Approval Voting and Arrow's Impossibility Theorem

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October 25, 2011

Abstract

Approval voting has attracted considerable interest among voting theorists, but they have rarely investigated it in the Arrovian framework of social welfare functions (SWF) and never connected it with Arrow's impossibility theorem. This note explores these two directions. Assuming that voters have dichotomous preferences, it first characterizes approval voting in terms of its SWF properties and then shows that these properties are incompatible if the social preference is also taken to be dichotomous. The positive result improves on some existing characterizations of approval voting in the literature, as well as on Arrow's and May's classic analyses of voting on two alternatives. The negative result corresponds to a novel and perhaps surprising version of Arrow's impossibility theorem.

JEL Classification Numbers: D71.

1 Introduction

Approval voting is the rule by which voters can cast votes for as many candidates as they wish, giving no more than one vote to each of them, and those candidates with the greatest vote total are elected. For the purpose of theoretical investigations, approval voting has been defined to be a *social*

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choice function, i.e., a mapping that associates a nonempty subset of candidates to every profile of individual preferences among these candidates. This formalism is suitable for voting rules generally because it connects well with strategic inquiries and the landmark Gibbard-Satterthwaite theorem. It has been applied to approval voting very effectively. Since Brams and Fishburn (1978) and Fishburn (1978) initiated the theoretical work on approval voting, it has mostly been concerned with strategy-proofness and related properties (see the surveys by Brams and Fishburn, 2002, and Xu, 2010). The present note departs from this tradition by defining approval voting as *social welfare function*, or as we will rather call it to avoid irrelevant welfaristic suggestions, a *collective preference function*, that is to say, a mapping that associates a collective preference among candidates to every profile of individual preferences among these candidates. This is of course Arrow's original formalism in social choice theory, by which he thought it possible to capture not only welfare criteria, but also voting rules (see 1963, ch. V). By adopting it here, we shift the theoretical focus from strategy-proofness to the related, but distinct condition of independence of irrelevant alternatives, as well as further Arrovian conditions such as positive responsiveness and the Pareto principle.

When approval voting is defined in terms of collective preference, rather than collective choice, it is natural to impose the restriction that individual preferences be *dichotomous*, i.e., have at most two indifference classes. This restriction will be upheld throughout, with two possibilities being investigated in turn: the *simply dichotomous* case, where only individual preferences are dichotomous, and the *twice dichotomous* case, where collective preferences also are. We present a characterization of approval voting in the former case, and one of dictatorship - thus an impossibility theorem - in the latter case. Although the conditions are not the same in the two results, we may interpret them as saying that the conditions for approval voting lead to an inconsistency when the domain and range are similarly restricted. The positive result connects with some characterizations of approval voting, especially Ju's (2011), which is the only one to employ a collective preference framework like ours, but we arguably go one step further into the Arrovian foundations of the voting rule. The negative result appears to be unprecedented and surprising when it is compared with the vast technical literature on Arrovian impossibilities.

2 Definitions and aggregative conditions

As usual, a *weak preference ordering* R means a binary relation that is transitive, reflexive and complete; equivalently R has an asymmetric part P and a symmetric part I, which satisfy the PP, PI, IP, II variants of transitivity. The statements xPy, xIy, xRy have their standard readings, i.e., "x is strictly preferred to y", "x is indifferent to y", "x is strictly preferred or indifferent to y". An *indifference class for* R is one of the equivalence classes generated by I, i.e., a set of the form $\{x \in X : xIx_0\}$ for some fixed $x_0 \in X$.

As a particular case of R, a *dichotomous weak preference ordering* has one or two indifference classes. It satisfies the PP variant of transitivity vacuously and (for a sufficient number of elements) the PI, IP and II variants non-vacuously. It can also be described in terms of its indifference classes directly. If R has two indifference classes, we denote by H and L the higher and lower one, respectively, and if R has only one indifference class, a case of complete indifference, we denote this class by C. The obvious translation rules are:

$$\begin{aligned} xPy &\Leftrightarrow x \in H, y \in L \\ xIy &\Leftrightarrow \text{either } x, y \in H \text{ or } x, y \in L \text{ or } x, y \in C. \end{aligned}$$

Define $\mathcal{O}$ to be the set of all weak preference orderings on X, and $\mathcal{D} \subset \mathcal{O}$ to be the set of dichotomous weak preference orderings on X.

The technical literature sometimes defines dichotomous preferences to have exactly two indifference classes, thus excluding complete indifference (e.g., Brams and Fishburn, 1978 and 2002). We show below that our results extends to this stronger definition at a modest cost.

