Problems in Geometry
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Problem 1 [BMOTC]
Prove that the medians from the vertices A and B of triangle ABC are mutually perpendicular if and only if $|BC|^2 + |AC|^2 = 5|AB|^2$.

Problem 2 [BMOTC]
Suppose that $\angle A$ is the smallest of the three angles of triangle ABC. Let D be a point on the arc $BC$ of the circumcircle of $ABC$ which does not contain A. Let the perpendicular bisectors of $AB$, $AC$ intersect $AD$ at $M$ and $N$ respectively. Let $BM$ and $CN$ meet at T. Prove that $BT + CT \leq 2R$ where $R$ is the circumradius of triangle $ABC$.

Problem 3 [BMOTC]
Let triangle $ABC$ have side lengths $a$, $b$ and $c$ as usual. Points $P$ and $Q$ lie inside this triangle and have the properties that $\angle BPC = \angle CPA = \angle APB = 120^\circ$ and $\angle BQC = 60^\circ + \angle A$, $\angle CQA = 60^\circ + \angle B$, $\angle AQB = 60^\circ + \angle C$. Prove that

$$(|AP| + |BP| + |CP|)^3, |AQ|, |BQ|, |CQ| = (abc)^2.$$  

Problem 4 [BMOTC]
The points $M$ and $N$ are the points of tangency of the incircle of the isosceles triangle $ABC$ which are on the sides $AC$ and $BC$. The sides of equal length are $AC$ and $BC$. A tangent line $t$ is drawn to the minor arc $MN$. Suppose that $t$ intersects $AC$ and $BC$ at $Q$ and $P$ respectively. Suppose that the lines $AP$ and $BQ$ meet at $T$.

(a) Prove that $T$ lies on the line segment $MN$.
(b) Prove that the sum of the areas of triangles $ATQ$ and $BTP$ is minimized when $t$ is parallel to $AB$.

Problem 5 [BMOTC]
In a hexagon with equal angles, the lengths of four consecutive edges are 5, 3, 6 and 7 (in that order). Find the lengths of the remaining two edges.
Problem 6 [BMOTC]

The incircle $\gamma$ of triangle $ABC$ touches the side $AB$ at $T$. Let $D$ be the point on $\gamma$ diametrically opposite to $T$, and let $S$ be the intersection of the line through $C$ and $D$ with the side $AB$. Show that $|AT| = |SB|$.

Problem 7 [BMOTC]

Let $S$ and $r$ be the area and the inradius of the triangle $ABC$. Let $r_A$ denote the radius of the circle touching the incircle, $AB$ and $AC$. Define $r_B$ and $r_C$ similarly. The common tangent of the circles with radii $r$ and $r_A$ cuts a little triangle from $ABC$ with area $S_A$. Quantities $S_B$ and $S_C$ are defined in a similar fashion. Prove that

$$\frac{S_A}{r_A} + \frac{S_B}{r_B} + \frac{S_C}{r_C} = \frac{S}{r}$$

Problem 8 [BMOTC]

Triangle $ABC$ in the plane $\Pi$ is said to be good if it has the following property: for any point $D$ in space, out of the plane $\Pi$, it is possible to construct a triangle with sides of lengths $|AD|$, $|BD|$ and $|CD|$. Find all good triangles.

Problem 9 [BMO]

Circle $\gamma$ lies inside circle $\theta$ and touches it at $A$. From a point $P$ (distinct from $A$) on $\theta$, chords $PQ$ and $PR$ of $\theta$ are drawn touching $\gamma$ at $X$ and $Y$ respectively. Show that $\angle QAR = 2\angle XAY$.

Problem 10 [BMO]

$AP$, $AQ$, $AR$, $AS$ are chords of a given circle with the property that

$$\angle PAQ = \angle QAR = \angle RAS.$$

Prove that

$$AR(AP + AR) = AQ(AQ + AS).$$

Problem 11 [BMO]

The points $Q$, $R$ lie on the circle $\gamma$, and $P$ is a point such that $PQ$, $PR$ are tangents to $\gamma$. $A$ is a point on the extension of $PQ$ and $\gamma'$ is the circumcircle of triangle $PAR$. The circle $\gamma'$ cuts $\gamma$ again at $B$ and $AR$ cuts $\gamma$ at the point $C$. Prove that $\angle PAR = \angle ABC$. 2
Problem 12 [BMO]

In the acute-angled triangle $ABC$, $CF$ is an altitude, with $F$ on $AB$ and $BM$ is a median with $M$ on $CA$. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that the triangle $ABC$ is equilateral.

Problem 13 [BMO]

A triangle $ABC$ has $\angle BAC > \angle BCA$. A line $AP$ is drawn so that $\angle PAC = \angle BCA$ where $P$ is inside the triangle. A point $Q$ outside the triangle is constructed so that $PQ$ is parallel to $AB$, and $BQ$ is parallel to $AC$. $R$ is the point on $BC$ (separated from $Q$ by the line $AP$) such that $\angle PRQ = \angle BCA$. Prove that the circumcircle of $ABC$ touches the circumcircle of $PQR$.

