

In Lemma 4 below, we characterize the global bidder's equilibrium bids. It can be found from $E\Pi_1^1 = E\Pi_1^2$. Note that these payoffs are changing as local bidders bidding for A are dropping out; that is, $m-1$ is changing. Therefore, the lemma below gives the global bidder's (updated) equilibrium drop out price as the local bidders change. We show, in the proof, that this updated price increases as local bidders drop out.

Lemma 4 *: Suppose that the average valuation of the global bidder is less than 1 and no local bidders have dropped out yet.*

If $v_{1A} > v_{1B}$, the global bidder[12] *will drop out of license B auction at the unique optimal drop-out price $p_1^* \in [0,1]$ that satisfies $E\Pi_1^1 = E\Pi_1^2$. Moreover,*

a) If $v_{1A} + \alpha < 1$, and $\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)dp_A + (v_{1B} - v_{1A}) < 0$, then $p_1^* < v_{1A}$ and the global bidder will stay in license A auction until v_{1A} (after dropping out from license B auction).

b) If $v_{1A} + \alpha < 1$, and $\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)dp_A + (v_{1B} - v_{1A}) > 0$, then $p_1^* > v_{1A}$ and the global bidder will also drop out of license A auction at p_1^*.

c) If $v_{1A} + \alpha > 1$, and $\int_{v_{1A}}^{1} G(p_A|p)dp_A + (v_{1B} + \alpha - 1) < 0$, then $p_1^* < v_{1A}$ and the global bidder will stay in license A auction until v_{1A} (after dropping out from license B auction).

d) If $v_{1A} + \alpha > 1$, and $\int_{v_{1A}}^{1} G(p_A|p)dp_A + (v_{1B} + \alpha - 1) > 0$, then $p_1^* > v_{1A}$ and the global bidder will also drop out of license A auction at p_1^*.

Proof. See the Appendix.

We are ready to summarize our Perfect Bayesian equilibrium.

Proposition 5 *(Perfect Bayesian Equilibrium) a) Local bidder $j = \{A, B\}$ $l = \{2, 3, ..m-1\}$ of each license will stay in the auction j until price reaches their valuation v_{kj}.*

b) A global bidder active only on license j will bid $v_{1j} + \alpha$, if he won license $k \neq j$. He will bid v_{1j} when he did not win license k.

c) When $v_{1A} > v_{1B}$ and the average valuation is less than one, the global bidder who is active on both licenses and facing $m - 1$ active local bidders on license A will drop out from license B at the price that equates equations 1 and 2.

d) When $v_{1A} < v_{1B}$ and the average valuation is less than one, the global bidder who is active on both licenses and facing $m - 1$ active local bidders on license B will drop out from license A at the price that equates equations 1 and 2 (symmetrically replaced v_{1A} with v_{1B}.

[12] If $v_{1A} < v_{1B}$, then the proposition has to be written symmetrically

e) If the average valuation is greater than one, the global bidder will stay in both auctions until price reaches his average valuation.

f) Out-of-equilibrium-path beliefs: When a bidder drops out, the other bidders will see this as an equilibrium behavior. Hence, there is no need to specify the out of equilibrium path beliefs.

At the beginning of the game, each bidder calculates its optimal drop-out price. For local bidders, the optimal drop out prices are their valuations. In equilibrium, it is optimal for the global bidder to stay in the auctions for both licenses up to his optimal drop-out price calculated in Lemma 4. When his average valuation exceeds 1, he will stay until this average valuation and win both licenses. When the price reaches the minimum of these optimal drop-out prices, that bidder drops out of license auction. If, for example, the highest local bidder for license B dropped out before the global bidder, the global bidder would continue to stay in the auction for license A until the price reaches $v_{1A}+\alpha$. At this price, he finds that the payoff from winning only license B is more than the payoff from winning both licenses even though it will enjoy synergy; hence, it drops out.

If the value of the licenses were identical (e.g. Albano et. al. (2001)), the global firm would drop out of both licenses at the same time. In this case, our Lemma 4 part b will be valid; that is, $p_1^* > v_{1A} = v_{1B}$, hence, the global bidder drops out from both licenses at the same time. This result coincides with Albano et. al. (2001),(2006) and Goerre and Yuanchuan (2010).

The following is a corollary of Lemma 4, and is an example for the optimal drop out price when $F(.)$ is a uniform distribution.

Corollary 6 *: Assume that valuations are drawn from a uniform distribution with a support* $[0, 1]$*. In addition, assume that* $v_{1A} > v_{1B}$ *(other case is symmetrically found by exchanging*

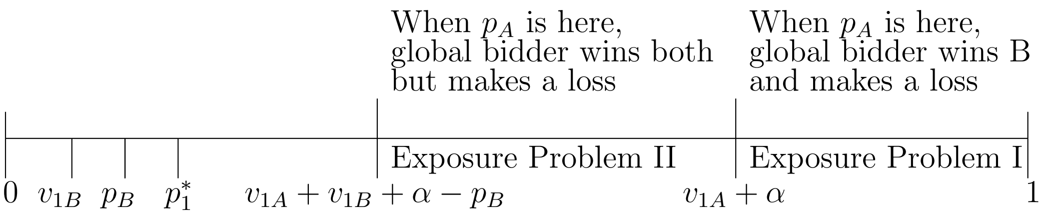


Figure 1: EXPOSURE PROBLEM

v_{1A} with v_{1B}), and there is one local bidder in each license.

