Word of Mouth Advertising and Strategic Learning in Networks [1]

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1 Introduction

Social networks representing the pattern of social interactions - who talks to or who observes whom- play a crucial role as a medium for the spread of information, ideas, diseases, products. Someone in the population may be struck with an infection or may adopt a new technology, and it can then either die out quickly or spread throughout the population, depending possibly on the location of the initial appearance, the structure of the network - for instance, how dense it is. The dynamics of adoption -the extent to which individuals are influenced by their neighbours, the impact of "word-of- mouth" communication- also play a role in determining the speed of diffusion. Given the large range of contexts in which social learning is important, it is not surprising that researchers from various disciplines have studied processes of diffusion from a variety of perspectives.[1]

For instance, the classic study of [3] describes the impact of the social structure on the prescription of a new drug by doctors. [7], [15] are amongst several papers that focus on social learning and the adoption of new technologies in agriculture in developing countries. Of greater relevance to the present paper is the use of social networks to promote new products through "word-of-mouth" communication and viral marketing.[2] The growing popularity of this form of advertising has resulted in a corresponding increase in the attention this has received both in the academic marketing literature as well as in the more popular media.

A particularly interesting form of word-of-mouth marketing, described in a New York Times magazine article by Rob Walker on December 5, 2004, provides the main motivation for this paper. Writing about how "word of mouth" marketing was superseding more traditional modes with fracturing market segments, he wrote about the "growing number of marketers organizing veritable armies of hired "trendsetters" or "influencers" or "street teams" to execute "seeding programs,", "viral marketing," "guerrilla marketing" " as "attempts to break the ...wall that used to separate the theater of commerce, persuasion and salesmanship from our actual day-to-day life." For example, the acquaintance who recommends a book to someone might actually not have read it but have been provided talking points by a paid agent of the publisher. In a somewhat different context, a government body hoping to spread a new technology might solicit important and connected individuals, in the focal community to recommend the new

[1]See Rogers(1995) for an account of recent research on diffusion.
[2]See [14], [11].

methods to people they know. It might well be known to the target segment of consumers that this type of activity is going on (especially after the appearance of the New York Times article!); this might lead to such recommendations from neighbors being received with some scepticism and possibly ignored. In such cases, even a good product and a good new technology might not diffuse as fast as it should or even diffuse at all, thus leading to a socially undesirable outcome. With the advent of web-based social networking sites, viral marketing has taken on added importance. See, for example, http://techcrunch.com/2008/04/21/facebook-publishes-insiders-guide-to-viral-marketing/, where several techniques are discussed including using the "News Feed" feature to convey information to friends of users and their friends and so on.

We study this problem by considering a finite number of potential buyers who are connected in a fixed network. There is a seller who can choose to 'seed" the network by paying an agent at any given node in the network to give a positive recommendation about the seller's product. The seller knows in advance whether his product is of good or bad quality. Buyers have ex ante beliefs about whether the product is good. Buyers are of two types. There is some probability that a buyer at a given node is an "innovator", who will try the new product immediately; with the complementary probability a buyer is normal in that she makes a rational decision on whether to buy or not. We assume that the ex ante belief is below the threshold required to induce the second type of buyers to purchase the product.

Buyers can also recommend buying the product to their neighbors in the network. Buyers who are not innovators and who receive recommendations from their neighbors have to form posterior beliefs about the quality of the product, and then decide whether to buy the product. Each purchase gives the seller a unit profit (prices are assumed to be fixed) and future payoffs are discounted; so if buying is optimal, buying now is better than waiting.

We model this process as a game of incomplete information and focus on the perfect Bayesian equilibrium of this game. Our principal interest is in studying the conditions under which the product of good quality will diffuse with probability one throughout the network in the fastest possible time - we call this the efficient diffusion equilibrium(EDE). In particular, what are the types of network structures which are conducive to the existence of an EDE?

In order to answer this question, we have to analyze the optimal marketing strategies of the two types of firms. This is related to a popular theme in the existing marketing literature. This literature suggests that it is optimal to initially target a few "influential" members of the population since these influential members can then more easily convince others in the network to buy the product.[3] A "naive" view is that highly connected individuals are more likely to be influential. However, [8] provide a more nuanced and interesting analysis by showing that "the optimality of targeting highly connected nodes depends very much on the content of social interaction."

Our analysis shows that the two types of firms will follow very different targeting strategies in equilibrium. A firm which produces a good quality product will want to buy off an agent which can reach out to all other agents in the shortest possible time. On the other hand, a firm producing a bad quality product is more myopic because it knows that its product will not sell beyond one period since no one other than the agent it has bought off will recommend the product. So, this firm will want to place its implant at nodes which have the highest number of connections. However, the bad quality firm must also ensure that its agent's recommendation is credible. For instance, if consumers know that the bad quality firm buys off a particular agent with probability one, then that agent's recommendation is less likely to be credible.

The optimal behavior of the two types of firm determine whether an EDE exists or not. Our analysis reveals some counter-intuitive results. For example, it turns out that if any individual is "too influential" in the sense of being connected to *everyone*, then an EDE cannot exist. This is becasue both types of firms would target such an individual, thereby destroying the credibility of her recommendation. Of course, efficient diffusion would typically be guaranteed in contexts where the credibility of recommendations is not at stake. It also turns out for somewhat subtler reasons that *larger* networks are more likely to support efficient diffusion.

We provide a partial characterization of networks which support EDEs. We also have some "comparative statics" results. In particular, we focus on the role of the network structure, as well as on the probability that consumers are innovators. Of course, there can be other types of inefficient equilibria and we illustrate the nature of these equilibria for the special case where the network is a *line*. We also briefly discuss the robustness of our results if the model is changed to allow for innovators to make *negative* recommendations. Obviously, the possibility of negative recommendations will lower the expected profits from employing an "implant" for the bad quality firm, and thus make the existence of an EDE more likely. However, conditional on the bad quality firm employing an implant, there is very little

[3]This has given rise to the development of algorithms to locate the most influential members in a network. See [13] and [10].

qualitative change in our results.

1.1 Related literature

There has been a voluminous literature on diffusion of innovation arising from different causes; for a game-theoretic analysis on how these different causes could lead to different observed patterns of diffusion, see Peyton Young [17]. However, Young does not explicitly consider a network structure.

The paper that considers the most closely related problem of a monopolist "seeding" a network in order to spread information about his product is [8]. This paper considers a monopolist who can choose a fraction x of the population (modelled as the unit interval [0,1]) at a cost of $c(x)$). Each individual picks a finite number of neighbors from the unit interval, so the neighbors of different individuals are in independent subsets (the probability of a common neighbour is 0). The number of neighbors varies according to a probability distribution; each individual can get information about the product either directly from the monopolist or indirectly from one of her neighbors, who is himself directly informed. The results obtained are essentially about properties of the degree distribution that facilitate spread of the monopolist's information.