Problem 14 [BMO]

$ABP$ is an isosceles triangle with $AB=AP$ and $\angle PAB$ acute. $PC$ is the line through $P$ perpendicular to $BP$ and $C$ is a point on this line on the same side of $BP$ as $A$. (You may assume that $C$ is not on the line $AB$). $D$ completes the parallelogram $ABCD$. $PC$ meets $DA$ at $M$. Prove that $M$ is the midpoint of $DA$.

Problem 15 [BMO]

In triangle $ABC$, $D$ is the midpoint of $AB$ and $E$ is the point of trisection of $BC$ nearer to $C$. Given that $\angle ADC = \angle BAE$ find $\angle BAC$.

Problem 16 [BMO]

$ABCD$ is a rectangle, $P$ is the midpoint of $AB$ and $Q$ is the point on $PD$ such that $CQ$ is perpendicular to $PD$. Prove that $BQC$ is isosceles.

Problem 17 [BMO]

Let $ABC$ be an equilateral triangle and $D$ an internal point of the side $BC$. A circle, tangent to $BC$ at $D$, cuts $AB$ internally at $M$ and $N$ and $AC$ internally at $P$ and $Q$. Show that $BD + AM + AN = CD + AP + AQ$.

Problem 18 [BMO]

Let $ABC$ be an acute-angled triangle, and let $D$, $E$ be the feet of the perpendiculars from $A$, $B$ to $BC$ and $CA$ respectively. Let $P$ be the point where the line $AD$ meets the semicircle constructed outwardly on $BC$ and $Q$ be the point where the line $BE$ meets the semicircle constructed outwardly on $AC$. Prove that $CP = CQ$. 

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Problem 19 [BMO]

Two intersecting circles $C_1$ and $C_2$ have a common tangent which touches $C_1$ at $P$ and $C_2$ at $Q$. The two circles intersect at $M$ and $N$, where $N$ is closer to $PQ$ than $M$ is. Prove that the triangles $MNP$ and $MNQ$ have equal areas.

Problem 20 [BMO]

Two intersecting circles $C_1$ and $C_2$ have a common tangent which touches $C_1$ at $P$ and $C_2$ at $Q$. The two circles intersect at $M$ and $N$, where $N$ is closer to $PQ$ than $M$ is. The line $PN$ meets the circle $C_2$ again at $R$. Prove that $MQ$ bisects $\angle PMR$.

Problem 21 [BMO]

Triangle $ABC$ has a right angle at $A$. Among all points $P$ on the perimeter of the triangle, find the position of $P$ such that $AP + BP + CP$ is minimized.

Problem 22 [BMO]

Let $ABCDEF$ be a hexagon (which may not be regular), which circumscribes a circle $S$. (That is, $S$ is tangent to each of the six sides of the hexagon.) The circle $S$ touches $AB$, $CD$, $EF$ at their midpoints $P$, $Q$, $R$ respectively. Let $X$, $Y$, $Z$ be the points of contact of $S$ with $BC$, $DE$, $FA$ respectively. Prove that $PY$, $QZ$, $RX$ are concurrent.

Problem 23 [BMO]

The quadrilateral $ABCD$ is inscribed in a circle. The diagonals $AC$, $BD$ meet at $Q$. The sides $DA$, extended beyond $A$, and $CB$, extended beyond $B$, meet at $P$. Given that $CD = CP = DQ$, prove that $\angle CAD = 60^\circ$.

Problem 24 [BMO]

The sides $a$, $b$, $c$ and $u$, $v$, $w$ of two triangles $ABC$ and $UVW$ are related by the equations

$$u(v + w - u) = a^2$$
$$v(w + u - v) = b^2$$
$$w(u + v - w) = c^2$$

Prove that triangle $ABC$ is acute-angled and express the angles $U$, $V$, $W$ in terms of $A$, $B$, $C$.  

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Problem 25 [BMO]

Two circles $S_1$ and $S_2$ touch each other externally at $K$; they also touch a circle $S$ internally at $A_1$ and $A_2$ respectively. Let $P$ be one point of intersection of $S$ with the common tangent to $S_1$ and $S_2$ at $K$. The line $PA_1$ meets $S_1$ again at $B_1$ and $PA_2$ meets $S_2$ again at $B_2$. Prove that $B_1B_2$ is a common tangent to $S_1$ and $S_2$.

Problem 26 [BMO]

Let $ABC$ be an acute-angled triangle and let $O$ be its circumcentre. The circle through $A$, $O$ and $B$ is called $S$. The lines $CA$ and $CB$ meet the circle $S$ again at $P$ and $Q$ respectively. Prove that the lines $CO$ and $PQ$ are perpendicular.