$$p_1^* = \begin{cases} \frac{1}{2}\{v_{1B} + \alpha + 1 - (v_{1B}^2 + 1 - 2v_{1B} - \alpha^2 + 2v_{1B}\alpha + 2\alpha - 4v_{1A}\alpha)^{\frac{1}{2}}\}, \\ \qquad\qquad \textit{if } 0 < v_{1A} < 1 - \alpha \textit{ and } 2(1 - v_{1A})(v_{1A} - v_{1B}) > \alpha^2; \\ \frac{1}{3}\{v_{1A} + v_{1B} + \alpha + 1 - ((v_{1A} + v_{1B} + \alpha + 1)^2 - 3(v_{1A} + \alpha)^2 - 6v_{1B})^{\frac{1}{2}}\}, \\ \qquad\qquad \textit{if } 0 < v_{1A} < 1 - \alpha \textit{ and } 2(1 - v_{1A})(v_{1A} - v_{1B}) \leq \alpha^2; \\ \frac{1}{2}\{v_{1B} + \alpha + 1 - \{(v_{1B} + \alpha + 1)^2 - 4(v_{1A} + v_{1B} + \alpha) + 2 + 2v_{1A}^2\}^{\frac{1}{2}}\}, \\ \qquad\qquad \textit{if } 1 - \alpha \leq v_{1A} < 1 \textit{ and } 1 + v_{1A} > 2(v_{1B} + \alpha); \\ \frac{2(v_{1A} + v_{1B} + \alpha) - 1}{3}, \\ \qquad\qquad \textit{if } 1 - \alpha \leq v_{1A} < 1 \textit{ and } 1 + v_{1A} \leq 2(v_{1B} + \alpha). \end{cases} \tag{5}$$

The optimal drop-out price is a function that takes a unique value defined in the corollary above. For example, case $0 < v_{1A} < 1 - \alpha$ and $2(1 - v_{1A})(v_{1A} - v_{1B}) > \alpha^2$ implies that $p_1^* < v_{1A}$.

2.1 Exposure Problem

We now can discuss the exposure problem with the help of Figure 1. In the first type of exposure problem, the global bidder may win license B at a price above his stand alone valuation (i.e., $v_{1B} < p_B < p_1^*$) and lose the other license (i.e., $p_A > v_{1A} + \alpha$. This is the type of exposure problem Chakraborty (2004) focuses on. In the second type of exposure problem, the global bidder wins both licenses but incurs a loss. This is the case when he wins license B at $v_{1B} < p_B < p_1^*$ and wins license A at $v_{1A} + \alpha > p_A > v_{1A} + \alpha + v_{1B} - p_B$. Note that if he wins license A at the price $v_{1A} + \alpha + v_{1B} - p_B$, his payoff is zero. The global bidder stays in the auction for license A in order to minimize its loss from winning only license B even if the price passes $v_{1A} + \alpha + v_{1B} - p_B$. If objects were homogenous, the second type of exposure problem would not be observed since the bidder would drop out of both license auctions at the same time.

Table 1: **PROBABILITY OF EXPOSURE PROBLEM**

α	**Percentage of Exposure Problem 1** One Local Bidder	**Percentage of Exposure Problem 2** One Local Bidder	**Total Percentage** One Local Bidder	**Percentage of Inefficiency** One Local Bidder
Beta Distribution with parameters $\alpha = 1$ and $\beta = 4$				
0.2	2.85	0.64	3.48	7.84
0.4	2.29	2.43	4.72	5.45
0.6	0.54	1.59	2.13	2.15
0.8	0.04	0.62	0.66	0.66
Uniform Distribution				
0.2	1.17	0.40	1.57	3.29
0.4	1.13	1.07	2.20	4.22
0.6	0.69	1.51	2.20	4.01
0.8	0.36	1.25	1.61	2.62
Beta Distribution with parameters $\alpha = 2$ and $\beta = 2$				
0.2	0.86	0.29	1.16	5.10
0.4	0.80	1.31	2.12	4.75
0.6	0.82	2.31	3.14	3.88
0.8	0.22	1.65	1.86	1.92
Beta Distribution with parameters $\alpha = 4$ and $\beta = 1$.				
0.2	0.34	0.77	1.11	2.72
0.4	0.20	0.93	1.13	1.95
0.6	0.00	0.50	0.50	0.64
0.8	0.00	0.26	0.26	0.28

2.2 Simulations

In the following, through simulations, we determine the probability of the occurrence of exposure problems under various environments. As we noted, we say that exposure problem occur when the global bidder wins one or both licenses with a loss ex-post. We have used MATLAB to write our simulation code. This code first draws the valuations for both the global and the local bidders from a given distribution function. One set of valuations correspond to one auction. We, then, calculate the optimal drop-out price of the global bidder; local bidders' drop-out prices are their valuations. If some local bidders valuations are lower than the global bidder, the global bidder's drop-out price is updated as these local bidders drop out from the auction. If the global bidder does not win the first license, then no exposure problem occurs. If he wins the first license above its valuation for that license, then we calculate his updated price for the remaining license (unless the global bidder drops out from both licenses at the same time). We next determine whether the global bidder will win the remaining license at a positive profit (no exposure problem), at a loss (exposure problem II) or lose the remaining license (exposure problem I). Dividing the number of each of these events to the number of draws yields the probability of each event.[13]

We use four different distribution functions to draw valuations: uniform, beta distribution with $\alpha = \beta = 2$, beta distribution with $\alpha = 1$ and $\beta = 4$, and beta distribution with $\alpha = 4$ and $\beta = 1$. The second distribution is a mean preserving spread of the uniform distribution. The third one is first order stochastically dominated by the uniform distribution, while the fourth one first order stochastically dominates the uniform distribution. We use one local bidder on each license.[14] We run simulations for four different synergy levels: 0.2, 0.4, 0.6, 0.8.[15] The 0.2 represents for small synergy, 0.4 and 0.6 represents middle synergy, and 0.8 represents a large synergy level. We report the results in Table 1.

We find that the global bidder may face exposure problem with probability 4.33 per cent, if the valuations are drawn from beta distribution with parameters $\alpha = 1$ and $\beta = 4$, and the synergy level is equal to 0.4.

Table 1 also shows that the exposure problem occurs with the smallest probability among all these different distributions when the synergy level is 0.8. This is expected since the global bidder can bid very high for the remaining license (in most cases more than 1) but not face exposure problem much.

[13] In our simulation, to simplify calculations, we only consider cases in which the global bidder values the license B more than license A. After 15000 draws, we select only the valuations where the global bidder's valuation for license A is greater than license B. Hence, we are left with approximately 7500 draws. We used UNIX system of the University of Manitoba, and our laptops for the simulations. In the UNIX machine, it took more than four days to run each code.

[14] Given the complexity of the code we use, we feel that using one local bidder in our simulations are enough to draw reasonable conclusions though this can be extended to two local bidders in each license. Using three local bidders or more would extremely complicate the code since one has to keep track of updated prices every time a local bidder drops off.