Unlike our work, Galeotti and Goyal do not consider a seller with private information or Bayesian buyers who take into account how their current state of information reveals what is happening in unobserved parts of the network. Our results relate primarily to different notions of centrality in a given network structure, rather than to the average degree or the variance of the degree distribution as in Galeotti and Goyal.

Our work is also related to social learning in networks, as in [1], [2].[4] The main difference between our work and these papers is that we have a strategic seller who has private information about quality and is trying to manipulate the diffusion,, whilst none of the other papers do. Also our buyers are fully rational Bayesians; even [1] do not model buyers who infer events in unobserved parts of the network, as our buyers do. It is not surprising therefore that our equilibria are quite different.

The next section describes the formal model. We then illustrate the workings of our model when the network structure is a line. This is followed by a partial characterization result on the type of networks which sustain an EDE. Our comparative statics results are contained in the subsequent

[4]See [9] for an illuminating survey of papers on learning in networks. Other related papers are[4], [5], [6].

section. The following sections describe the nature of inefficient equilibria on the line and extensions to the basic model.

2 The model

In this section, we describe the basic model. The set $N = \{1, 2, \ldots, n\}$ represents the set of consumers. The structure of interactions between the set of consumers is represented by means of a graph g in which the nodes are elements of N and $ij \in g$ if consumers i and j can communicate with each other. There is a firm F, which is interested in selling its product. The product is either of type $G(ood)$ or $B(ad)$. Firm F knows the type of its product.

All buyers have an initial probability p that the product is of the good type. There are *two* types of consumers. Buyers of the first type - we refer to them as the *innovators*- get utility g' from the G product and utility $-b'$ from the B product. These numbers satisfy

$$g'p - b'(1-p) > 0.$$

Since a buyer's payoff is 0 if she does not buy the product, the innovators will want to buy the product immediately. On the other hand, the second type of buyers get utilities g and b from the G and B type products. These numbers are such that

$$p < \bar{p} \equiv \frac{b}{b+g}$$

So, the second type of consumer will not buy the good unless she revises her probability belief about the good in subsequent periods.

Each consumer buys the product at most once. The firm gets 1 unit for each item purchased and 0 if an item is not purchased. There are no capacity constraints on the number of items sold.

Future payoffs are discounted by δ for F and for the consumers.

The time line: Nature draws the type of F and this is revealed only to F. F chooses a site i to place one "implant" at a "small" cost c or decides not to use any implants. The implant, if any, is paid to pass on a recommendation to his neighbors in g. If i is not an implant, she can be an "innovator" in which case she tries the new product immediately. The probability that any site i is an innovator is $\rho < 1$ and the event that "i is an innovator" is independent of other events "$j \neq i$ is an innovator". All this takes place, in sequence, at $t = 0$. At $t = 1$, any i who is an innovator, and who has obtained a utility of g', makes a recommendation to his neighbors.[5] The neighbors receiving the recommendation might choose to buy the product or not to buy it. An implant may choose also to make a recommendation. At $t = \tau$, any site who is either an implant or has tried the product and found it good, after receiving a recommendation in $\tau - 1$, can make a recommendation to neighbors. A site i does not observe if neighbors have received recommendations or have chosen to buy the product- she only observes whether recommendations are made by the neighbors themselves.

There is no exogenous time limit on the game; however, since there are a finite number of neighbors and each speaks at most once, the game must end in finite time.

Updating Beliefs

Let α and β be the mixed equilibrium strategies of types G and B respectively. That is, α_i is the probability with which type G uses consumer i as an implant. Suppose consumer i receives a recommendation from her neighbor $i-1$ in period 1. If she receives no other recommendation, what is the probability that the product is G? Let us denote this by $\eta^1_{i,i-1}$, where the superscript refers to the time the recommendation is received and the subscripts to the recipient and the sender of the recommendation.

Let $N_i(g) = \{j \in N | ij \in g\}$ denote the set of neighbors of i, and $d_i(g) = |N_i(g)|$ be the *degree* of i in g. (Henceforth, whenever there is no ambiguity about g, we will simply write d_i, d_j, etc.)

Now, conditional on G, $i-1$ would be able to recommend the product if it is either an innovator or an implant. This has a probability $\alpha_{i-1} + (1-\alpha_{i-1})\rho$. It must also be the case that (since i has received no other recommendation) none of i's other neighbors contains either a "good" implant or an innovator. Recall that the absence of an innovator at a site is independent of the presence or absence of innovators at other sites. Also, if the innovator is at $i-1$, then it cannot be at any of i's other neighbors. Noting that the probability that $i-1$ contains a bad implant is $(1-p)\beta_{i-1}$, the posterior belief of i that the product is good follows from Bayes Theorem, and is

$$\eta^1_{i,i-1} = \frac{p(1-\rho)^{d_i - 1}\left[\alpha_{i-1} + (1 - \sum_{j\in N_i}\alpha_j - \alpha_i)\rho\right]}{p(1-\rho)^{d_i - 1}\left[\alpha_{i-1} + (1 - \sum_{j\in N_i}\alpha_j - \alpha_i)\rho\right] + (1-p)\beta_{i-1}} \qquad (1)$$

[5] By "recommendation", we mean a positive recommendation. An innovator who finds that the product is bad does not make a negative recommendation. An implant always makes a recommendation, but may choose the time at which she makes the recommendation for strategic reasons. We discuss negative recommendations later in the paper.

Some "special" cases illustrate the nature of the updating process. Suppose that the type G firm uses a pure strategy so that for some site m, $\alpha_m = 1$. Suppose i is a neighbor of m, but receives only one recommendation from some $j \neq m$. Then, i must conclude that j is a bad implant - if the product had been good, then there would have been a good implant at m who would then have passed on a recommendation to her. This argument generalizes even when the type G firm uses a strategy whose support is some set M containing more than one node. Suppose now that i is a *common* neighbor of all nodes in M. Again, if i does not receive a recommendation from some member of M, she will conclude that any other recommendation comes from a bad implant. Next, suppose again that $\alpha_m = 1$ and that m receives a recommendation from some neighbor. Of course, such recommendations are not credible to m - she would have been used as an implant by the type G form if the product was good. These inferences are confirmed by equation (1) - in all cases, the numerator is 0.

Of course, if i receives a recommendation from *two* or more neighbors, then i concludes that the product is G with probability one- if there is a bad implant at j, then there cannot be a bad implant at $j' \neq j$.

Suppose next that i receives a recommendation from $i-1$ in some period $t > 1$, but no recommendation from any other neighbor. If i has not received any recommendations before period t and receives one from $i-1$ in period t, this can happen because the product is Bad, there is an implant at $i-1$ and the implant chooses to speak at period t. Alternatively, the product is Good, $i-1$ heard a recommendation from one of her neighbors in the previous period, but none of i's other neighbors received a recommendation from any of their neighbors in period $t-1$. Explicit computations of these probabilities are hard to describe since these depend on the structure of the network. But, notice that some recommendations are easy to dismiss. For instance, suppose g is a *line*, and let i be an extreme point of g, with degree one. Then, any recommendation from i coming in period $t > 1$ is not taken seriously by i's neighbor since i could not have received a recommendation in period $t-1$.