Problem 27 [BMO]

Two circles touch internally at $M$. A straight line touches the inner circle at $P$ and cuts the outer circle at $Q$ and $R$. Prove that $\angle QMP = \angle RPM$.

Problem 28 [BMO]

$ABC$ is a triangle, right-angled at $C$. The internal bisectors of $\angle BAC$ and $\angle ABC$ meet $BC$ and $CA$ at $P$ and $Q$, respectively. $M$ and $N$ are the feet of the perpendiculars from $P$ and $Q$ to $AB$. Find the measure of $\angle MCN$.

Problem 29 [BMO]

The triangle $ABC$, where $AB < AC$, has circumcircle $S$. The perpendicular from $A$ to $BC$ meets $S$ again at $P$. The point $X$ lies on the segment $AC$ and $BX$ meets $S$ again at $Q$. Show that $BX = CX$ if and only if $PQ$ is a diameter of $S$.

Problem 30 [BMO]

Let $ABC$ be a triangle and let $D$ be a point on $AB$ such that $4AD = AB$. The half-line $l$ is drawn on the same side of $AB$ as $C$, starting from $D$ and making an angle of $\theta$ with $DA$ where $\theta = \angle ACB$. If the circumcircle of $ABC$ meets the half-line $l$ at $P$, show that $PB = 2PD$. 
Problem 31 [BMO]

Let $BE$ and $CF$ be the altitudes of an acute triangle $ABC$, with $E$ on $AC$ and $F$ on $AB$. Let $O$ be the point of intersection of $BE$ and $CF$. Take any line $KL$ through $O$ with $K$ on $AB$ and $L$ on $AC$. Suppose $M$ and $N$ are located on $BE$ and $CF$ respectively, such that $KM$ is perpendicular to $BE$ and $LN$ is perpendicular to $CF$. Prove that $FM$ is parallel to $EN$.

Problem 32 [BMO]

In a triangle $ABC$, $D$ is a point on $BC$ such that $AD$ is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and $CD = AB$. Prove that $\angle A = 72^\circ$.

Problem 33 [Putnam]

Let $T$ be an acute triangle. Inscribe a rectangle $R$ in $T$ with one side along a side of $T$. Then inscribe a rectangle $S$ in the triangle formed by the side of $R$ opposite the side on the boundary of $T$, and the other two sides of $T$, with one side along the side of $R$. For any polygon $X$, let $A(X)$ denote the area of $X$. Find the maximum value, or show that no maximum exists, of $\frac{A(R) + A(S)}{A(T)}$ where $T$ ranges over all triangles and $R$, $S$ over all rectangles as above.

Problem 34 [Putnam]

A rectangle, $HOMF$, has sides $HO=11$ and $OM=5$. A triangle $ABC$ has $H$ as the orthocentre, $O$ as the circumcentre, $M$ the midpoint of $BC$ and $F$ the foot of the altitude from $A$. What is the length of $BC$?

Problem 35 [Putnam]

A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Problem 36 [Putnam]

Let $A$, $B$ and $C$ denote distinct points with integer coordinates in $R^2$. Prove that if $(|AB| + |BC|)^2 < 8[ABC] + 1$ then $A$, $B$, $C$ are three vertices of a square. Here $|XY|$ is the length of segment $XY$ and $[ABC]$ is the area of triangle $ABC$. 

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Problem 37 [Putnam]

Right triangle $ABC$ has right angle at $C$ and $\angle BAC = \theta$; the point $D$ is chosen on $AB$ so that $|AC| = |AD| = 1$; the point $E$ is chosen on $BC$ so that $\angle CDE = \theta$. The perpendicular to $BC$ at $E$ meets $AB$ at $F$. Evaluate $\lim_{\theta \to 0} |EF|$. 

Problem 38 [BMO]

Let $ABC$ be a triangle and $D$, $E$, $F$ be the midpoints of $BC$, $CA$, $AB$ respectively. Prove that $\angle DAC = \angle ABE$ if, and only if, $\angle AFC = \angle ADB$.

Problem 39 [BMO]

The altitude from one of the vertex of an acute-angled triangle $ABC$ meets the opposite side at $D$. From $D$ perpendiculars $DE$ and $DF$ are drawn to the other two sides. Prove that the length of $EF$ is the same whichever vertex is chosen.

Problem 40

Two cyclists ride round two intersecting circles, each moving with a constant speed. Having started simultaneously from a point at which the circles intersect, the cyclists meet once again at this point after one circuit. Prove that there is a fixed point such that the distances from it to the cyclists are equal all the time if they ride: (a) in the same direction (clockwise); (b) in opposite direction.

Problem 41

Prove that four circles circumscribed about four triangles formed by four intersecting straight lines in the plane have a common point. (Michell’s Point).

Problem 42

Given an equilateral triangle $ABC$. Find the locus of points $M$ inside the triangle such that $\angle MAB + \angle MBC + \angle MCA = \frac{\pi}{2}$.