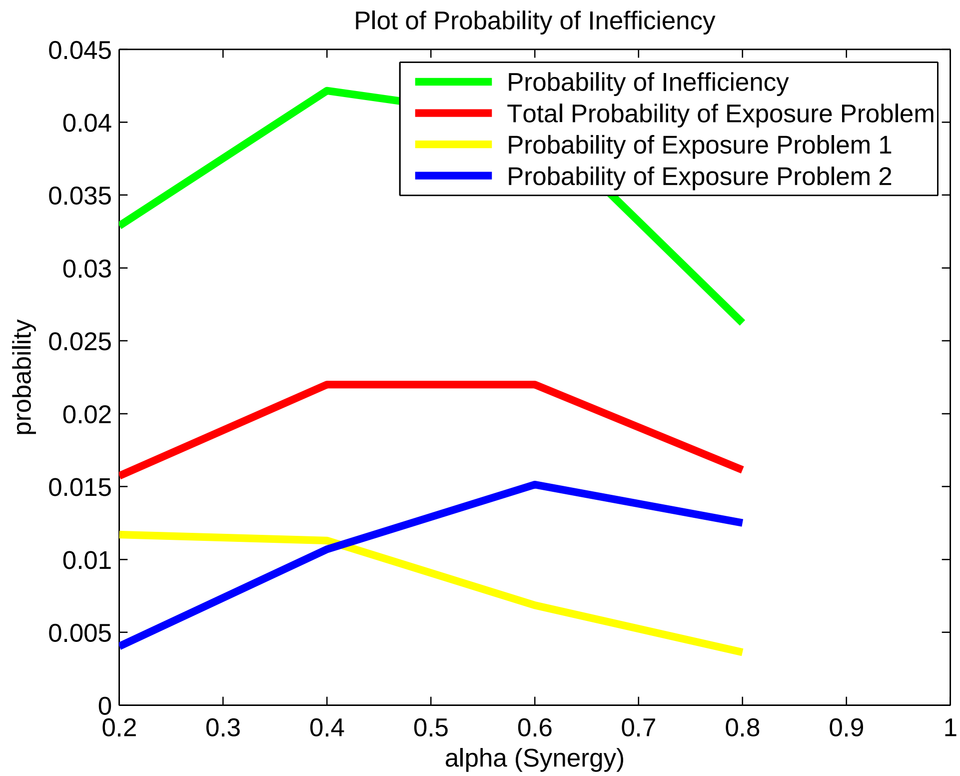
[15] We also write codes for a finer synergy level of 0.1,0.2,...0.9 but we do not report them since no new insight is learned from this.

In our simulations, Exposure problem I occurred most often when the synergy level is 0.4. In this case, the global bidder overbids to enjoy the middle level synergy, so he is very likely to make a (substantial) loss when he wins the first license. His optimal drop-out price for the remaining license is generally below 1; hence, the risk of losing the second license is high, and this is the reason for observing exposure problem I for the synergy level of 0.4. Exposure problem II generally occurs the most when the synergy level is 0.6. After winning the first license, the global bidder will stay in the remaining license auction for a higher drop out price; hence, rather than exposure problem I, exposure problem II is likely to occur.

Of the uniform and beta distribution with parameters $\alpha = \beta = 2$, we see that exposure problem is more likely to occur in the latter one. This is expected since the valuations are more likely to be on the extreme sides. Hence, local bidders are more likely to have high valuations for the remaining license, and force the global bidder to drop out or make a loss.

Of the two beta distribution, the one with $\alpha = 4$ and the one with $\alpha = 1$, we observe that the exposure problem occurs much less frequently with the first one. The reason is that with the former one, the valuations of the global bidder is likely to be higher; hence, even with a small synergy level, their optimal updated drop out price after winning one license is more than one in most cases. Therefore, it is less likely to have exposure problem. On the other hand, when the global bidder's updated optimal price is less than one, it is also more likely that the local bidder's valuation will be high so in such cases, one may see exposure problem. As the synergy increases, this case is less likely to happen.

In Figure 2.2, we show that, for uniform distribution, the allocation is inefficient 8 per cent of the time when $\alpha = 0.2$. As α increases, global bidder wins both licenses more often without facing exposure problem that much, and this is the efficient outcome. Hence, we observe inefficient allocation only 3 per cent of the time for $\alpha = 0.8$. Most of the inefficiency is due to exposure problem.



3 Comparison with Vickrey Clarke Groves Auction

In this section, we will compare the revenue of our auction with the revenue of Vickrey Clarke Groves (VCG) auction. VCG auction maximizes the expected payment of each agent among all mechanisms for allocating multiple objects that are efficient, incentive compatible, and individually rational.[16] In this auction, the seller will let the bidders bid on license A, license B and the whole package license A and B.

We, will first calculate the payment of each winner for all cases; that is, calculate the revenue of the seller. The payment of a winner (say player i) in this auction is the difference between the social welfare of the others if the bidder did not participate in the auction (denote this as $W^{-i}(x^{-i})$ where x^{-i} denote the bid of all players other than player i), and the welfare of the others when he participated in the auction, and bid truthfully (denote

[16]See proposition 16.2 of Krishna (2010)

this as $W^{-i}(x)$, where x denote the bid of all players.) since truthful bidding is the weakly dominated strategy.

The table below shows the valuations of the bidders. To give an example of how payments are calculated, let us assume that $v_{2A} + v_{1B} > v_{1A} + v_{1B} + \alpha > v_{2A} + v_{3B}$. VCG auction will allocate license A to local bidder A, and license B to global bidder. Payment of local bidder A is $(W^{-2}(x^{-2})) - W^{-2}(x) = (v_{1A} + v_{1B} + \alpha) - v_{1B} = v_{1A} + \alpha$. If local bidder A does not participate in the auction, then global bidder will win the package; hence, the welfare of the others $W^{-2}(x^{-2}) = v_{1A} + v_{1B} + \alpha$. When it participates in the auction, global bidder gets only license B, and local bidder B gets nothing; hence, the welfare of others in this case is $W^{-2}(x) = v_{1B}$.