Given a set of candidates X with $\mid X \mid \geq 3$ and a finite set of voters $N = \{1, ..., n\}$ with $n \geq 2$, we define a *collective preference function* (*CPF*) to be a mapping

$$F : (R_1, ..., R_n) \longmapsto R.$$

Throughout, the domain of F will be dichotomous, with two cases considered in succession, i.e., $F : \mathcal{D}^n \to \mathcal{O}$ and $F : \mathcal{D}^n \to \mathcal{D}$; we call them the *simply dichotomous* and *twice dichotomous* case, respectively.

Correspondingly, the *CPF* associated with approval voting can be stated in two different ways.

Notation 1 *For any profile* $(R_1, ..., R_n)$ *in* $\mathcal{O}$ *or* $\mathcal{D}$, *we put* $N(xP_iy) = \{i : xP_iy\}$ *and* $n(xP_iy) = |\{i : xP_iy\}|$ *(similarly for* R_i, I_i*), and* $N(x \in H_i) =$

$\{i : x \in H_i\}$ *and* $n(x \in H_i) = |\{i : x \in H_i\}|$ *(similarly for* L_i*,* C_i*), and we put*

$$Max(R_1, ..., R_n) = \{x \in X : N(x \in H_i) \geq N(y \in H_i), \forall y \in X\}.$$

Definition 1 *F is approval voting* if, for all* $(R_1, ..., R_n) \in \mathcal{D}^n$*,* $xPy \Leftrightarrow n(x \in H_i) > n(y \in H_i)$*.*

Approval voting* trivially defines an ordering for any profile, and it has range $\mathcal{O}$, not $\mathcal{D}$, if $|X| \geq 3$. However, we may force the $\mathcal{D}$ range in redefining the *CPF* thus.

Definition 2 *F is approval voting** if, for every* $(R_1, ..., R_n) \in \mathcal{D}^n$*,*

- *if* $Max(R_1, ..., R_n) \neq X$*, then*

$$H = Max(R_1, ..., R_n) \text{ and } L = X \diagdown Max(R_1, ..., R_n);$$

- *if* $Max(R_1, ..., R_n) = X$*, then* $R = CI$*.*

It is trivial, but useful to notice that under dichotomous individual preferences, plurality voting has the same *CPF* as approval voting*. Arrow (1963, p. 58) defined the *CPF* of *majority voting* by the condition that, for all $(R_1, ..., R_n)$ in the domain, xRy if

$$n(xR_iy) \geq n(yR_ix), \text{ or equivalently } n(xP_iy) \geq n(yP_ix),$$

and this can be taken to define the *CPF* of *plurality voting* when there are more than two alternatives. If the domain is $\mathcal{D}^n$ rather than $\mathcal{O}^n$, this definition collapses into that just given for approval voting*. Since the latter always defines an ordering, we have a quick proof of the well-known fact - first observed by Inada (1964) - that plurality voting is transitive when individual preferences are dichotomous.

Now to the properties that *CPF* may satisfy. We begin by listing those which Arrow (1963) famously declared to be mutually inconsistent.

Condition 1 *Independence of irrelevant alternatives (IIA): For all* $(R_1, ..., R_n)$*,* $(R'_1, ..., R'_n) \in \mathcal{D}^n$ *and all* $x, y \in X$*, if* $xR_iy \Leftrightarrow xR'_iy$ *and* $yR_ix \Leftrightarrow yR'_ix$ *for all* $i \in N$*, then* $xRy \Leftrightarrow xR'y$*.*

Condition 2 *Weak Pareto (WP): For all* $(R_1, ..., R_n) \in \mathcal{D}^n$ *and all* $x, y \in X$*, if* xP_iy *for all* $i \in N$*, then* xPy.

Condition 3 *Non-dictatorship (ND): There is no* $j \in N$ *such that for all* $(R_1, ..., R_n) \in \mathcal{D}^n$ *and all* $x, y \in X$*, if* xP_jy*, then* xPy.

Both approval voting* and approval voting** satisfy *WP*. However, only the former, not the latter, satisfies *IIA*. The following 3-alternative, 2-individual profiles (R_1, R_2), (R_1', R_2') illustrates the failure: xP_1yI_1z, zP_2yI_2x and $xI_1'z$ $P_1'y$, $zP_2'yI_2'x$; by approval voting**, $x, z \in H, y \in L$, so xPy, and $z \in H', x, y \in L'$, so $xI'y$, contradicting *IIA*. Since approval voting* satisfies *ND*, it follows that Arrow's impossibility theorem fails because of the dichotomous domain (this is in effect Inada's observation). The existence of approval voting** also fits with Arrow's impossibility theorem, but this time, because IIA is violated. The final section will show that this violation is not accidental, but forced by the unusual range restriction.