Problem 43

In a triangle $ABC$, on the sides $AC$ and $BC$, points $M$ and $N$ are taken, respectively and a point $L$ on the line segment $MN$. Let the areas of the triangles $ABC$, $AML$ and $BNL$ be equal to $S$, $P$ and $Q$, respectively. Prove that
\[ S^{\frac{1}{3}} \geq P^{\frac{1}{3}} + Q^{\frac{1}{3}}. \]

**Problem 44**

For an arbitrary triangle, prove the inequality \( \frac{bc \cos A}{b+c} + a < p < \frac{bc+a^2}{a} \), where \( a, b, \) and \( c \) are the sides of the triangle and \( p \) its semiperimeter.

**Problem 45**

Given in a triangle are two sides: \( a \) and \( b \) (\( a > b \)). Find the third side if it is known that \( a + h_a \leq b + h_b \), where \( h_a \) and \( h_b \) are the altitudes dropped on these sides (\( h_a \) the altitude drawn to the side \( a \)).

**Problem 46**

One of the sides in a triangle \( ABC \) is twice the length of the other and \( \angle B = 2\angle C \). Find the angles of the triangle.

**Problem 47**

In a parallelogram whose area is \( S \), the bisectors of its interior angles are drawn to intersect one another. The area of the quadrilateral thus obtained is equal to \( Q \). Find the ratio of the sides of the parallelogram.

**Problem 48**

Prove that if one angle of a triangle is equal to 120°, then the triangle formed by the feet of its angle bisectors is right-angled.

**Problem 49**

Given a rectangle \( ABCD \) where \(|AB| = 2a, |BC| = a\sqrt{2}\). With \( AB \) is diameter a semicircle is constructed externally. Let \( M \) be an arbitrary point on the semicircle, the line \( MD \) intersect \( AB \) at \( N \), and the line \( MC \) at \( L \). Find \(|AL|^2 + |BN|^2\).

**Problem 50**

Let \( A, B \) and \( C \) be three points lying on the same line. Constructed on \( AB, BC \) and \( AC \) as diameters are three semicircles located on the same side of the line. The centre of a circle touching the three semicircles is found at a distance \( d \) from the line \( AC \). Find the radius of this circle.
Problem 51

In an isosceles triangle $ABC$, $|AC| = |BC|$, $BD$ is an angle bisector, $BDEF$ is a rectangle. Find $\angle BAF$ if $\angle BAE = 120^\circ$.

Problem 52

Let $M_1$ be a point on the incircle of triangle $ABC$. The perpendiculars to the sides through $M_1$ meet the incircle again at $M_2$, $M_3$, $M_4$. Prove that the geometric mean of the six lengths $M_iM_j$, $1 \leq i \leq j \leq 4$, is less than or equal to $r\sqrt{3}$, where $r$ denotes the inradius. When does the equality hold?

Problem 53 [AMM]

Let $ABC$ be a triangle and let $I$ be the incircle of $ABC$ and let $r$ be the radius of $I$. Let $K_1$, $K_2$ and $K_3$ be the three circles outside $I$ and tangent to $I$ and to two of the three sides of $ABC$. Let $r_i$ be the radius of $K_i$ for $1 \leq i \leq 3$. Show that

$$r = \sqrt{r_1r_2} + \sqrt{r_2r_3} + \sqrt{r_3r_1}$$

Problem 54 [Prithwijit’s Inequality]

In triangle $ABC$ suppose the lengths of the medians are $m_a$, $m_b$ and $m_c$ respectively. Prove that

$$\frac{am_a + bm_b + cm_c}{(a+b+c)(m_a + m_b + m_c)} \leq \frac{1}{3}$$

Problem 55 [Loney]

The base $a$ of a triangle and the ratio $r(<1)$ of the sides are given. Show that the altitude $h$ of the triangle cannot exceed $\frac{a}{1-r^2}$ and that when $h$ has this value the vertical angle of the triangle is $\frac{\pi}{2} - 2\tan^{-1}r$.

Problem 56 [Loney]

The internal bisectors of the angles of a triangle $ABC$ meet the sides in $D$, $E$ and $F$. Show that the area of the triangle $DEF$ is equal to $\frac{2\Delta abc}{(a+b)(b+c)(c+a)}$.

Problem 57 [Loney]

If $a$, $b$, $c$ are the sides of a triangle, $\lambda a$, $\lambda b$, $\lambda c$ the sides of a similar triangle inscribed in the former and $\theta$ the angle between the sides $a$ and $\lambda a$, prove that $2\lambda \cos \theta = 1$. 