A	B	AB
v_{1A}	v_{1B}	v_{1A}+v_{1B}+α
v_{2A}	0	v_{2A}
0	v_{3B}	v_{3B}

Payment of the global bidder is: $(W^{-1}(x^{-1})) - W^{-1}(x) = (v_{2A} + v_{3B}) - v_{2A} = v_{3B}$. If the global bidder does not participate in the auction, local bidder A and B wins each license;hence, the welfare of others is the term inside the parenthesis. When the global bidder participates in the auction, local bidder A wins license A but the welfare of local bidder B is zero; hence the welfare of others in this case is just v_{2A}.

The total revenue of the seller in this case will be $v_{1A} + \alpha + v_{3B}$.

We summarize the revenue of the seller for all cases in the following proposition. [17]

Proposition 7 *Suppose that there is one global bidder and one local bidder bidding for each license. In the VCG auction, the seller's revenue will be as follows depending on the valuations of the bidders.*

CASE I: Suppose that the valuations are such that $v_{1A} + v_{1B} + \alpha < v_{2A} + v_{3B}$.

[17] This result can easily be extended to one global and many local bidders case. Since, we use one global bidder and one local bidder in each license in the simulations, to save the notation, we give the result for a special case.

A) (Local bidders win each license) And suppose that $v_{1A} < v_{2A}$ and $v_{1B} < v_{3B}$. There are four sub cases to consider.

i) $v_{3B} > v_{1B} + \alpha$ and $v_{2A} > v_{1A} + \alpha$, then the revenue is $v_{1A} + v_{1B}$.

ii) $v_{3B} < v_{1B} + \alpha$ and $v_{2A} < v_{1A} + \alpha$, then the revenue is $2(v_{1A} + v_{1B} + \alpha) - v_{3B} - v_{2A}$.

iii) $v_{3B} > v_{1B} + \alpha$ and $v_{2A} < v_{1A} + \alpha$, then the revenue is $2v_{1A} + v_{1B} + \alpha - v_{2A}$.

iv) $v_{3B} < v_{1B} + \alpha$ and $v_{2A} > v_{1A} + \alpha$, then the revenue is $v_{1A} + 2v_{1B} + \alpha - v_{3B}$.

B) (Local bidder wins A, and global bidder wins B) And suppose that $v_{2A} > v_{1A}$ and $v_{3B} < v_{1B}$. Then, the revenue is $v_{1A} + \alpha + v_{3B}$.

C) (Local bidder wins B, and global bidder wins A) And suppose that $v_{2A} < v_{1A}$ and $v_{3B} > v_{1A}$. Then, the revenue is $v_{1B} + \alpha + v_{2A}$

CASE II: Suppose that the valuations are such that $v_{1A} + v_{1B} + \alpha > v_{2A} + v_{3B}$.

A) (Global bidder wins both licenses) And suppose that $v_{1A} < v_{2A}$ and $v_{1B} < v_{3B}$. Then, the revenue is $v_{2A} + v_{3B}$.

B) (Global bidder wins both licenses) And suppose that $v_{1A} > v_{2A}$ and $v_{1B} > v_{3B}$. Then, the revenue is $v_{2A} + v_{3B}$.

C) (Global bidder wins license A, local bidder B wins license B) And suppose that $v_{2A} < v_{1A}$ and $v_{3B} > v_{1B} + \alpha$. Then, the revenue is $v_{2A} + v_{1B} + \alpha$.

D) (Global bidder wins license B, local bidder A wins license A) And suppose that $v_{2A} > v_{1A} + \alpha$ and $v_{3B} < v_{1B}$. Then, the revenue is $v_{3B} + v_{1A} + \alpha$.

We compare the revenue of the simultaneous ascending auction with those of the VCG auction through simulation methods. Our results are summarized in Figure 3. We run the simulations for $\alpha = 0.2, 0.4, 0.6, 0.8$. When $\alpha = 0.2$, the revenue is 10 per cent higher in the simultaneous ascending auction.[18]. When $\alpha = 0.8$, global bidder must be winning the auction most of the time, and they pay a total of $v_{2A} + v_{3B}$. The expected payment would be 1 in the uniform distribution, which we observe in Figure 3.

[18] In the 2008 Canadian spectrum auction, the revenue was 4 billion dollars so 10 per cent is a significant number in our view.