3 The collective preference function of approval voting

Here we characterize approval voting* in terms of slightly reinforced Arrovian conditions. First, this function obviously satisfies a vast strengthening of *ND*:

Condition 4 *Anonymity (A): For all* $(R_1, ..., R_n) \in \mathcal{D}^n$ *and all permutations* σ *of* $\{1, ..., n\}$, $F(R_1, ..., R_n) = F(R_{\sigma(1)}, ..., R_{\sigma(n)})$.

Second, approval voting* satisfies Pareto conditions besides *WP*:

Condition 5 *Pareto indifference (PI): For all* $(R_1, ..., R_n) \in \mathcal{D}^n$ *and all* $x, y \in X$*, if* xI_iy *for all* $i \in N$*, then* xIy.

Condition 6 *Strict Pareto (SP): For all* $(R_1, ..., R_n) \in \mathcal{D}^n$ *and all* $x, y \in X$*, if* xR_iy *for all* $i \in N$ *and* xP_iy *for some* i*, then* xPy.

It turns out that approval voting* can be characterized by adding *IIA* to these three conditions, or rather to the first two and *WP*, because this set will be shown to entail *SP*.

Theorem 1 *A collective preference function* $F : \mathcal{D}^n \to \mathcal{O}$ *is approval voting* if and only if it satisfies IIA, A, PI, and WP.*

The sufficiency part relies on three lemmas. The first derives a classic condition that strengthens *IIA* and is sometimes taken to be primitive by *CPF* theory.

Condition 7 *Neutrality (N): For all* $(R_1, ..., R_n)$, $(R'_1, ..., R'_n) \in \mathcal{D}^n$ *and all* $x, y, z, w \in X$, *if* $xR_iy \Leftrightarrow zR'_iw$ *and* $yR_ix \Leftrightarrow wR'_iz$ *for all* $i \in N$, *then* $xRy \Leftrightarrow zR'w$.

Lemma 1 *If* $F : \mathcal{D}^n \to \mathcal{O}$ *satisfies IIA and PI, it satisfies N.*

Notation 2 *Instead of* ($xR_iy \Leftrightarrow xR'_iy$ *and* $yR_ix \Leftrightarrow yR'_ix$) *and of* ($xRy \Leftrightarrow xR'y$ *and* $yRx \Leftrightarrow yR'x$), *we write* $xR_iy \approx xR'_iy$ *and* $xRy \approx xR'y$.

Proof. Consider first the case of four distinct $x, y, z, w \in X$. By assumption, $(R_1, ..., R_n)$ and $(R'_1, ..., R'_n) \in \mathcal{D}^n$ are s.t. $xR_iy \approx zR'_iw$ for all $i \in N$. Take $(\overline{R}_1, ..., \overline{R}_n) \in \mathcal{D}^n$ s.t. $xR_iy \approx x\overline{R}_iy$ for all $i \in N$, and s.t. $x\overline{I}_iz$ and $y\overline{I}_iw$ for all $i \in N$. Thus, by construction, $z\overline{R}_iw \approx zR'_iw$ for all $i \in N$. Suppose that xRy. Then, $x\overline{R}y$ follows from *IIA*, $z\overline{R}w$ from *PI*, and finally $zR'w$ from *IIA*.

Related proofs take care of the two cases in which there are three distinct elements among $x, y, z, w \in X$, $x \neq y$, and the position of the common element is the same in the two pairs, i.e., $x = z$ or $y = w$. Now, suppose that the common element changes position, i.e., $x = w$ or $y = z$. We give a proof for the former case. By assumption, $(R_1, ..., R_n)$ and $(R'_1, ..., R'_n) \in \mathcal{D}^n$ are s.t. $xR_iy \approx zR'_ix$ for all $i \in N$. Take $(\overline{R}_1, ..., \overline{R}_n) \in \mathcal{D}^n$ s.t. $xR_iy \approx z\overline{R}_iy$ for all $i \in N$. From one of the cases with unchanged positions, $xRy \Leftrightarrow z\overline{R}y$. By construction, $zR'_ix \approx z\overline{R}_iy$ for all $i \in N$, so from the other case, $zR'x \Leftrightarrow z\overline{R}y$, and finally $xRy \Leftrightarrow zR'x$.

If there are three distinct elements among $x, y, z, w \in X$, and $x = y$, or $z = w$, N reduces to *PI*.