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Problem 58

Let \(a, b\) and \(c\) denote the sides of a triangle and \(a + b + c = 2p\). Let \(G\) be the median point of the triangle and \(O, I\) and \(I_a\) the centres of the circumscribed, inscribed and escribed circles, respectively (the escribed circle touches the side \(BC\) and the extensions of the sides \(AB\) and \(AC\)), \(R, r\) and \(r_a\) being their radii, respectively. Prove that the following relationships are valid:

(a) \(a^2 + b^2 + c^2 = 2p^2 - 2r^2 - 8Rr\)
(b) \(|OG|^2 = R^2 - \frac{a^2+b^2+c^2}{9}\)
(c) \(|IG|^2 = \frac{p^2 + 5r^2 - 16Rr}{9}\)
(d) \(|OI|^2 = R^2 - 2Rr\)
(e) \(|OI_a|^2 = R^2 + 2Rr_a\)
(f) \(|II_a|^2 = 4R(r_a - r)\)

Problem 59

\(MN\) is a diameter of a circle, \(|MN| = 1\), \(A\) and \(B\) are points on the circles situated on one side of \(MN\), \(C\) is a point on the other semicircle. Given: \(A\) is the midpoint of semicircle, \(MB = \frac{3}{5}\), the length of the line segment formed by the intersection of the diameter \(MN\) with the chords \(AC\) and \(BC\) is equal to \(a\). What is the greatest value of \(a\)?

Problem 60

Given a parallelogram \(ABCD\). A straight line passing through the vertex \(C\) intersects the lines \(AB\) and \(AD\) at points \(K\) and \(L\), respectively. The areas of the triangles \(KBC\) and \(CDL\) are equal to \(p\) and \(q\), respectively. Find the area of the parallelogram \(ABCD\).

Problem 61 [Loney]

Three circles, whose radii are \(a, b\) and \(c\), touch one another externally and the tangents at their points of contact meet in a point; prove that the distance of this point from either of their points of contact is \(\sqrt{\frac{abc}{a+b+c}}\).

Problem 62 [Loney]

If a circle be drawn touching the inscribed and circumscribed circles of a triangle and the side \(BC\) externally, prove that its radius is \(\frac{a}{2} \tan^2 \frac{A}{2}\).
Problem 63

Characterize all triangles $ABC$ such that

$$AI_a : BI_b : CI_c = BC : CA : AB$$

where $I_a$, $I_b$, $I_c$ are the vertices of the excentres corresponding to $A$, $B$, $C$ respectively.

Problem 64

On the sides $AB$ and $BC$ of triangle $ABC$, points $K$ and $M$ are chosen such that the quadrilaterals $AKMC$ and $KBMN$ are cyclic, where $N = AM \cap CK$. If these quadrilaterals have the same circumradii then find $\angle ABC$.

Problem 65 [AMM]

Let $B'$ and $C'$ be points on the sides $AB$ and $AC$, respectively, of a given triangle $ABC$, and let $P$ be a point on the segment $B'C'$. Determine the maximum value of

$$\min([B'PB], [C'PC'])$$

where $[F]$ denotes the area of $F$.

Problem 66 [AMM]

For each point $O$ on diameter $AB$ of a circle, perform the following construction. Let the perpendicular to $AB$ at $O$ meet the circle at point $P$. Inscribe circles in the figures bounded by the circle and the lines $AB$ and $OP$. Let $R$ and $S$ be the points at which the two incircles to the curvilinear triangles $AOP$ and $BOP$ are tangent to the diameter $AB$. Show that $\angle RPS$ is independent of the position of $O$.

Problem 67

Let $E$ be a point inside the triangle $ABC$ such that $\angle ABE = \angle ACE$. Let $F$ and $G$ be the feet of the perpendiculars from $E$ to the internal and external bisectors, respectively, of angle $BAC$. Prove that the line $FG$ passes through the mid-point of $BC$. 

Problem 68

Let $A$, $B$, $C$ and $D$ be points on a circle with centre $O$ and let $P$ be the point of intersection of $AC$ and $BD$. Let $U$ and $V$ be the circumcentres of triangles $APB$ and $CPD$, respectively. Determine conditions on $A$, $B$, $C$ and $D$ that make $O$, $U$, $P$ and $V$ collinear and prove that, otherwise, quadrilateral $OUPV$ is a parallelogram.

Problem 69 [AMM]

Let $R$ and $r$ be the circumradius and inradius, respectively of triangle $ABC$.

(a) Show that $ABC$ has a median whose length is at most $2R - r$.
(b) Show that $ABC$ has an altitude whose length is at least $2R - r$.

Problem 70 [AMM]

Let $ABCD$ be a convex quadrilateral. Prove that if there is point $P$ in the interior of $ABCD$ such that

$$\angle PAB = \angle PBC = \angle PCD = \angle PDA = 45^\circ$$

then $ABCD$ is a square.

Problem 71 [AMM]

Let $M$ be any point in the interior of triangle $ABC$ and let $D$, $E$ and $F$ be points on the perimeter such that $AD$, $BE$ and $CF$ are concurrent at $M$. Show that if triangles $BMD$, $CME$ and $AMF$ all have equal areas and equal perimeters then triangle $ABC$ is equilateral.