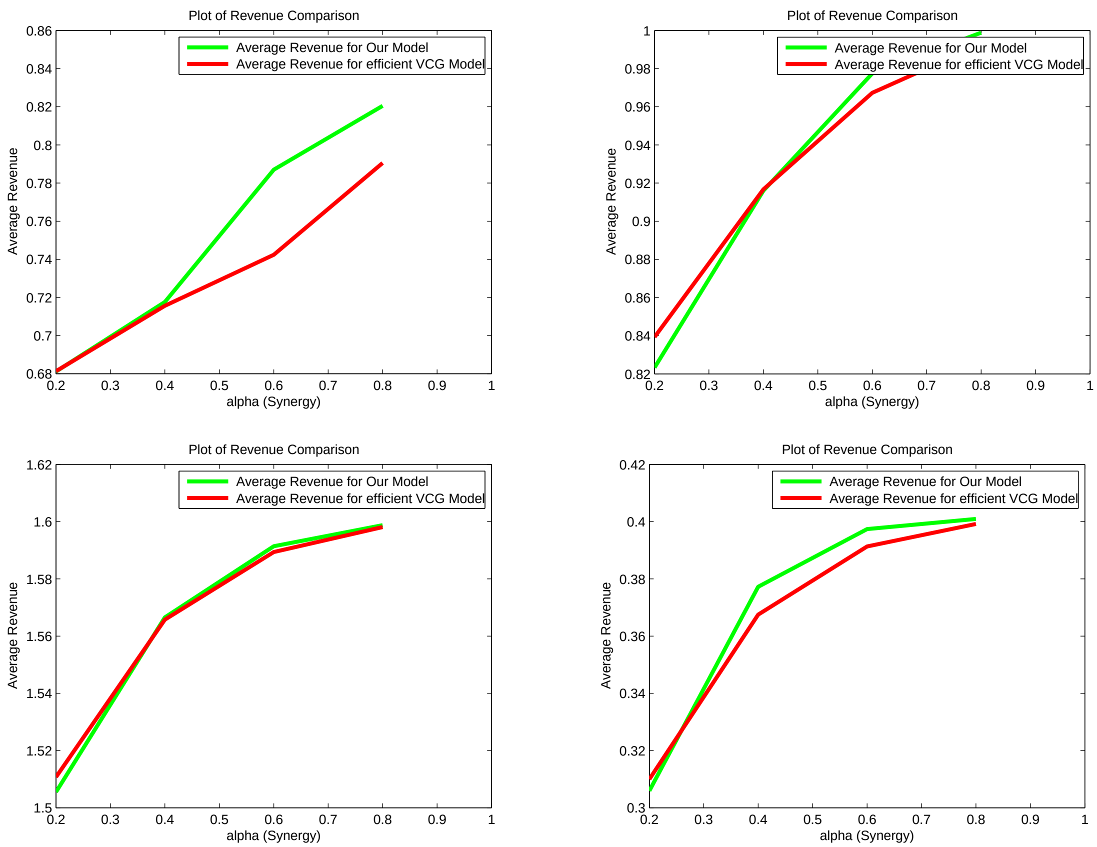


Figure 2:
Uniform Distribution, Top Left;
Beta Distribution with $\alpha = 2$, $\beta = 2$, Top Right;
Beta Distribution with $\alpha = 4$, $\beta = 1$, Bottom Left;
Beta Distribution with $\alpha = 1$, $\beta = 4$, Bottom Right

4 More Than One Global Bidder

In this section, we will analyze the case where there is more than one global bidder. The difficulty in this case arises from the fact that a global bidder's optimal drop out price depends on the other global bidders' optimal drop out price. Then, to calculate the optimal drop out prices, one should solve more than one non-linear equations simultaneously. To make things worse, the global bidder should know the distribution of the other global bidder's drop out price when making its own drop out price calculation. This makes deriving an analytical result impossible.

To make our case, assume that there are two global bidders, Firm 1 and Firm 2, and one local bidder on each of license A and license B. Everything else is the same as the previous section. Firm 1 will make a similar calculation as we discussed in one global bidder case –except that the price would depend on the drop out price of the second global bidder which we denote as p_2^*– Specifically, he should make the following calculation while deciding to stay in the license B auction or not. This calculation is done based on a history in which none of the other bidders have dropped out yet.

$$E\Pi_1^1 = Max\{0, E\Big[(v_{1A} - [\int_p^{v_{1A}} Max\{v_{2A}, v_{3A}\}P(p_2^* < v_{4B})dG(Max\{v_{2A}, v_{3A}\}|p) + \tag{6}$$

$$\int_p^{v_{1A}} Max\{(v_{2A} + \alpha), v_{3A}\}P(p_2^* > v_{4B})dG(Max\{(v_{2A} + \alpha), v_{3A}\}|p)]|p_2^*\Big]\} \tag{7}$$

$$E\Pi_1^2 = \int_p^{Min\{v_{1A}+\alpha,1\}} (V_1 - p - p_A)dG(Max\{v_{2A}, v_{3A}\}|p) + \int_{Min\{v_{1A}+\alpha,1\}}^1 (v_{1B} - p)dG(Max\{v_{2A}, v_{3A}\}|p) \tag{8}$$

The explanation of equation 6 is as follows. If this global bidder drops out from license B before the other global and local bidders, then it will continue on license A auction as a local bidder. Hence, it can only derive a benefit of v_{1A} if it wins the license. The price it will pay for license A depends on the result of whether the local or the other global bidder wins license B. If the local bidder B wins license B (this happens with probability $P(p_2^* < v_{4B})$

and we integrate this in the range p to v_{1A}), then the price of license A will be the maximum of local bidder A's and the other global bidder's valuations.

Equation 7 analyze the case in which the other global bidder wins license B. This happens with probability $P(p_2^* > v_{4B})$ (in the range p to v_{1A}), then it can enjoy synergy and will bid until $v_{2A} + \alpha$ for license A. Then, the price global bidder 1 will pay is $Max\{(v_{2A} + \alpha), v_{3A}\}$.

Equation 8 analyzes the case in which Firm 1 wins license B, and then continue optimally on license A. This is the same as "one global bidder" case. The only difference is that the other global bidder is now a local bidder; hence, there is one more local bidder in the license A auction compared to the "one global case."

Since p_2^* is not known, the global bidder must use its distribution!

By subtracting the second equation from the first one and equating it to zero, we will define an implicit function of p_1^* and p_2^* as

$$F_1(p_1^*, p_2^*) = 0$$

Now, we can similarly (symmetrically) define two equations for global bidder 2, and calculate $F_2(p_1^*, p_2^*) = 0$. To find the optimal drop out price, one has to solve these two non-linear equations simultaneously for p_1^* and p_2^*. However, the question of how to find distribution of p_i^* which is not known makes the solution impossible. We are not clear what simulation techniques may solve this problem so we leave this as an open problem.

5 Conclusion and Discussion

We showed the optimal bidding strategies of global bidders when there are moderate synergies and the licenses are heterogeneous. We also analyzed exposure problem.

We were able to show exposure problem can occur even when the global bidder wins all licenses. Literature has not studies heterogeneous license case with moderate synergies since it is technically challenging when one uses more than one global bidder. With this paper, we fill this gap. One of our contributions is to write a complicated code to calculate the probability of exposure problem. Our simulation results show that the exposure problem may be minor for some distributions but may be up to 4.3 per cent for some others.

Extending the results to n global bidders would be very complicated since the optimal strategies of global bidders (optimal drop out prices) should be determined jointly which in turn would depend on how many local and how many global bidders are still in the auction. Moreover, one has to know the distribution of the other global bidder's optimal drop out price while calculating the optimal drop out price! We leave this as an open problem and followed the literature that use only one global bidder (e.g. Kagel and Levin (2005)).

Our other contribution is comparing the revenue and the efficiency properties of the simultaneous ascending auction with those of the VCG auction. We show that when synergy level is small ($\alpha = 0.2$), the simultaneous ascending auction generates approximately 10 per cent more revenue but allocates licenses inefficiently 8 per cent of the time.

6 Appendix

Proof of Lemma 4:

We will prove that there is a unique optimal drop out price by solving $E\Pi_1^1 = E\Pi_1^2$. We have four cases.

Case I: In this case, we will assume $v_{1A}+\alpha < 1$ and $\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)dp_A + (v_{1B} - v_{1A}) < 0$ implies $p_1^* < v_{1A}$ (which in turn implies $E\Pi_1^1 > 0$).