However, although an explicit description is cumbersome, the term $\eta^t_{i,i-1}$ is well-defined.

Efficient Diffusion Equilibrium

Notice that in the absence of any implants of either type, the presence of an innovator guarantees that the good product will diffuse through the entire network. Since the probability that there is at least one innovator is $1-(1-\rho)^n$, this is also the probability that all the n buyers will eventually purchase the product if it is good. Of course, if the the firm whose product is bad decides to employ an implant, then the probability of complete diffusion may be lower since recommendations may not be accepted with probability one in equilibrium.

We are particularly interested in PBE which we will refer to as the *efficient diffusion equilibrium* (EDE). An EDE will have two properties -(i) the good product will diffuse throughout the network with probability one, and (ii) the diffusion will occur in the least number of periods amongst all equilibria in which diffusion occurs with probability one. Notice that since there is discounting, the good type in this equilibrium is also interested in the speed of diffusion. Of course, the bad type does not care about the speed of diffusion since no consumer will recommend the bad product.

Clearly, the good product will diffuse through the entire network with probability one only if there is at least one sequence of recommendations which is accepted with probability one. Also, the speed of diffusion is maximized if the good implant is located optimally. The optimal site(s) for a Good implant is related to a measure of centrality of network structures. Let d_{ij} denote the *geodesic distance* between i and j in the graph g. That is, d_{ij} is the length of the shortest path between i and j.

Definition 1 *A node i maximizes decay centrality in a graph g if $\sum_{j \neq i} \delta^{d(i,j)} \geq \sum_{j \neq k} \delta^{d(k,j)}$ for all $k \in N$.*

If (α, β) form part of an EDE, then the support of α must be contained in the set of nodes which maximize decay centrality. While it is not easy to compute this set in general graphs, the set is easily identified in special cases. For instance, if g is a line, then the median(s) must be maximizing decay centrality. Or if g is a star, then the hub is obviously the node maximizing decay centrality.

Of course, we also need to identify when G and B will decide to use an implant. This must depend on a comparison of the increase in expected profit resulting from an implant and the cost c incurred by employing an implant. We assume that the cost is in the range which makes it profitable to employ just one implant.

3 An Example

In this section, we provide an informal discussion of a specific network structure, - the *line* when n is odd, in order to illustrate the conditions required for an efficient diffusion equilibrium.

So, let g be a line, with the sites ordered so that 1 and n are the end-points of the line having degree one, while all other sites have degree 2. Since n is odd, the unique median maximizes decay capacity. Hence, in any PBE (α, β), where α and β denote the equilibrium strategies of the Good and Bad types respectively, $\alpha_m = 1$. Now, if this PBE is to be an EDE, then the recommendation coming from m has to be accepted with probability one. That is, $m-1$ and $m+1$ must accept any recommendation coming from m with probability one. Since the bad type can always mimic the Good type, this implies that wherever the bad implant is placed, her recommendation must be accepted by two neighbors.

But, of course, $\beta_m \neq 1$. For, if $\beta_m = 1$, then from equation (1), $\eta^1_{m-1,m} = \eta^1_{m+1,m} = p$ and neither of m's neighbors would buy the product after receiving a recommendation from m- which is a contradiction. So, while the support of β can *include* m, it cannot coincide with $\{m\}$. Over what set of nodes can the bad type "distribute" β? The answer of course is that the support of β must be contained in those nodes which have an effective degree of 2 since $\bar{d}_m = 2$. This must be the set

$$S \equiv N \setminus \{1, m-2, m-1, m+1, m+2, n\}$$

It is clear that 1 and n cannot be in S since they have only one neighbor. Although $m-2$ and $m+2$ have degree 2, notice that $m-1$ (respectively $m+1$) will not believe a single recommendation from $m-2$ (respectively from $m+2$). Of course, $m-1$ and $m+1$ cannot sell to m. Also, notice that if $n < 9$, then m will be the sole member of S, and there will not be any EDE.

It is also clear when the bad type will want to use an implant when there is an EDE. Let the implant be placed at i. Since i can be an innovator with probability ρ (in which case he would buy the product anyway), the effective cost of an implant at i is

$$\rho + c$$

The benefit is the additional probability that $i-1$ and $i+1$ buy the product. With probability $(1-\rho)^2$, neither are implants. With probability $2(1-\rho)\rho$, one of the two is an implant. Hence, the benefit is

$$2(1-\rho)^2\delta + 2(1-\rho)\rho\delta = 2\delta(1-\rho)$$

So,the net gain of an implant for B is given by

$$2\delta(1-\rho) - \rho - c$$

where c is the cost of an implant.[6] So, B will use an implant if

$$c \leq c^B(\delta, \rho) \equiv 2\delta(1-\rho) - \rho$$

Not surprisingly, the higher the value of ρ, the lower is the expected gain from employing an implant.

Consider G. Let $\underline{\pi}^G(\delta, \rho)$ denote the profit of G in the absence of an implant, and $\bar{\pi}^G(\delta, \rho)$ denote the profit of G if an implant is used.[7]

Then, G will use an implant if

$$c \leq c^G(\delta, \rho) \equiv \bar{\pi}^G(\delta, \rho) - \underline{\pi}^G(\delta, \rho)$$

The expected profits of G with or without an implant increases in ρ. However, notice that the *difference* between the two levels of profit decreases as $\delta \to 1$ since the speed of diffusion is less important as δ increases. In the limit,

$$\lim_{\delta \to 1} (\bar{\pi}^G(\delta, \rho) - \underline{\pi}^G(\delta, \rho)) = n - 1 - n[1 - (1-\rho)^n]$$

On the other hand, an increase in δ increases the value of an implant to B. So, for "high" δ, it may well be the case that only the bad type uses an implant!

Let $c_m = \min\left(c^B(\delta, \rho), c^G(\delta, \rho)\right)$.

Suppose $\beta_i > 0$ for some i. It is easy to place an upper bound on how high β_i can be in equilibrium. Since i's recommendation must be accepted with probability one by both her neighbors, their updated probability that the product is good cannot fall below $\bar{p}$. Equation 1 and the fact that $\alpha_m = 1$ now readily yield these upper bounds.

$$\bar{\beta}_m = \frac{p(1-\bar{p})}{\bar{p}(1-p)}, \text{and for } i \neq m, \bar{\beta}_i = \rho\frac{p(1-\bar{p})}{\bar{p}(1-p)}$$

It is now easy to describe what an EDE looks like when $n \geq 9$ is odd, and the cost of an implant does not exceed c_m.

(i) The type G firm puts its implant at the median of the line with probability one.