Problem 72

The perpendiculars $AD$, $BE$, $CF$ are produced to meet the circumscribed circle in $X$, $Y$, $Z$ prove that

$$\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} = 4$$

Problem 73 [AMM]

Given an odd positive integer $n$, let $A_1$, $A_2$, ..., $A_n$ be a regular polygon with circumcircle $\Gamma$. A circle $O_i$ with radius $r$ is drawn externally tangent to $\Gamma$ at $A_i$ for $i = 1, 2, \ldots, n$. Let $P$ be any point on $\Gamma$ between $A_n$ and $A_1$. A circle $C$ (with any radius) is drawn externally tangent to $\Gamma$ at $P$. Let $t_i$ be the length of the common external tangent between the circles $C$ and $O_i$. Prove that

$$\sum_{i=1}^{n} (-1)^i t_i = 0.$$
Problem 74 [INMO]

The circumference of a circle is divided into eight arcs by a convex quadrilateral $ABCD$, with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by $p$, $q$, $r$, $s$ in counter-clockwise direction starting from some arc. Suppose $p + r = q + s$. Prove that $ABCD$ is a cyclic quadrilateral.

Problem 75 [INMO]

In an acute-angled triangle $ABC$, points $D$, $E$, $F$ are located on the sides $BC$, $CA$, $AB$ respectively such that

\[
\frac{CD}{CE} = \frac{CA}{CB} \cdot \frac{AE}{AF} = \frac{AB}{AC} \cdot \frac{BF}{BD} = \frac{BC}{BA}.
\]

Prove that $AD$, $BE$, $CF$ are the altitudes of $ABC$.

Problem 76

In trapezoid $ABCD$, $AB$ is parallel to $CD$ and let $E$ be the mid-point of $BC$. Suppose we can inscribe a circle in $ABED$ and also in $AECD$. Then if we denote $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$ prove that:

\[
a + c = \frac{b}{3} + d, \quad \frac{1}{a} + \frac{1}{c} = \frac{3}{b}.
\]

Problem 77 [BMO]

Let $ABC$ be a triangle with $AC > AB$. The point $X$ lies on the side $BA$ extended through $A$ and the point $Y$ lies on the side $CA$ in such a way that $BX = CA$ and $CY = BA$. The line $XY$ meets the perpendicular bisector of side $BC$ at $P$. Show that

\[
\angle BPC + \angle BAC = 180^\circ
\]

Problem 78 [Loney]

If $D$, $E$, $F$ are the points of contact of the inscribed circle with the sides $BC$, $CA$, $AB$ of a triangle, show that if the squares of $AD$, $BE$, $CF$ are in arithmetic progression, then the sides of the triangle are in harmonic progression.
Problem 79 [Loney]

Through the angular points of a triangle straight lines making the same angle \( \alpha \) with the opposite sides are drawn. Prove that the area of the triangle formed by them is to the area of the original triangle as \( 4 \cos^2 \alpha : 1 \).

Problem 80 [Loney]

If \( D, E, F \) be the feet of the perpendiculars from \( ABC \) on the opposite sides and \( \rho, \rho_1, \rho_2, \rho_3 \) be the radii of the circles inscribed in the triangles \( DEF, AEF, BFD, CDE \), prove that \( r^3 \rho = 2R\rho_1\rho_2\rho_3 \).

Problem 81 [Loney]

A point \( O \) is situated on a circle of radius \( R \) and with centre \( O \) another circle of radius \( \frac{3R}{2} \) is described. Inside the crescent-shaped area intercepted between these circles a circle of radius \( \frac{R}{8} \) is placed. Show that if the small circle moves in contact with the original circle of radius \( R \), the length of arc described by its centre in moving from one extreme position to the other is \( \frac{7}{12} \pi R \).

Problem 82 [Crux]

A Gergonne cevian is the line segment from a vertex of a triangle to the point of contact, on the opposite side, of the incircle. The Gergonne point is the point of concurrency of the Gergonne cevians.

In an integer triangle \( ABC \), prove that the Gergonne point \( \Gamma \) bisects the Gergonne cevian \( AD \) if and only if \( b, c, \frac{|3a-b-c|}{2} \) form a triangle where the measure of the angle between \( b \) and \( c \) is \( \frac{\pi}{3} \).

Problem 83

Prove that the line which divides the perimeter and the area of a triangle in the same ratio passes through the centre of the incircle.