First, we show that there exists a unique solution that makes equations 1 and 2 equal, and this is the optimal drop out price p_1^*. We define a new function, $J(p) = E\Pi_1^1 - E\Pi_1^2$. To prove uniqueness, we will show that this function is monotonically increasing and it is negative when $p = v_{1B}$ (by lemma 2 p cannot be less than v_{1B}) and is positive when $p = v_{1A}$. Hence, there must be a unique root at the interval $v_{1B} < p < v_{1A}$.

$J(p,m) = \int_p^{v_{1A}} (v_{1A} - p_A)g(p_A|p)dp_A - \int_p^{v_{1A}+\alpha} (V_1 - p - p_A)g(p_A|p)dp_A$
$- \int_{v_{1A}+\alpha}^1 (v_{1B} - p)g(p_A|p)dp_A$.
By using $(v_{1B} - p)\int_p^1 g(p_A|p)dp_A = v_{1B} - p$, we can re-write it as
$\int_p^{v_{1A}} (v_{1A} - p_A)g(p_A|p)dp_A - \int_p^{v_{1A}+\alpha} (v_{1A} + \alpha - p_A)g(p_A|p)dp_A - (v_{1B} - p)$

By using integration by parts twice (and using $dv = g(p_A|p)dp_A$), we have

$= (v_{1A} - p_A)G(p_A|p) \mid_p^{v_{1A}} - \int_p^{v_{1A}} G(p_A|p)d(v_{1A} - p_A)$

$- (v_{1A} + \alpha - p_A)G(p_A|p) \mid_p^{v_{1A}+\alpha} + \int_p^{v_{1A}+\alpha} G(p_A|p)d(v_{1A} + \alpha - p_A) - (v_{1B} - p)$

$= \int_p^{v_{1A}} G(p_A|p)dp_A - \int_p^{v_{1A}+\alpha} G(p_A|p)dp_A - (v_{1B} - p)$

We take partial derivative of $J(p, m)$ with respect to p, we have,

$\frac{\partial J(p,m)}{\partial p} = \frac{\partial}{\partial p}[-\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)] + 1 > 0$

It is positive since the term $\frac{\partial}{\partial p}[\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)]$ is negative. As the lower limit of the integral increases, the value of the expression decreases (does not increase) if the term inside is non-negative which is true since it is a cumulative distribution function. We must also show that $\frac{\partial G(p_A|p)}{\partial p} \leq 0$ to prove this. While one can easily see that this is correct (as p increases the cumulative distribution conditional on p decreases), we will give a formal proof by using Leibniz's rule when necessary.

$\Leftrightarrow \frac{\partial G(p_A|p)}{\partial p} = \frac{\partial[(\frac{\int_p^{p_A} f(v)dv}{\int_p^1 f(v)dv})^{m-1}]}{\partial p}$

$= -(m-1)f(p)\frac{(\int_p^{p_A} f(v)dv)^{m-2}}{(\int_p^1 f(v)dv)^{m-1}} + (m-1)f(p)\frac{(\int_p^{p_A} f(v)dv)^{m-1}}{(\int_p^1 f(v)dv)^{m}}$

$= \frac{(m-1)f(p)(\int_p^{p_A} f(v)dv)^{m-2}}{(\int_p^1 f(v)dv)^{m-1}}[-1 + \frac{\int_p^{p_A} f(v)dv}{\int_p^1 f(v)dv}]$

$= \frac{(m-1)f(p)(\int_p^{p_A} f(v)dv)^{m-2}}{(\int_p^1 f(v)dv)^{m-1}}[-1 + F(p_A|p)] < 0$ (≤ 0 only if $p_A = 1$).

Thus, $J(p, m)$ is monotonically increasing function of p, when $v_{1B} \leq p < v_{1A}$.

If $p = v_{1B}$, then $J(v_{1B}) = \int_{v_{1B}}^{v_{1A}} G(p_A|\alpha)dp_A - \int_{v_{1B}}^{v_{1A}+\alpha} G(p_A|v_{1B})dp_A$

$= -\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|v_{1B})dp_A < 0.$

If $p = v_{1A}$, $J(v_{1A}) = 0 - \int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)dp_A - (v_{1B} - v_{1A}) > 0$, then our assumption $\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)dp_A + (v_{1B} - v_{1A}) < 0$ implies that $J(p = v_{1A}) > 0$.

Hence, there is a unique root in the interval $v_{1B} < p < v_{1A}$.

Next, we show that as the number of active firms in license A auction decreases, the optimal drop out price will increase. We will use the implicit function theorem for this:

$\Leftrightarrow \frac{dp_1^*}{dm} = -\frac{\frac{\partial J(p_1^*,m)}{\partial m}}{\frac{\partial J(p_1^*,m)}{\partial p_1^*}} < 0.$

We have already shown that $\frac{\partial J(p_1^*,m)}{\partial p_1^*} > 0$.

Since $J(p, m) = \int_p^{v_{1A}} G(p_A|p)dp_A - \int_p^{v_{1A}+\alpha} G(p_A|p)dp_A - (v_{1B} - p)$.

We take partial derivative of $J(p,m)$ with respect to m, that is,

$\frac{\partial J(p,m)}{\partial m} = \int_{p}^{v_{1A}} \frac{\partial G(p_A|p)}{\partial m} dp_A - \int_{p}^{v_{1A}+\alpha} \frac{\partial G(p_A|p)}{\partial m} dp_A$

$= -\int_{v_{1A}}^{v_{1A}+\alpha} \frac{\partial G(p_A|p)}{\partial m} dp_A = -\int_{v_{1A}}^{v_{1A}+\alpha} \ln(F(p_A|p))G(p_A|p)dp_A > 0.$

Since $\frac{\partial G(p_A|p)}{\partial m} = \ln(F(p_A|p))G(p_A|p) < 0$. Hence, we show that $\frac{\partial G(p_A|p)}{\partial m} > 0$ holds.

By the implicit function theorem, we show that the optimal drop out price increases as the number of local firms, m, decreases.