(ii) The support of β is contained in S, and for each i in the support of β, $\beta_i \leq \rho\bar{\beta}_m$, $\beta_m \leq \bar{\beta}_m$ and $\beta_i = 0$ elsewhere.

(iii) The implant (irrespective of type) "speaks" in period 1.

[6] Since the implant is placed "today" and the implant's recommendation is accepted tomorrow, the revenue has to be discounted.

[7] These expressions are difficult to compute for general n.

(iv) Recommendations received from each site in the support of β are accepted with probability one in period 1.

(v) In subsequent periods $t > 1$, a site $i < m$ accepts a recommendation from $i + 1$ with probability one if the equilibrium response of $i + 1$ was to accept the recommendation from $i+2$ in period $t-1$. Similarly, a site $i > m$ accepts a recommendation from $i-1$ if the equilibrium response of $i-1$ was to accept the recommendation from $i - 2$ in period $t - 1$

As we have mentioned before, Property (ii) follows from the fact that recommendations will not be accepted unless the updated belief that the product is good reaches the threshold value of $\bar{p}$. Property (iv) is consistent with this since a site that receives a recommendation (and is not an innovator or an implant) is at least indifferent between buying and not buying and strictly prefers buying if the inequality is strict.

For property (iii), note that if the product is bad, then the implant can only hope to persuade two neighbors to buy the product - her recommendation will not be passed on. If the product is good, then the product will diffuse through the entire population given (v). Since an implant's recommendation is accepted with probability one by both neighbors, the implant gets the same outcome as early as possible in either case.

Property (i) follows by noting that type G cannot be indifferent between m and any other site. At m, the implant will obtain an expected payoff of $\delta.2+\delta^2.2+....$ for $m-1$ terms. At $m-k$, say, the payoff will be $\delta.2+\delta^2.2+...\delta^t2$ for $m - k - 1$ terms and $\delta^{m-k}.1 + ..$ for an additional $2k$ terms, thus taking $m + k - 1$ periods to diffuse completely rather than $m - 1$ periods, if the good implant locates at m.[8] Thus the speed of diffusion is higher by locating at m, since there are two new buyers in each period for every period the diffusion continues, whilst at $m - k$, there is only one buyer for every period after $k - 1$.

The argument for (v) includes the following: Since the B implant speaks in period 1 in equilibrium, any recommendation after period 1 must come from a relayed recommendation from an innovator or a good implant at some $i \neq m$ or m respectively. If $t \geq m - 1$, no recommendation will occur along the equilibrium path but it is assumed that off-equilibrium beliefs also induce acceptance of the recommendation.

We note that the B implant will not deviate to speaking after $m-1$, even with this belief, because of discounting and the acceptance probabilities of 1 even if B speaks early.

[8]This calculation does not take ρ into account, but it is obvious this wouldn't change the ranking because sites are independently innovators or non-innovators

Remark 1 *We note again that* $\mid S \mid$ *must be large enough for the B implant to put "small amounts" of probability at each site in S in his randomization. For example, if* $\mid S \mid$ *is very small so that there is no distribution that makes it possible for* $\beta_i \leq \rho\bar{\beta}_m$ *for all sites in S and* $\beta_m \leq \bar{\beta}_m$*, then this equilibrium will not exist. On the other hand, notice that if a smaller "line" supports an EDE, then so must a larger line. In other words, efficient diffusion is more likely the larger the number of potential customers!*

Of course, inefficient equilibria can exist when n is not large enough. These will typically involve the bad implant delaying his recommendation for strategic reasons. In such equilibria, recommendations will not be accepted with probability one. We will get back to this issue in a later section.

4 A Partial Characterization Result

In this section, we describe a sufficient condition for an EDE to exist, and then show that this condition is necessary for certain types of network structures.

Say that a link $ij \in g$ is *critical* if $g - ij$ has more components than g. That is, if a critical link is removed from a connected network g, then the network g no longer remains connected. Say that a node i is *critical* in g if *all* links of i are critical. Of course, if the network is a tree, then all links are critical, and so all nodes other than the leaves of the tree are critical.

Fix some set $M \subset N$. For any node $i \notin M$, let $\bar{N}_i(M, g) = N_i(g) \setminus (\cap_{j \in M} N_j(g))$, where $N_j(g)$ is the set of neighbors of j in g. Let $\bar{d}_i(M, g) = |\bar{N}_i(M, g)|$. We will refer to $\bar{d}_i$ as the *adjusted* degree of a node $i \notin M$. For any $m \in M$, let $\bar{d}_m = d_m = |N_m(g)|$. That is, if $i \in M$, then the adjusted degree of i coincides with $|N_i(g)|$.

Suppose α denotes the mixed strategy employed by the type G firm in any equilibrium. Let M be the support of α. Then $\bar{N}_i(M, g)$ is the set of potential neighbors of i who may possibly believe that the product is good after receiving a single recommendation from i in period 1. To see this, notice that if the product is good, then a positive recommendation *must* come from some member of M. The absence of such a recommendation signals that any other recommendation comes from a bad implant. So, any k who is a common neighbor of *all* sites in M will find a recommendation credible only if some member of M sends a recommendation.

In order to simplify the notation, we will simply write $\bar{N}_i, \bar{d}_i$ whenever the absence of M and g will not cause any confusion.

Given some $M \subset N$, partition nodes into sets $S_1, \ldots, S_K$ such that S_1 is the set of nodes maximizing $\bar{d}_i$, S_2 is the set of nodes with the next highest value of $\bar{d}_i$, and so on.

For every node i, let h_i be the site in $\bar{N}_i$ which maximizes degree, and d_{h_i} be its degree. Define $\bar{\beta}_i$ to be the value of β which sets $\eta^1_{h_i,i} = \bar{p}$.

Remark 2 *Notice that each $\bar{\beta}_i$ depends on the vector α via equation 1. We will not explicitly indicate this dependence in order to simplify the notation.*

Notice that equation 1 also implies that

$$\eta^1_{h_i,i} \leq \eta^1_{j,i} \text{ for all } j \in \bar{N}_i$$

Let $D(g)$ be the set of nodes maximizing decay centrality in g.

The next theorem identifies a sufficient condition for a network structure to support an EDE. It also shows that if all nodes in $D(g)$ are critical and (somewhat loosely speaking) have highest adjusted degree, then this condition is also necessary. These conditions are satisfied by the line, and so this theorem will include the line as a special case.

Theorem 1 *Suppose the cost of an implant is sufficiently low for both types of the firm to use an implant. Then, an EDE exists if there is some α with support M contained in the set $D(g)$ such that*

$$\sum_{i \in S_1(M)} \bar{\beta}_i \geq 1 \tag{2}$$

Conversely, an EDE does not exist if every $m \in D(g)$ is critical, for all $M \subseteq D(g)$, $D(g) \subset S_1(M)$ and equation 2 does not hold.