Problem 84

Let \( m_a, m_b, m_c \) and \( w_a, w_b, w_c \) denote, respectively, the lengths of the medians and angle bisectors of a triangle. Prove that

\[
\sqrt{m_a} + \sqrt{m_b} + \sqrt{m_c} \geq \sqrt{w_a} + \sqrt{w_b} + \sqrt{w_c}.
\]
Problem 85

A quadrilateral has one vertex on each side of a square of side-length 1. Show that the lengths \(a, b, c\) and \(d\) of the sides of the quadrilateral satisfy the inequalities

\[2 \leq a^2 + b^2 + c^2 + d^2 \leq 4.\]

Problem 86 [Purdue Problem of the Week]

Given a triangle \(ABC\), find a triangle \(A_1B_1C_1\) so that

1. \(A_1 \in BC, B_1 \in CA, C_1 \in AB\)
2. the centroids of triangles \(ABC\) and \(A_1B_1C_1\) coincide

and subject to (1) and (2) triangle \(A_1B_1C_1\) has minimal area.

Problem 87

Prove that if the perpendiculars dropped from the points \(A_1, B_1\) and \(C_1\) on the sides \(BC, CA\) and \(AB\) of the triangle \(ABC\), respectively, intersect at the same point, then the perpendiculars dropped from the points \(A, B\) and \(C\) on the lines \(B_1C_1, C_1A_1\) and \(A_1B_1\) also intersect at one point.

Problem 88

Drawn through the intersection point \(M\) of medians of a triangle \(ABC\) is a straight line intersecting the sides \(AB\) and \(AC\) at points \(K\) and \(L\), respectively, and the extension of the side \(BC\) at a point \(P\) (\(C\) lying between \(P\) and \(B\)). Prove that

\[
\frac{1}{|MK|} = \frac{1}{|ML|} + \frac{1}{|MP|}
\]

Problem 89

Prove that the area of the octagon formed by the lines joining the vertices of a parallelogram to the midpoints of the opposite sides is 1/6 of the area of the parallelogram.

Problem 90

Prove that if the altitude of a triangle is \(\sqrt{2}\) times the radius of the circumscribed circle, then the straight line joining the feet of the perpendiculars dropped from the foot of this altitude on the sides enclosing it passes through the centre of the circumscribed circle.
Problem 91

Prove that the projections of the foot of the altitude of a triangle on the sides enclosing this altitude and on the two other altitudes lie on one straight line.

Problem 92

Let \(a, b, c\) and \(d\) be the sides of an inscribed quadrilateral (\(a\) is opposite to \(c\)), \(h_a, h_b, h_c\) and \(h_d\) the distances from the centre of the circumscribed circle to the corresponding sides. Prove that if the centre of the circle is inside the quadrilateral, then

\[ ah_c + ch_a = bh_d + dh_b \]

Problem 93

Prove that three lines passing through the vertices of a triangle and bisecting its perimeter intersect at one point (called Nagell’s point). Let \(M\) denote the centre of mass of the triangle, \(I\) the centre of the inscribed circle, \(S\) the centre of the circle inscribed in the triangle with vertices at the midpoints of the sides of the given triangle. Prove that the points \(N, M, I\) and \(S\) lie on a straight line and \(|MN| = 2|IM|, |IS| = |SN|\).

Problem 94 [Loney]

If \(\Delta_0\) be the area of the triangle formed by joining the points of contact of the inscribed circle with the sides of the given triangle whose area is \(\Delta\) and \(\Delta_1, \Delta_2\) and \(\Delta_3\) the corresponding areas for the escribed circles prove that

\[ \Delta_1 + \Delta_2 + \Delta_3 - \Delta_0 = 2\Delta \]

Problem 95

Prove that the radius of the circle circumscribed about the triangle formed by the medians of an acute-angled triangle is greater than 5/6 of the radius of the circle circumscribed about the original triangle.

Problem 96

Let \(K\) denote the intersection point of the diagonals of a convex quadrilateral \(ABCD\), \(L\) a point on the side \(AD\), \(N\) a point on the side \(BC\), \(M\) a point on the diagonal \(AC\), \(KL\) and \(MN\) being parallel to \(AB\), \(LM\) parallel to \(DC\). Prove that \(KLMN\) is a parallelogram and its area is less than 8/27 of the area of the quadrilateral \(ABCD\) (Hattori’s Theorem).
Problem 97

Two triangles have a common side. Prove that the distance between the centres of the circles inscribed in them is less than the distance between their non-coincident vertices (Zalgaller’s problem).

Problem 98

Prove that the sum of the distances from a point inside a triangle to its vertices is not less than $6r$, where $r$ is the radius of the inscribed circle.

Problem 99

Given a triangle. The triangle formed by the feet of its angle bisectors is isosceles. Is the given triangle isosceles?

Problem 100

Prove that the perpendicular bisectors of the line segments joining the intersection points of the altitudes to the centres of the circumscribed circles of the four triangles formed by four arbitrary straight lines in the plane intersect at one point (Herwey’s point).