If there are two distinct elements, say x and y, which do not exchange positions, N reduces to *IIA*. Otherwise, suppose that $(R_1, ..., R_n)$ and $(R'_1, ..., R'_n) \in \mathcal{D}^n$ are s.t. $xR_iy \approx yR'_ix$ for all $i \in N$. Take $z \neq x, y$ and $(\overline{R}_1, ..., \overline{R}_n) \in \mathcal{D}^n$ s.t. $xR_iy \approx x\overline{R}_iz$ for all $i \in N$. It follows that $xRy \Leftrightarrow x\overline{R}z$. Take $(\overline{\overline{R}}_1, ..., \overline{\overline{R}}_n) \in \mathcal{D}^n$ s.t. $yR'_ix \approx y\overline{\overline{R}}_iz$ for all $i \in N$. It follows that $yR'x \Leftrightarrow y\overline{\overline{R}}z$. Now, by construction, $x\overline{R}_iz \approx y\overline{\overline{R}}_iz$ for all $i \in N$, whence $x\overline{R}z \Leftrightarrow y\overline{\overline{R}}z$. Combining the equivalences, one gets $xRy \Leftrightarrow yR'x$, as desired. ∎

The second lemma derives another condition that is also sometimes taken to be primitive. We give it in two versions, the former being weaker than the latter.

Condition 8 *Positive responsiveness 1 (PR1): For all* $(R_1, ..., R_n)$, $(R'_1, ..., R'_n) \in \mathcal{D}^n$ *and all* $x, y \in X$, *if* $xP_iy \Rightarrow xP'_iy$ *and* $xI_iy \Rightarrow xR'_iy$ *for all* $i \in N$, *and* yP_ix *and* xR'_iy, *or* xI_iy *and* xP_iy, *for some* i, *then* $xPy \Rightarrow xP'y$.

Condition 9 *Positive responsiveness 2 (PR2): For all* $(R_1, ..., R_n)$, $(R'_1, ..., R'_n) \in \mathcal{D}^n$ *and all* $x, y \in X$, *if* $xP_iy \Rightarrow xP'_iy$ *and* $xI_iy \Rightarrow xR'_iy$ *for all* $i \in N$, *and* yP_jx *and* xR'_jy, *or* xI_jy *and* xP_jy, *for some* i, *then* $xRy \Rightarrow xP'y$.

Lemma 2 *If* $F : \mathcal{D}^n \to \mathcal{O}$ *satisfies N and SP, it satisfies PR2.*

Proof. To derive *PR2*, we first assume that $(R_1, ..., R_n)$, $(R'_1, ..., R'_n) \in \mathcal{D}^n$ and $x, y \in X$ meet the antecedent condition without any full reversal of strict preference, i.e., without any j s.t. yP_jx and xP'_jy.

Take $z \neq x, y$ and $(\overline{R}_1, ..., \overline{R}_n) \in \mathcal{D}^n$ so defined: for all $i \in N$,

- if xP_iy and xP'_iy, then $x\overline{P}_iy\overline{I}_iz$; if yP_ix and yP'_ix, then $z\overline{I}_iy\overline{P}_ix$; if xI_iy and xI'_iy, then $x\overline{I}_iy\overline{I}_iz$;
- if yP_ix and xI'_iy, then $z\overline{P}_ix\overline{I}_iy$; if xI_iy and xP'_iy, then $z\overline{I}_ix\overline{P}_iy$.

Thus, for all $i \in N$, $xR_iy \approx x\overline{R}_iz$, $z\overline{R}_iy$, and $xR'_iy \approx x\overline{R}_iy$. We also note by inspecting the possibilities that for some i, $z\overline{P}_iy$. Now, suppose that xRy. Then, $x\overline{R}z$ by *N*, and because *SP* entails that $z\overline{P}y$, it follows that $x\overline{P}y$, hence $xP'y$ by *IIA*. This completes the proof of *PR2* in the case just considered.

If $(R_1, ..., R_n)$, $(R'_1, ..., R'_n) \in \mathcal{D}^n$ and $x, y \in X$ meet the antecedent of *PR2* in full generality, take $(R''_1, ..., R''_n) \in \mathcal{D}^n$ as in the the remaining part of the proof of *PR1*. Suppose that xIy. By the case just proved, $xP''y$, and again by this case, $xP'y$. ■

The third lemma shows that *SP* can be replaced by *WP* in the characterization of approval voting*.