Proof. Suppose there is α such that equation 2 is satisfied . Let α be the strategy employed by the good type, and choose β such that

$$\beta_i \leq \bar{\beta}_i \text{ for all } i \in S_1$$

and

$$\beta_i = 0 \text{ if } i \notin S_1$$

The response strategies are straightforward. All sites accept all recommendations in all periods.

It is easy to check that these strategies constitute a PBE. Clearly, the type G firm has no incentive to deviate since her implant is at some site maximizing decay centrality and recommendations are accepted with probability one. Similarly, the type B firm has no incentive to deviate since she obtains a payoff of $\bar{d}_1|S_1| - c$, conditional on no innovators in $\bar{N}_1$. Clearly, no other site can yield a higher payoff. The response decisions are optimal because (i) any recommendation coming from a site not in the union of the supports of α and β must be coming from an innovator, and (ii) the updated belief of any i after receiving a recommendation from a potential implant is at least as large as the threshold value $\bar{p}$.

Consider now the necessity of this condition. First, the type G firm must be choosing an α with support contained in $D(g)$ in any EDE. Fix any such α and suppose $\alpha_m > 0$. Since m is critical in g, all neighbors of m must accept a recommendation from m with probability one in an EDE. So, the type B firm by placing her implant at some site where $\alpha_i > 0$ can obtain $|S_1|$ "hits". Hence, in equilibrium, the support of β must be contained in S_1. Moreover, if $\beta_i > 0$, every member of $\bar{N}_i(g)$ has to accept the recommendation of i. Hence, the maximum probability weight that the type B firm can put on i cannot exceed $\bar{\beta}_i$. This is not possible if equation 2 does not hold. ■

Equation 2 is easy to interpret. If there are a sufficient number of nodes maximizing adjusted degree, then an EDE is easy to support since the type B firm has enough "space" to distribute his probability. Why is equation 2 not necessary without additional conditions? Suppose, for simplicity that $\alpha_m = 1$ for some node in $D(g)$, but $\alpha_m \notin S_1$, but say in S_2. Also, assume that equation 2 does not hold. It is possible then to have another equilibrium in which (i) the type B uses a mixed strategy over nodes in $S_1 \cup S_2$, (ii) the probability of acceptance of recommendations coming from nodes in S_1 is adjusted below one so as to ensure that the expected payoff from an implant located in S_1 is the same as that from an implant in S_2.[9] The freedom to distribute some probability weight over nodes in S_2 may now help in ensuring existence of equilibrium. Instead of formally deriving a sufficient condition for this type of equilibrium, we illustrate such an equilibrium in the example below.

Example 1 *Let $n = K(K+1)$ where $k > 3$. Denote $I = \{i_1, \ldots, i_K\}$, and let each $i_k \in I$ be the hubs of K stars $g_1, \ldots, g_K$, each g_k having $K+1$ peripheral sites. Finally, let site 1 be connected to each site in I, and to no other site. Also, no site i_k in I is connected to any site in the other stars.*

[9] However, the value of δ cannot be too high. If δ is high, then the low discounting may induce the type B implant at some site in S_1 to strategically postpone her recommendation to a later period.

So,

$$g = (\bigcup_{i \in I} g_i) \cup \{1i_1, \ldots, 1i_K\}$$

We want to choose values of ρ *and* δ *such that* 1 *is a member of* $D(g)$. *Then,* $\bar{d}_1 = K$. *But, notice that for each* $i_k \in I$, $\bar{d}_{i_k} = K+1$ *since none of the peripheral sites in* g_{i_k} *are connected to* 1.

Suppose the type G firm places an implant at 1. *Then, the expected benefit from the implant will be*

$$B_1 = (1-\rho)\delta\left[\Gamma_K(1+\delta\Gamma_{K+1})\right]$$

where for any k

$$\Gamma_k = \sum_{i=1}^{k} \binom{k}{i} i(1-\rho)^i \rho^{k-i}$$

On the other hand, if the type G firm places an implant at any of the sites in $\{i_1, \ldots, i_K\}$, *then the expected payoff is*

$$B_2 = (1-\rho)\delta\left[\Gamma_{K+1} + (1-\rho)\left(1+\delta\Gamma_{K-1}(1+\delta\Gamma_{K+1})\right)\right]$$

Evaluating the two expressions at $\rho = 0$, *it is easy to check that*

$$B_1 > B_2 \text{ if } \delta > \frac{2}{K^2-1}$$

That is, if $\delta > \frac{2}{K^2-1}$, *then* 1 *maximizes decay capacity for* $\rho = 0$ *and hence for "small" values of* ρ. *Assume that* $\rho > 0$ *is such that* $B_1 > B_2$. *Also,*

$$\delta \in \left(\frac{2}{K^2-1}, \frac{\Gamma_K}{\Gamma_{K+1}}\right)$$

Let

$$\bar{\beta}_1 = \frac{(1-\rho)^K p(1-\bar{p})}{\bar{p}(1-p)}, \bar{\beta}_i = \frac{p\rho(1-\bar{p})}{\bar{p}(1-p)} \text{for each } i \in I,$$

Using equation 1, it is easy to verify that if $\alpha_1 = 1$, *then for each* $i \in I \cup \{1\}$, $\eta^1_{k,i} = \bar{p}$ *if* $\beta_i = \bar{\beta}_i$. *Suppose*

$$\sum_{i \in I} \bar{\beta}_i < 1, \text{ but } \sum_{i \in I} \bar{\beta}_i + \bar{\beta}_1 \geq 1$$

Notice that equation 2 is not satisfied. However, the pair of strategies (α, β) *along with response decisions specified below is an EDE.*

Let

(i) $\alpha_1 = 1$,

(ii) $\beta_i = \bar{\beta}_i$ *for each* $i \in I$, $\beta_1 = 1 - \sum_{i \in I} \bar{\beta}_i$.

(iii) Each $i \in I$ *accepts recommendation from* 1 *with probability one in period 1.*

(iv) Each $i \in g_{i_k}, i \neq i_k$ *accepts a recommendation from* i_k *with probability* $\frac{\Gamma_K}{\Gamma_{K+1}}$ *in period 1 and with probability 1 in period 2.*

(v) Both types of implants make their recommondations in period 1.