Since $\frac{\partial J(p,m)}{\partial p} > 0$ and $\frac{\partial J(p,m)}{\partial m} > 0$, we have, $\frac{dp_1^*}{dm} = -\frac{\frac{\partial F(p_1^*,m)}{\partial m}}{\frac{\partial F(p_1^*,m)}{\partial p_1^*}} < 0$

Case II: In this case, we will assume that $v_{1A}+\alpha < 1$ and $\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|v_{1A})dp_A + (v_{1B} - v_{1A}) > 0$ which implies $p_1^* > v_{1A}$. And this condition in turn implies that $E\Pi_1^1 = 0$.

Now let $J(p,m) = E\Pi_1^1 - E\Pi_1^2$

$J(p,m) = 0 - \int_{p}^{v_{1A}+\alpha}(V_1 - p - p_A)g(p_A|p)dp_A - \int_{v_{1A}+\alpha}^{1}(v_{1B} - p)g(p_A|p)dp_A$

$= -\int_{p}^{v_{1A}+\alpha} G(p_A|p)dp_A - (v_{1B} - p).$

When $p \geq v_{1A}$, we take partial derivative of $J(p,m)$ with respect to p, we have,

$\frac{\partial J(p,m)}{\partial p} = -\frac{\partial}{\partial p}[\int_{p}^{v_{1A}+\alpha} G(p_A|p)dp_A] + 1 > 0$, since $\frac{\partial G(p_A|p)}{\partial p} < 0$.

Thus, $J(p,m)$ is monotonically increasing function of p, when $v_{1A} \leq p \leq v_{1A} + \alpha$.

Our assumption $\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|v_{1A})dp_A + (v_{1B} - v_{1A}) < 0$ implies that $J(p = v_{1A}) < 0$. If $p = v_{1A}+\alpha$, then $J(v_{1A}+\alpha) = 0 - 0 - (v_{1B} - v_{1A} + \alpha) > 0$. Thus, there is a unique solution, p_1^*, in the interval $(v_{1A}, v_{1A} + \alpha)$.

Next, we show that when the number of active firms in license A auction decreases, this optimal drop out price will increase.

We take partial derivative of $J(p,m)$ with respect to m, we have,

$\frac{\partial J(p,m)}{\partial m} = -\int_{p}^{v_{1A}+\alpha} \ln(F(p_A|p))G(p_A|p)dp_A > 0.$

Since $\frac{\partial J(p,m)}{\partial p} > 0$ and $\frac{\partial J(p,m)}{\partial m} > 0$, we have,

$\frac{dp_1^*}{dm} = -\frac{\frac{\partial J(p_1^*,m)}{\partial m}}{\frac{\partial J(p_1^*,m)}{\partial p_1^*}} < 0$

Case III: In this case, we will assume that $v_{1A} + \alpha > 1$ and $\int_{v_{1A}}^{1} G(p_A|v_{1A})dp_A + (v_{1B} + \alpha - 1) < 0$ which implies $v_{1A} \leq p_1^*$. And this condition in turn implies that $E\Pi_1^1 > 0$.

Now let $J(p,m) = E\Pi_1^1 - E\Pi_1^2$

$J(p,m) = \int_{p}^{v_{1A}}(v_{1A} - p_A)g(p_A|p)dp_A - \int_{p}^{1}(V_1 - p - p_A)g(p_A|p)dp_A$

$= (v_{1A} - p_A)G(p_A|p) \mid_p^{v_{1A}} - \int_p^{v_{1A}} G(p_A|p)d(v_{1A} - p_A)$

$- (v_{1A} + v_{1B} + \alpha - p - p_A)G(p_A|p) \mid_p^1 + \int_p^1 G(p_A|p)d(v_{1A} + v_{1B} + \alpha - p - p_A)$

$= \int_p^{v_{1A}} G(p_A|p)dp_A - (v_{1A} + v_{1B} + \alpha - p - 1) - \int_p^1 G(p_A|p)dp_A$

$= -(v_{1A} + v_{1B} + \alpha - p - 1) - \int_{v_{1A}}^1 G(p_A|p)dp_A$

We take partial derivative of $J(p, m)$ with respect to p, we have,

$\frac{\partial J(p,m)}{\partial p} = \frac{\partial}{\partial p}[-\int_{v_{1A}}^1 G(p_A|p)] + 1 > 0$

It is positive since the term $\frac{\partial}{\partial p}[\int_{v_{1A}}^1 G(p_A|p)]$ is negative. And we have shown that $\frac{\partial G(p_A|p)}{\partial p} \leq 0$. Thus, $J(p, m)$ is monotonically increasing function of p, when $v_{1B} \leq p < v_{1A}$.

If $p = v_{1B}$, then $J(v_{1B}) = -\int_{v_{1A}}^1 G(p_A|v_{1B})dp_A - (v_{1A} + \alpha - 1) < 0$.

If $p = v_{1A}$, $J(v_{1A}) = 0 - \int_{v_{1A}}^1 G(p_A|v_{1A})dp_A - (v_{1B} + \alpha - 1) > 0$, then our assumption $\int_{v_{1A}}^1 G(p_A|v_{1A})dp_A + (v_{1B} + \alpha - 1) < 0$ implies that $J(p = v_{1A}) > 0$.

Hence, there is a unique root in the interval $v_{1B} < p < v_{1A}$.

Next, we skip to show that as the number of active firms in license A auction decreases, the optimal drop out price will increase, since we have done this in Case I.

Case IV: In this case, we will assume that $v_{1A} + \alpha > 1$ and $\int_{v_{1A}}^1 G(p_A|v_{1A})dp_A + (v_{1B} + \alpha - 1) > 0$ which implies $p_1^* > v_{1A}$. And this condition in turn implies that $E\Pi_1^1 = 0$.

Now let $J(p, m) = E\Pi_1^1 - E\Pi_1^2$

$J(p, m) = 0 - \int_p^1 G(p_A|p)dp_A - (v_{1A} + v_{1B} + \alpha - p - 1)$.

When $p > v_{1A}$, we take partial derivative of $J(p, m)$ with respect to p, we have,

$\frac{\partial J(p,m)}{\partial p} = -\frac{\partial}{\partial p}[\int_p^1 G(p_A|p)dp_A] + 1 > 0$, since $\frac{\partial G(p_A|p)}{\partial p} < 0$.