Lemma 3 *If* $F : \mathcal{D}^n \to \mathcal{O}$ *satisfies N, WP and A, it satisfies SP.*

Proof. Fix x, y and a sequence of profiles $(R^k_1, ..., R^k_n) \in \mathcal{D}^n$, $k \in \{1, ..., n-1\}$, s.t.

$$xP^k_iy, 1 \leq i \leq k \text{ and } xI^k_iy, k+1 \leq i \leq n.$$

In view of *N* and *A*, it is sufficient to show that *SP* holds for this pair and this sequence. The proof goes by induction on k.

- $k = 1$. Suppose that yR^1x; we reach a contradiction with *WP* by showing that yR^lx holds for all $l \in \{1, ..., n\}$, hence in particular for $l = n$. The initial supposition covers the case $l = 1$; now suppose we have proved that yR^lx for some $l \in \{1, ..., n-1\}$. Let us take the profiles $(R_1^{l+1}, ..., R_n^{l+1}) \in \mathcal{D}^n$, and for some $z \neq x, y$, $(\overline{R}_1^{l+1}, ..., \overline{R}_n^{l+1}) \in \mathcal{D}^n$ s.t.

$$\begin{aligned} x\overline{R}_i^{l+1}y &\approx xR_i^{l+1}y \text{ for all } i \in N, \\ z\overline{I}_i^{l+1}y \text{ for all } i &\neq l+1 \text{ and } z\overline{I}_{l+1}^{l+1}x\overline{P}_{l+1}^{l+1}y. \end{aligned}$$

  Then, we apply N in a comparison with $(R_1^l, ..., R_n^l)$, and N and A in a comparison with $(R_1^1, ..., R_n^1)$, to get $z\overline{R}^{l+1}x$ and $y\overline{R}^{l+1}z$, hence $y\overline{R}^{l+1}x$. So we have proved by induction that yR^lx holds for all $l \in \{1, ..., n\}$, as desired.

- Suppose *SP* holds for $(R_1^k, ..., R_n^k)$ and consider $(R_1^{k+1}, ..., R_n^{k+1})$. In the previous definition of $(\overline{R}_1^{l+1}, ..., \overline{R}_n^{l+1})$, replace l by k; this defines a profile $(\overline{R}_1^{k+1}, ..., \overline{R}_n^{k+1})$ that, by comparison with $(\overline{R}_1^k, ..., \overline{R}_n^k)$, satisfies $x\overline{P}^{k+1}z$ by *SP*, and by comparison with $(\overline{R}_1^1, ..., \overline{R}_n^1)$, satisfies $z\overline{P}^{k+1}y$ by A and *SP*. Hence $x\overline{P}^{k+1}y$, as was to be proved.

■

Proof. (End) Suppose that there is some $F : \mathcal{D}^n \to \mathcal{O}$ that is not approval voting*. Then, there are $(R_1, ..., R_n) \in \mathcal{D}^n$ and $x, y \in X$ s.t. either (i) $n(x \in H_i) = n(y \in H_i)$ and xPy, or (ii) $n(x \in H_i) > n(y \in H_i)$ and yRx.

In case (i), there are three groups of individuals, i.e., $N(xP_iy)$, $N(yP_ix)$, $N(xI_iy)$ with $n(xP_iy) = n(xP_iy)$. The first two groups are non-empty by *PI*. We may take a permutation σ that interchanges them and leaves the third group unchanged; by A, the resulting profile $(R_{\sigma(1)}, ..., R_{\sigma(n)})$ has the collective preference $xP_\sigma y$. Now, observing that for all $i \in N$, $xR_iy \approx yR_{\sigma(i)}x$, we apply N to the profile to get the contradiction that $yP_\sigma x$.

In case (ii), the three groups of individuals $N(xP_iy), N(yP_ix), N(xI_iy)$ are s.t. $n_1 = n(xP_iy) > n_2 = n(xP_iy)$. The second group is non-empty by *SP*, and from the inequality, the first group also is. Take a permutation σ that interchanges n_2 individuals in $N(xP_iy)$ with those in $N(yP_ix)$ and leaves the position of any others unchanged; by A, the resulting profile $(R_{\sigma(1)}, ..., R_{\sigma(n)})$ has the collective preference $yR_\sigma x$. Now, modify this profile into $(R_1', ..., R_n')$

by putting $yP_i'x$ if $i = \sigma(i)$ is any of the remaining $n_1 - n_2$ individuals of $N(xP_iy)$ and leaving any other individual's preference the same. Given this reinforcement of strict preference for y, *PR2* entails that $yP'x$. However, $(R_1', ..., R_n')$ also modifies $(R_1, ..., R_n)$ in such a way that N entails that $xR'y$, a contradiction. ■

Let us briefly check the logical independence of each condition in Theorem 1. That *PI* is independent of the others follows from considering the *CPF* corresponding to approval voting* complemented with any tie-breaking method; such a function will satisfy all conditions but *PI*. Similarly, that *IIA* is independent of the others follows from considering the *CPF* that corresponds to equal cumulative voting, which is another refinement of approval voting: each voter has one vote and divides it evenly among the candidates he approves of; candidates with the greatest total of fractional votes are elected. The independence of A results from considering dictatorial *CPF*.