To check that these constitute an equilibrium, first notice that the type G *firm has no incentive to deviate since (i)* 1 *maximizes decay capacity, (ii) the implant speaks immediately and the product diffuses throughout the network in 2 periods in view of (iii) and (iv) above. Consider now the type* B *implant. Her expected payoff from any site* $i \in I$ *is* Γ_K *given the acceptance probabilities of the periphwral sites in the star. This is also the expected payoff from the implant at* 1 *since* 1*'s recommendation is accepted with probability one. Finally, note that since* $\delta \leq \frac{\Gamma_K}{\Gamma_{K+1}}$, *the implant at* $i \in I$ *has no incentive to strategically postpone her recommendation to period 2, even though her recommendation in this period would be accepted with probability one.*

If $m \in S_1$ but is not critical, then there could be an equilibrium of the following kind. The good product may diffuse throughout the network with probability one even if some neighbors of m who do not constitute critical links with m refuse m's recommendations - the fact that some link mi is not critical obviously implies that there is some path from m to i not involving the link mi. An implication of this is that the type B firm needs fewer customers in equilibrium. Now, suppose each node $i \in S_1$ has one neighbor in $\bar{N}_i(g)$ with very high degree, h_i, while the others have relatively low degree. Then, one option for the type B firm is to put probability weights $\tilde{\beta}_i > \bar{\beta}_i$ on each i such that all nodes in $\bar{N}_i(g)$ except h_i accept i's recommendation. In other words, the freedom to dispense with h_i as a customer helps to raise the probability that type B can put on each node i in S_1 and so it may be possible to support an EDE even when equation 2 is not satisfied.

5 Comparative Statics

In this section, we discuss the role of different parameters and the network structure is sustainign an EDE.

It is easy to check that an increase in p makes it more likely that an EDE exists since it allows the type B firm to place more probability weight on nodes and still satisfy the requirement that the updated beliefs reach the threshold value of $\bar{p}$. In other words, if an EDE exists for some value of p and then p increases, there must continue to be an EDE. In what follows, we focus on the role of the network structure and of ρ.

5.1 The Role of the Network Structure

We first show that no site i can be too well-connected if the network is to sustain an EDE. In particular, no network which contains a star encompassing all nodes can support an EDE.

Theorem 2 *Suppose g contains a star as a subgraph encompassing all nodes. Then, g cannot support an EDE.*

Proof. Let $M = \{i \in N | d_i(g) = n-1\}$. If g contains a star, this set is non-empty. Then, all members of M maximize decay centrality. Let $M' \subset M$ be the support of α. Take any site $i \notin M'$. Then, $\bar{d}_i = 0$ since any site $j \neq i$ is connected to all sites in M'. So, the support of β is contained in M'. Take any node $i \in M'$ such that $\beta_i \geq \alpha_i$. Then, it follows from equation 1 that

$$\eta^1_{j,i} \leq p < \bar{p}$$

So, no neighbor of the bad implant at i buys the product after receiving a recommendation from i. Since the bad type is indifferent between all sites in the support of β, no site in the support of β can get her recommendation accepted. This implies that only the good type employs an implant. However, this cannot be an equilibrium since the bad type would then deviate and place an implant at some site in M'. ∎

In some cases, we will make the following assumption.

Assumption S: There is a unique node m maximizing decay capacity.

Several network structures satisfy this assumption - for example, the line when n is odd, or the star.

Under Assumption S, we can also place an upper bound on the degree of m.

Theorem 3 *Let Assumption S hold, with m the unique node maximizing decay centrality in g. Then, if g is to support an EDE, $d_m \leq \frac{n-1}{2}$.*

Proof. Let g support an EDE, and let $d_m = k$. Then, for all $i \neq m$, $\bar{d}_i \leq n - k - 1$. Suppose $d_m > \bar{d}_i$ for all $i \neq m$. Then the bad type would prefer to put her implant at m since recommendations from m are accepted with probability one. But, if $\beta_m = 1$, then $\eta^1_{m,j} = p < \bar{p}$ and this is not consistent with m's recommendations being accepted. Hence, $d_m = k \leq \bar{d}_i \leq n - k - 1$ for some $i \neq m$. This implies $d_m \leq \frac{n-1}{2}$. ■

Theorems 2 and 3 describe some network structures which cannot support an EDE. In particular, nodes maximizing decay centrality cannot be too well-connected since their connections tend to reduce the adjusted degree of those nodes which are "close" to them. In an intuitive sense, it is also clear that "larger" networks are more conducive to supporting EDE. The next example shows that an asymmetric network where the site maximizing decay centrality has low degree may actually facilitate efficient diffusion.

Example 2 *Let n be odd and $n \geq 9$. Choose any site, say m, and divide $N \setminus \{m\}$ into two equally-sized disjoint subsets N_1, N_2. Let i_1, k_1 be in N_1 and i_2, k_2 be in N_2. The network is the following:*

(i) m is connected to just i_1, i_2.

(ii) No connection between nodes in N_1 and N_2.

(iii) i_1 is connected to everyone in N_1 except k_1, and i_2 is connected to all nodes in N_2 except k_2.

(iv) All nodes in N_1 (except i_1, k_1) are connected to everyone in N_1.

(v) All nodes in N_2 (except i_2, k_2) are connected to everyone in N_2.

The adjusted degree of all nodes $i \neq m$ is $\frac{n-5}{2}$, while that of m is 2. So, $n \geq 9$ ensures that the adjusted degree of $i \neq m$ is at least as high as that of m.

Let $\alpha_m = 1, \beta_i = 1/(n-1)$ for all $i \neq m$. Responses are as follows:

(i) Sites i_1 and i_2 only accept recommendations from m.

(ii) Site m does not accept any recommendations.

(iii) All other sites accept recommendations from all other sites.

This will be an equilibrium if $\beta_i = 1/(n-1)$ ensures that the posterior probability is at least as large as $\bar{p}$.

Our conjecture is that given all other parameters, if this network of size n does not support an EDE, then *no network* of size n or smaller will support

5.2 The Influence of Innovators

Recall that ρ is the probability of an innovator. Here, we discuss the role of ρ on an EDE. In order to simplify the discussion, we assume throughout this section that Assumption S is satisfied. Thus, the type G firm must be using a pure strategy of placing her implant at m with probability one, where m is the unique node in $D(g)$.

Consider first the case where $\rho = 0$. This is a particularly stark case to analyze, where the equilibrium of the previous section does not exist-specifically only m can speak in the first period. A bad implant at other sites will have to wait to speak. For instance, a neighbor of m can speak in period 1 by pretending to have received a recommendation from m in period 0, a site at a distance of 2 from m can speak in period 2 and so on.

The specific structure of the network will determine whether the good product will diffuse throughout the network.

Theorem 4 *Suppose $\rho = 0$, Assumption S is satisfied*[10] *and the unique node maximizing decay centrality is both critical and is in S_1. Then, g cannot support an EDE.*

Proof. Let m be the unique node maximizing decay capacity. If an EDE exists, $\alpha_m = 1$. Also, if m is critical, then all of m's neighbors must accept m's recommendations. If some neighbor j does not accept m's recommendation with probability one, then the criticality of m implies that the good product will not diffuse to some segment of the network. Since $m \in S_1$, the type B firm can get $|S_1|$ acceptances by putting an implant at m. Of course, $\beta_m = 1$ is not possible since all neighbors of m would then not revise their beliefs about the product. On the other had,when $\rho = 0$, recommendations from no other site are credible initially. Hence, the expected payoff to type B from an implant at $i \neq m$ is strictly less than the payoff at m if the recommendation from m were credible. This shows that the type B firm does not have an equilibrium strategy. ■

This theorem of course immediately implies that when $\rho = 0$, the line or, more generally, a tree where the node maximizing decay capacity also has maximal adjusted degree cannot support an EDE. It is easy to construct examples of trees which support an EDE if m does not have maximal adjusted degree. Consider the following example.