Thus, $J(p, m)$ is monotonically increasing function of p, when $v_{1A} \leq p \leq 1$.

Our assumption $\int_{v_{1A}}^1 G(p_A|v_{1A})dp_A + (v_{1B} + \alpha - 1) > 0$ implies that $J(p = v_{1A}) < 0$. If $p = 1$, then $J(1) = 0 - 0 - (v_{1B} + v_{1A} + \alpha - 2) > 0$. Since $v_{1B} + v_{1A} + \alpha < 2$. Thus, there is a unique solution, p_1^*, in the interval $(v_{1A}, 1)$.

We also skip to show that when the number of active firms in license A auction decreases, this optimal drop out price will increase which has been proven in Case II.

■

Proof of Proposition 7: Suppose that there is one global bidder and one local bidder bidding for each license. In the VCG auction, the seller's revenue will be as follows depending on the valuations of the bidders.

CASE I: Suppose that the valuations are such that $v_{1A} + v_{1B} + \alpha < v_{2A} + v_{3B}$.

A) (Local bidders win each license) And suppose that $v_{1A} < v_{2A}$ and $v_{1B} < v_{3B}$. There are four sub cases to consider.

i) $v_{3B} > v_{1B} + \alpha$ and $v_{2A} > v_{1A} + \alpha$, then the revenue is $v_{1A} + v_{1B}$.

since $W(x^{-2}) - W^{-2}(x) = v_{1A} + v_{3B} - (0 + v_{3B}) = v_{1A}$ and $W(x^{-3}) - W^{-3}(x) = v_{1B} + v_{2A} - v_{2A} = v_{1B}$

ii) $v_{3B} < v_{1B} + \alpha$ and $v_{2A} < v_{1A} + \alpha$, then the revenue is $2(v_{1A} + v_{1B} + \alpha) - v_{3B} - v_{2A}$.

since $W(x^{-2}) - W^{-2}(x) = v_{1A} + v_{1B} + \alpha - (0 + v_{3B}) = v_{1A} + v_{1B} + \alpha - v_{3B}$ and $W(x^{-3}) - W^{-3}(x) = v_{1A} + v_{1B} + \alpha - (0 + v_{2A}) = v_{1A} + v_{1B} + \alpha - v_{2A}$

iii) $v_{3B} > v_{1B} + \alpha$ and $v_{2A} < v_{1A} + \alpha$, then the revenue is $2v_{1A} + v_{1B} + \alpha - v_{2A}$.

since $W(x^{-2}) - W^{-2}(x) = v_{1A} + v_{3B} - (0 + v_{3B}) = v_{1A}$, and $W(x^{-3}) - W^{-3}(x) = v_{1A} + v_{1B} + \alpha - (0 + v_{2A}) = v_{1A} + v_{1B} + \alpha - v_{2A}$

iv) $v_{3B} < v_{1B} + \alpha$ and $v_{2A} > v_{1A} + \alpha$, then the revenue is $v_{1A} + 2v_{1B} + \alpha - v_{3B}$.

since $W(x^{-2}) - W^{-2}(x) = v_{1A} + v_{1B} + \alpha - (0 + v_{3B}) = v_{1A} + v_{1B} + \alpha - v_{3B}$ and $W(x^{-3}) - W^{-3}(x) = v_{1B} + v_{2A} - v_{2A} = v_{1B}$

B) (Local bidder wins A, and global bidder wins B) And suppose that $v_{2A} > v_{1A}$ and $v_{3B} < v_{1B}$. Then, the revenue is $v_{1A} + \alpha + v_{3B}$.

since $W(x^{-2}) - W^{-2}(x) = v_{1A} + v_{1B} + \alpha - (0 + v_{1B}) = v_{1A} + \alpha$, and $W(x^{-1}) - W^{-1}(x) = v_{3B} + v_{2A} - v_{2A} = v_{3B}$

C) (Local bidder wins B, and global bidder wins A) And suppose that $v_{2A} < v_{1A}$ and $v_{3B} > v_{1B}$. Then, the revenue is $v_{1B} + \alpha + v_{2A}$

since $W(x^{-3}) - W^{-3}(x) = v_{1A} + v_{1B} + \alpha - (0 + v_{1A}) = v_{1B} + \alpha$, and $W(x^{-1}) - W^{-1}(x) = v_{3B} + v_{2A} - v_{3B} = v_{2A}$

CASE II: Suppose that the valuations are such that $v_{1A} + v_{1B} + \alpha > v_{2A} + v_{3B}$.

A) (Global bidder wins both licenses) And suppose that $v_{1A} < v_{2A}$ and $v_{1B} < v_{3B}$. Then, the revenue is $v_{2A} + v_{3B}$.

since $W(x^{-1}) - W^{-1}(x) = v_{2A} + v_{3B} - (0 + 0) = v_{2A} + v_{3B}$.

B) (Global bidder wins both licenses) And suppose that $v_{1A} > v_{2A}$ and $v_{1B} > v_{3B}$. Then, the revenue is $v_{2A} + v_{3B}$.

since $W(x^{-1}) - W^{-1}(x) = v_{2A} + v_{3B} - (0 + 0) = v_{2A} + v_{3B}$.

C) (Global bidder wins license A, local bidder B wins license B) And suppose that $v_{2A} < v_{1A}$ and $v_{3B} > v_{1B} + \alpha$. Then, the revenue is $v_{2A} + v_{1B} + \alpha$.

since $W(x^{-1}) - W^{-1}(x) = v_{2A} + v_{3B} - (v_{3B} + 0) = v_{2A}$ and $W(x^{-3}) - W^{-3}(x) = v_{1A} + v_{1B} + \alpha - (v_{1A} + 0) = v_{1B} + \alpha$.

D) (Global bidder wins license B, local bidder A wins license A) And suppose that $v_{2A} > v_{1A} + \alpha$ and $v_{3B} < v_{1B}$. Then, the revenue is $v_{3B} + v_{1A} + \alpha$.

since $W(x^{-1}) - W^{-1}(x) = v_{2A} + v_{3B} - (v_{2A} + 0) = v_{3B}$ and $W(x^{-2}) - W^{-2}(x) = v_{1A} + v_{1B} + \alpha - (v_{1B} + 0) = v_{1A} + \alpha$.

■

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