Inspection of the proofs shows that the complete indifference ordering $CI \in \mathcal{D}$ does not occur as an auxiliary profile unless$\mid X \mid = 3$ (see the relevant step in the proof of Lemma 2). Thus, if we put $\mathcal{D}^- = \mathcal{D} \setminus \{CI\}$, we have an immediate extension of the theorem.

Remark 1 *If* $\mid X \mid \geq 4$*, the theorem also holds for* $F : (\mathcal{D}^-)^n \to \mathcal{D}$.

With this extension at hand, we can redefine dichotomous preferences to have *exactly* two indifference classes, as in some of the earlier work on approval voting (e.g., Brams and Fishburn, 1978, 2002).

To the best of our knowledge, only Ju (2011, Theorem 2) has characterized approval voting in terms of *CPF*. Assuming the simply dichotomous framework, he shows that plurality voting, hence implicitly approval voting*, is that *CPF* which satisfies *A*, *N*, plus a positive responsiveness and a non-trivality condition, to be defined now.

Condition 10 *Positive responsiveness 3 (PR3): For all* $(R_1, ..., R_n)$, $(R_1', ..., R_n') \in \mathcal{D}^n$ *and all* $x, y \in X$, *if* $xP_iy \Rightarrow xP_i'y$ *and* $xI_iy \Rightarrow xR_i'y$ *for all* $i \in N$, *and* yP_ix *and* $xR_i'y$, *or* xI_iy *and* xP_iy, *for some* i, *then* $xRy \Rightarrow xR'y$.

Condition 11 *Non-triviality (NT):* F *does not have the constant value* CI.

PR3 is weaker than *PR2* and by itself incomparable with *PR1*. However, we can obtain Ju's significant characterization from ours by an added step of reasoning. Since *N* entails *IIA* and *PI*, it is sufficient to show that *N*, *PR3*, and *NT* entail *WP*.

Corollary 1 *(Ju, 2011) A collective preference function $F : \mathcal{D}^n \to \mathcal{O}$ is approval voting* if and only if it satisfies N, A, PR3 and NT.*

Proof. To be shown that WP holds. Given $(R_1, ..., R_n) \in \mathcal{D}^n$ and $x, y \in X$ satisfying xP_iy, $i \in N$, take $z \neq x, y$, $(R'_1, ..., R'_n) \in \mathcal{D}^n$ s.t. $zI'_ixP'_iy$, $i \in N$, and $(R''_1, ..., R''_n) \in \mathcal{D}^n$ s.t. $zP''_ixI''_iy$, $i \in N$. N (or PI) leads to $xI''y$, $PR3$ to $xR'y$, and IIA to xRy. It remains to exclude that xIy. If this indifference holds, we can interchange x and y in $(R_1, ..., R_n)$ and obtain $(\overline{R}_1, ..., \overline{R}_n) \in \mathcal{D}^n$ with $y\overline{P}_ix$, $i \in N$, and thus $y\overline{I}x$ by N. Now, take any profile whatever: by an application of $PR3$ based on either $(R_1, ..., R_n)$ or $(\overline{R}_1, ..., \overline{R}_n)$, it follows that x and y are collectively indifferent. This would hold for any choice of $x, y \in X$, hence violate NT. ■

Conversely, one may recover our characterization from Ju's once Lemmas 1 and 2 have been established - his theorem then playing the role of Lemma 3 and the End of Proof. As this further comparison suggests, the two equivalent sets of conditions for approval voting* are not quite at the same axiomatic level, ours going one step further into the preference-theoretic analysis of approval voting.

The other available characterizations of this voting rule are choice-theoretic, hence not directly comparable with ours. However, Vorsatz (2007, Theorem 1) has elaborated on Fishburn's (1978) in the simply dichotomous case and characterized the social choice function of approval voting* in terms of four conditions, i.e., Anonymity, Neutrality, Strategyproofness and Strict monotonicity, which are clearly reminiscent of Ju's. Roughly, Neutrality corresponds with *N*, Strategyproofness with *IIA*, and Strict monotonicity with *PR3*.