Example 3 *Let $n \geq 9$ and n be odd. Let g be as follows. Individual 1 has just 2 links to 2 and 3. Divide the set $\{4, \ldots, n\}$ into two equal subsets, let*

[10] This assumption can be relaxed but is maintained here for ease of exposition.

2 be connected to all individuals in the first subset, each of whom have no other link. Similarly, let,individual 3 be connected to all agents in the second subset, each of whom have no other link.

Assume that

$$\delta \geq \max\left(\frac{4}{n-3}, \frac{n-5}{n-3}\right)$$

First, notice that if type G places his implant at 1 and all subsequent recommendations are accepted with probability one, then his payoff is $2\delta + (n-3)\delta^2 - c$. On the other hand, if he places his implant at 2 or 3, then his payoff is $(((n-3)/2)+1)\delta + \delta^2 + ((n-3)/2)\delta^3 - c$. The inequality $\delta \geq \frac{n-5}{n-3}$ ensures that the first sum is at least as large.

Second, assume also that p and $\bar{p}$ are such that the type B can put probability $1/2$ on each of the nodes 2 and 3, and still make a credible recommendation in period 1. That is, the neighbors of 2 and 3 have to infer whether sites 2 and 3 have received a recommendation from the implant of type G placed at 1 in the previous period or whether it is the type B implant who is speaking in period 1, having strategically kept silent in period 0. A probability weight of $\beta_2 = \beta_3 = 1/2$ brings their updated belief that the recommendation is being passed on from 1 reach the threshold,given the initial parameter values.

Then, the following is an EDE. The type G puts his implant at 1 with probability 1, the type B randomizes between 2 and 3 with equal probability. The type G implant speaks immediately while the type B implant speaks in period 1. All recommendations are accepted with probability one.

These constitute an equilibrium because if $\delta \geq \frac{4}{n-3}$, then the type B implant has no incentive to deviate and place his implant at 1 - he gets $\delta^2(n-3)/2 - c$ in equilibrium, whereas he would get $2\delta - c$ by placing his implant at 1. The response decisions are optimal because updated beliefs are not below the threshold.

How does a positive ρ impact on the possibility of efficient diffusion? Fix all other parameters and the network structure g. Now, ρ influences the nature of equilibrium in two ways. First, the higher the value of ρ, the lower is the net gain from having an implant for both firm types. So, there will be some value of $\bar{\rho}$ such that if $\rho \geq \bar{\rho}$,then one or both types of firm F will refrain from employing an implant.

Second, the value of ρ influences the updating process according to equation 1. Suppose ρ changes. How does this affect $\eta^1_{i,i-1}$ for a fixed value of β_i and distribution α? An increase in ρ makes it more likely that $i-1$ is an innovator, but also makes it less likely that none of the *other* neighbors of i are innovators. These effects move in opposite directions and an unambiguous answer is difficult to provide.

Suppose, however, that Assumption S holds, and an EDE exists where type B's equilibrium strategy places no probability weight on m, the the unique node maximizing decay centrality. Then, the trade-offs are somewhat easier to discern.

In this case, the expression for $\eta^1_{i,j}$ simplifies considerably for $j \neq m$ and becomes

$$\eta^1_{i,j} = \frac{p\rho(1-\rho)^{d_i-1}}{p\rho(1-\rho)^{d_i-1} + (1-p)\beta_i}$$

Now,

$$\text{sign } \frac{\partial \eta^1_{i,j}}{\partial \rho} = \text{ sign } \frac{\partial p\rho(1-\rho)^{d_i-1}}{\partial \rho}$$

Hence,

$$\frac{\partial \eta^1_{i,j}}{\partial \rho} \geq 0 \text{ iff } (1 - d_i\rho) \geq 0$$

So, if the initial value of ρ for which an equilibrium exists is "low", then an increase in ρ does not decrease $\eta^1_{i,j}$ and so the same strategies continue to be an equilibrium. On the other hand, an increase in ρ at higher values decreases $\eta^1_{i,j}$ and so the same strategy may not be an equilibrium. These suggest that there is some threshold value of ρ above which equilibrium of this type is not possible.

6 Inefficient Equilibria

From the previous sections, it is clear that some network structures may not support efficient diffusion with probability one. What is the nature of the equilibria in this case? We examine this issue in this section. We assume that the network structure is a line and that n is odd. These assumptions are not necessary but simplify the exposition quite considerably.

We now turn our attention to the problem mentioned in the first remark in the section above.

Recall that

$$S = N \setminus \{1, n, m-1, m+1, m-2, m+2\}$$

is the set of sites which can possibly have two consumers, and that an EDE may not be possible if S contains too few members for the type B firm to distribute its probability weight. There are two ways out of this non-existence problem. First, it is possible that the equilibrium acceptance probabilities of consumers are relatively low,and so the type B firm is indifferent between entering and not entering. Second, the implant of type B may decide to speak "over time" for strategic reasons, thereby allow it to assign greater probability weights on each node in S. We describe how to construct both type of equilibria.

Consider $i \in S$. Suppose $i = 2$, and there is an implant at 2. Suppose this implant at 2 makes a recommendation in period 2. Then, 3 can believe that site 1 is an innovator and has passed on a positive recommendation in period 1, which is then passed on by 2 in period 2. But, suppose 2 makes a recommendation in period 3. Then, 3 knows that 2 must be an implant who has not made a recommendation in an earlier period.

In general, suppose $i < m$, and is at a distance of k_l from 1 and k_r from m with $k_r < k_l$. Then, if i makes a recommendation in period $t = k_r$, then $i + 1$ will know that in the G case, a positive recommendation should have come from the right in no more $k_r - 1$ periods.

Hence, for each $i \in S_2$, there is a time period i_T up to which a bad implant can make recommendations and still be have a positive probability of getting her recommendation accepted by both her neighbors. Note that $m_T = 1$.

For each $i \in S$, let s_i^t denote the probability with which the implant at i makes a recommendation in period t for $1 \leq t \leq i_T$. In equilibrium, the implant mixes over the timing of the recommendation only if the sum of the acceptance probabilities of his neighbors in period $t - 1$ equals δ times the corresponding sum in period $(t + 1)$.

$$\eta_i^t = \frac{p\rho(1-\rho)^{2t-1}}{p\rho(1-\rho)^{2t-1} + (1-p)\beta_i s_i^t}$$

Using the values of η_i^t, assuming that each $\eta_i^t = \bar{p}$, we get

$$\bar{p}\left[p\rho(1-\rho)^{2t-1} + (1-p)\beta_i s i^t\right] = p\rho(1-\rho)^{2t-1}$$

So,

$$\beta_i s_i^t = \frac{p\rho(1-\rho)^{2t-1}(1-\bar{p})}{\bar{p}(1-p)}$$

Hence,

$$\frac{s_i^t}{s_i^{t-1}} = (1-\rho)^2$$

and so

$$s_i^t = (1-\rho)^{2t-1} s_i^1 \tag{3}$$

Since the sum of $\{s_i^t\}$ equals 1, this determines s_i^1.

Now, consider $i \notin S$. If $k_l < k_r$, then equation 3 continues to hold. If $k_r < k_l$, then since $\alpha_m = 1$, we get

$$\frac{s_i^T}{s_i^{T-1}} = (1-\rho)^2/\rho$$

Let

$$\tilde{\beta}_i s_i^1 = \bar{\beta}_i \tag{4}$$

Case 1: $\sum_{i \in N} \tilde{\beta}_i \leq 1$.

In this case, set $\beta_i = \tilde{\beta}_i$ for each $i \in N$. Adjust the acceptance probabilities so that the net payoff is zero. Then, $(1-\tilde{\beta}_i)$ is the probability that the bad implant does not enter.

Case 2: $\sum_{i \in N} \tilde{\beta}_i > 1$.

Let $S_1 = N \setminus S$. Write

$$\sum_{i \in N} \tilde{\beta}_i = \sum_{i \in S_2} \tilde{\beta}_i + \sum_{i \in S_1} \tilde{\beta}_i$$

Let $S_1^1 = \{i \in S_1 | i_T \geq j_T \text{ for all } j \in S_1\}$. For each $i \in S_1^1$, suppose $(r_i^1, \ldots, r_i^{i_T - 1})$ be such that $r_i^t > s_i^t$, while maintaining

$$r_i^t = (1-\rho)^2 r_i^1$$

Also, choose $\hat{\beta}_i$ so that

$$\hat{\beta}_i r_i^1 = \bar{\beta}_i$$

Since $r_i^1 > s_i^1$, $\hat{\beta}_i < \tilde{\beta}_i$.

Also, note that $\eta_i^{i_T} > \bar{p}$. Hence, i's recommendation must be accepted with probability one in this period. So, this gives the expected payoff as $\delta^{i_T - 1}$. Adjust all acceptance probabilities to give this expected payoff for i other periods and at other sites.

Now, r_i^1 can be raised continuously till $r_i^{i_T} = 0$. For each choice of r_i^1, look at the sum of $\sum_{i \in N \setminus S_1^1} \tilde{\beta}_i + \sum_{i \in S_1^1} \hat{\beta}_i$. If the sum is less than or equal to 1, then stop. Otherwise, repeat the process for nodes in S_1 to reduce the length over which they speak to $i_T - 1$, and so on. Stop if the corresponding sum is 1, or else take all $\hat{\beta}_i$ in S_1 to 0. Now, repeat the process for sites in S^2. Since $\hat{\beta}_i$ can be reduced to $\bar{\beta}_i$, there must be a point at which $\sum_{i \in S_2} \hat{\beta}_i \leq 1$.

7 Extensions

The model as presented here is, of course, a stylised one. We consider some possible extensions.

7.1 Negative recommendations

We have assumed that recommendations can only be positive. The intuition motivating this is that a person can observe someone using the product over some period of time; if she tries it and finds it unacceptable, she will discard it. To consider the chicken sausage example, someone who dislikes the particular brand will not display the package at pot luck parties with her neighbours, unless she is acting as an implant for the firm. Also, individuals may suffer a loss of face if they have to admit that they have been "duped" into buyng a bad product, and so may not pass on information about bad products.

However, one needs to consider negative recommendations as well if only to consider the robustness of the model. It is easy to check that theorems 2 and 3 continue to remain valid. Theorem 4 will remain valid for regular networks such as the *circle*.

It also turns out that the possibility of negative recommendations will actually simplify calculations in one respect in that the probability calculations would not now depend on the potential recipient's degree. So, the analogue of equation 1 will now be

$$\eta^1_{i,i-1} = \frac{p\left[\alpha_{i-1} + (1 - \sum_{j \in N_i} \alpha_j - \alpha_i)\rho\right]}{p\left[\alpha_{i-1} + (1 - \sum_{j \in N_i} \alpha_j - \alpha_i)\rho\right] + (1-p)\beta_{i-1}} \tag{5}$$

However, it would complicate expected payoff calculations for B, where a high degree recipient of an implant's (positive) recommendation would be more likely to have countervailing negative information than one of low degree. Such a problem would not arise for regular graphs, but the general issue is illustrated below.

So, suppose B places implants at i and j with some positive probability. Then, his expected payoff from i is

$$E_i = \sum_{k \in \bar{N}_i} (1-\rho)^{d_k - 1} P_k$$

where P_k is the probability with which the offer is accepted by k. The updated belief for all $k \in \bar{N}_i(g)$ will now be the same - it will just depend on β_i and not on the degrees of k.

So, since $E_i = E_j$, we need

$$\sum_{k \in \bar{N}_j} (1-\rho)^{d_k - 1} P_k = \sum_{k \in \bar{N}_i} (1-\rho)^{d_k - 1} P_k$$

The specification of a general sufficient condition is now more difficult because the derivation of the support of β is now more complicated. However, the qualitative result that a larger network is more conducive for efficient diffusion remains unchanged.

The possibility of negatiev recommendations may also help in sustaining efficient diffusion, particularly in dense networks where nodes have high degree. This is because the expected payoff of the type B firm will now be lower- and it will be lower the larger is the degrees of different sites. So, the type B firm may simply not employ implants.

7.2 Multiple implants

If the firm can choose multiple implants, the qualitative features of the analysis will be similar. Clearly it does not make sense for the multiple implants to have overlapping supports (for the firm's randomised strategies). This suggests that for large networks, the firm will partition the networks in such a way as to have one implant randomly located (for B) in each element of the partition. If δ is close to 1 and an EDE exists as above, there is very little incentive for G to incur the cost of an additional implant, since this can only speed up the diffusion and the benefit from this might be low compared to the cost. Therefore, for low discounting, we would expect to have several B implants but only one G implant. This suggests that the B implants would either have to rely on a relatively high ρ for credibility or speak only at sufficiently late time periods for the message from a supposed good implant to have reached the particular site concerned.

8 Conclusions

Our motivation was to explore some of the implications of viral marketing within a social network. Consumers are aware that firms may "seed" the network, and also know that both "good" and "bad" quality firms may take recourse to this form of advertising. In the presence of imperfections regarding the quality of a specific product, consumers cannot take recommendations from their social neighbours at face value - the credibility of recommendations is at stake. A crucial ingredient of our analysis is that customers are rational, and update beliefs using Bayes Rule. Within this framework, we show that a priori notions about what network structure is conducive to efficient diffusion may be misleading. In particular, "small" networks and highly-connected agents may actually deter the diffusion of the good product.

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