On a different tack, a comparison is in order with May's (1952) classic result on majority voting on two candidates. Expanding on Arrow's "possibility theorem" for this case, May characterized it by three conditions, which are *N*, *A* and *PR2* when translated into the present framework. This is a modest corollary to Theorem 1. As the latter makes clear, the relevant cardinality restriction bears on the voters' sets of equivalence classes, and not on the set of candidates.

4 From approval voting to dictatorship

In this section, we shift to the twice dichotomous case and demonstrate that the conditions characterizing approval voting* in the simply dichotomous case now characterize dictatorship. We actually prove the more powerful result that Arrow's theorem with its initial conditions holds in the twice dichotomous case.

Proposition 1 *No collective preference function $F : \mathcal{D}^n \to \mathcal{D}$ satisfies IIA, WP and ND.*

The proof goes through four lemmas.

Condition 12 *Pareto Preference (PP): For all $(R_1, ..., R_n) \in \mathcal{D}^n$ and all $x, y \in X$, if xR_iy for all $i \in N$, then xRy.*

Lemma 4 *If a collective preference function $F : \mathcal{D}^n \to \mathcal{D}$ satisfies IIA and WP, it satisfies PI and PP.*

Proof. Consider $(R_1, ..., R_n) \in \mathcal{D}^n$ and $x, y \in X$ s.t. xR_iy for all $i \in N$. Take $z \neq x, y$ and $(R'_1, ..., R'_n) \in \mathcal{D}^n$ s.t. xP'_iz and $xR_iy \approx xR'_iy$. (If xP_iy, $y \in L'_i$ and if xI_iy, $y \in H'_i$.) If xI_iy for all $i \in N$, then xI'_iy for all $i \in N$. In this case, *WP* entails that $xP'z$ and $yP'z$, so $xI'y$ since F has range $\mathcal{D}$, and xIy by *IIA*. This completes the derivation of *PI*. In the general case where xR_iy for all $i \in N$, *WP* entails that $xP'z$, and the $\mathcal{D}$ range that $xR'y$, whence xRy by *IIA.* This completes the derivation of *PP*. ■

Lemma 5 *If a collective preference function $F : \mathcal{D}^n \to \mathcal{D}$ satisfies IIA and PI, it satisfies N.*

Proof. Same as for Lemma 1. (The $\mathcal{D}$ range plays no role in this proof.) ■

Lemma 6 *If a collective preference function $F : \mathcal{D}^n \to \mathcal{D}$ satisfies N and WP, it satisfies PR1.*

Proof. As in the proof of Lemma 2, we begin by assuming that $(R_1, ..., R_n)$, $(R'_1, ..., R'_n) \in \mathcal{D}^n$ and $x, y \in X$ meet the antecedent condition without any full reversal of strict preference. With $z \neq x, y$ and $(\overline{R}_1, ..., \overline{R}_n) \in \mathcal{D}^n$ as defined in this proof, we have again that, for all $i \in N$, $xR_iy \approx x\overline{R}_iz$, $z\overline{R}_iy$, and $xR'_iy \approx x\overline{R}_iy$. Since N implies *IIA*, *PP* holds by Lemma 4, hence $z\overline{R}y$. Now if xPy, then $x\overline{P}z$ by N, and from the last fact, $x\overline{P}y$, whence $xP'y$ by *IIA*. The end of the proof parallels that of Lemma 2. ■

A group $M \subseteq N$ is said to be *decisive* if for all pairs $(x, y) \in X^2$ and all profiles $(R_1, ..., R_n) \in \mathcal{D}^n$,

$$xP_iy, i \in M \Longrightarrow xPy,$$

and it is said to be *semi-decisive on the pair* $(x, y) \in X^2$ *in the profile* $(R_1, ..., R_n) \in \mathcal{D}^n$ if

$$xP_iy, i \in M, \; yR_ix, i \in N \setminus M, \text{ and } xPy.$$

Lemma 7 *If a collective preference function* $F : \mathcal{D}^n \to \mathcal{D}$ *satisfies N and PR1, any group* $M \subseteq N$ *that is semi-decisive on a pair* $(x, y) \in X^2$ *in a profile* $(R_1, ..., R_n) \in \mathcal{D}^n$ *is decisive.*

Proof. Take M as specified, partition $N \setminus M$ into $G_1 = \{i \mid yP_ix\}$ and $G_2 = \{i \mid yI_ix\}$, and consider the profiles $(R'_1, ..., R'_n) \in \mathcal{D}^n$ s.t.

$$xP'_iy, i \in M, \; yP'_ix, i \in G_1, \text{ and } yP'_ix \text{ for at least one } i \in G_2.$$

If we show that M is semi-decisive on the given (x, y) in any such profile $(R'_1, ..., R'_n)$, the conclusion that M is decisive will follow by *PR1* and N.

To establish the desired property that $xP'y$, take $z \neq x, y$ and $(R''_1, ..., R''_n)$ s.t. for all $i \in N$, $zR''_iy \approx xR_iy$ and $xR''_iy \approx xR'_iy$. N entails that $zP''y$ and would entail $yP''x$ if $yP'x$ held, but this is prohibited by the $\mathcal{D}$ range. Hence $xR'y$. It remains to show that $xI'y$ is impossible.

If $xI'y$, then $xI''y$ by N, and $zP''x$ follows. From *IIA*, any profile $(R'''_1, ..., R'''_n) \in \mathcal{D}^n$ s.t. for all $i \in N$, $zR'''_ix \approx zR''_ix$ satisfies $zP'''x$. The $\mathcal{D}^n$ domain contains such a profile with $xR'''_iy \approx xR_iy$ for all $i \in N$. (For a proof, see the construction:

- $x, z \in H'''_i$ and $y \in L'''_i$ if $i \in M$;
- $y \in H'''_i$ and $x, z \in L'''_i$ if $i \in G_1$;
- $xI'''_iyI'''_iz$ if $i \in G_2$ and xI'_iy;
- $z \in H'''_i$ and $x, y \in L'''_i$ if $i \in G_2$ and yP'_ix.)

Now, *IIA* entails $xP'''y$, which is impossible given the $\mathcal{D}$ range. This completes the proof that $xP'y$. ■

Define $V \subseteq 2^N$ to be the set of all decisive groups. To contradict *ND* is tantamount to showing that V contains a singleton, and we proceed to this last stage of the argument.

Proof. (End) The set V is non-empty because $N \in V$ in virtue of *WP*. Since N is finite, there exists in V a group of smallest cardinality M^*, which cannot be $\varnothing$ in virtue of *PP*, which is secured by Lemma 4. With Lemmas 5 and 6 at hand, the conclusion of Lemma 7 follows. We use it to show that M^* is a singleton.

Suppose by way of contradiction that $\mid M^* \mid \geq 2$, so that M^* can be partitioned into two non-empty groups M_1^*, M_2^*. We take $x, y, z \in X$ and a profile $(R_1, ..., R_n) \in \mathcal{D}^n$ with the following properties:

- for all $i \in M_1^*$, xP_iyI_iz;
- for all $i \in M_2^*$, zI_ixP_iy;
- for all $i \in N \setminus M^*$, $yP_i'xI_i'z$ if yP_ix and $yI_i'xP_i'z$ if yI_ix.

It follows, first, that xPy because $M^* = M_1^* \cup M_2^*$ is decisive, and second, that $yR'z$ because $zP'y$ would mean that M_2^* is semi-decisive on (z, y) in $(R_1, ..., R_n)$, hence decisive, and this would contradict the minimality of M^*. The two conclusions entail that $xP'z$, but this would mean that M_1^* is semi-decisive on (x, z) in $(R_1, ..., R_n)$, hence decisive, another contradiction with the minimality of M^*. ■

The formal argument has gone through the three stages of neutrality, positive responsiveness, and finally dictatorship, which are familiar from early proofs in social choice theory, but dichotomous individual preferences do not support those parts of the proofs which make active use of PP-transitivity, so that each stage here needs a special proof based on either the domain $\mathcal{D}^n$ or both this domain and the range $\mathcal{D}$ (in Lemmas 4 and 7). Relatedly, there is no use here for the *free triple property*, which is the main sufficient condition on an individual preference domain to derive an Arrovian impossibility. Social choice theory rarely, if ever, considers restrictions put on both the domain *and* range of the *CPF*; see the authoritative surveys by Gaertner (2002) and Le Breton and Weymark (2011).

At virtually each stage, we make flexible use of the voters' indifference relation, i.e., we interpret xI_iy as being sometimes $x, y \in H_i$, sometimes $x, y \in L_i$, depending on what is to be proved. The dictatorial conclusion would collapse without this convenience. To illustrate, let us add the restriction that H_i be a singleton, calling $\mathcal{D}^S$ the subset of weak dichotomous orderings that satisfies it. There exist non-dictatorial $F : (\mathcal{D}^S)^n \to \mathcal{D}$ that satisfy both *IIA* and *WP*, e.g., the following *CPF*: for all $(R_1, ..., R_n) \in \mathcal{D}^S$, if xP_iy for all $i \in N$, then xPy; otherwise,

$$F(R_1, ..., R_n) = CI.$$

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