Problem 101 [Crux]

Given triangle $ABC$ with $AB < AC$. Let $I$ be the incentre and $M$ be the mid-point of $BC$. The line $MI$ meets $AB$ and $AC$ at $P$ and $Q$ respectively. A tangent to the incircle meets sides $AB$ and $AC$ at $D$ and $E$ respectively. Prove that

$$\frac{AP}{BD} + \frac{AQ}{CE} = \frac{PQ}{2MI}$$

Problem 102 [Crux]

Let $ABC$ be a triangle with $\angle BAC = 60^\circ$. Let $AP$ bisect $\angle BAC$ and let $BQ$ bisect $\angle ABC$, with $P$ on $BC$ and $Q$ on $AC$. If $AB + BP = AQ + QB$, what are the angles of the triangle?

Problem 103

Prove that the sum of the squares of the distances from an arbitrary point in the plane to the sides of a triangle takes on the least value for such a point inside the triangle whose distances to the corresponding sides are proportional to these sides. Prove also that this point is the intersection point of the symmedians of the given triangle (Lemoine’s Point).
Problem 104

Given a triangle $ABC$. $AA_1$, $BB_1$ and $CC_1$ are its altitudes. Prove that Euler’s lines of the triangles $AB_1C_1$, $A_1BC_1$ and $A_1B_1C$ intersect at a point $P$ of the nine-point circles such that one of the line segments $PA_1$, $PB_1$, $PC_1$ is equal to sum of the other two (Thebault’s problem).

Problem 105

Let $M$ be an arbitrary point in the plane and $G$, the centroid of triangle $ABC$. Prove that

$$3|MG|^2 = |MA|^2 + |MB|^2 + |MC|^2 - \frac{1}{3}(|AB|^2 + |BC|^2 + |CA|^2)$$

(Leibnitz’s Theorem)

Problem 106

Let $ABC$ be a regular triangle with side $a$ and $M$ some point in the plane found at a distance $d$ from the centre of the triangle $ABC$. Prove that the area of the triangle whose sides are equal to the line segments $MA$, $MB$ and $MC$ can be expressed by the formula

$$S = \frac{\sqrt{3}}{12}|a^2 - 3d^2|$$

Problem 107 [Todhunter]

If $Q$ be any point in the plane of a triangle and $R_1, R_2, R_3$ the radii of the circles about $QBC$, $QCA$, $QAB$ prove that

$$(\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3})(-\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3})(\frac{a}{R_1} - \frac{b}{R_2} + \frac{c}{R_3})(\frac{a}{R_1} + \frac{b}{R_2} - \frac{c}{R_3}) = \frac{a^2b^2c^2}{R_1^2R_2^2R_3^2}$$

Problem 108 [Mathematical Gazette]

$PQRS$ is a quadrilateral inscribed in a circle with centre $O$. $E$ is the intersection of the diagonals $PR$ and $QS$. Let $F$ be the intersection of $PQ$ and $RS$ and $G$ the intersection of $PS$ and $QR$. The circle on $FG$ as diameter meets $OE$ at $X$. The perpendicular bisectors of $SX$ and $PX$ meet at $A$ and $B$, $C$, $D$ are defined similarly by cyclic change of letters.

(i) Prove that the tangents at $P$ and $Q$ and the line $OB$ are concurrent.
(ii) Prove that $PQ$, $AC$, $SR$, $FG$ are concurrent at $F$.
(iii) Prove that $AD$, $BC$, $FG$ are concurrent.

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Problem 109 [AMM]

Let \(X, Y, Z\) be three distinct points in the interior of an equilateral triangle \(ABC\). Let \(\alpha, \beta, \gamma\) be positive numbers adding up to \(\frac{\pi}{3}\) with the property that \(\angle XBA = \angle YAB = \alpha\), \(\angle YCB = \angle ZBC = \beta\) and \(\angle ZAC = \angle XCA = \gamma\). Find the angles of triangle \(XYZ\) in terms of \(\alpha, \beta, \gamma\).

Problem 110 [Todhunter]

If \(O\) be the centre of the circle inscribed in a triangle \(ABC\) and \(r_a, r_b, r_c\) the radii of the circles inscribed in the triangles \(OBC, OCA, OAB\), show that

\[
\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} = 2\left(\cot\left(\frac{A}{4}\right) + \cot\left(\frac{B}{4}\right) + \cot\left(\frac{C}{4}\right)\right)
\]

Problem 111 [BMO]

Let \(P\) be an internal point of triangle \(ABC\) and let \(\alpha, \beta, \gamma\) be defined by

\[
\alpha = \angle BPC - \angle BAC \\
\beta = \angle CPA - \angle CBA \\
\gamma = \angle APB - \angle ACB
\]

Prove that

\[
P A^{\sin(\angle BAC)} \sin(\alpha) = P B^{\sin(\angle CBA)} \sin(\beta) = P C^{\sin(\angle ACB)} \sin(\gamma)
\]

Problem 112

Let \(ABC\) be a triangle with incentre \(I\) and inradius \(r\). Let \(D, E, F\) be the feet of the perpendiculars from \(I\) to the sides \(BC, CA, AB\) respectively. If \(r_1, r_2, r_3\) are the radii of circles inscribed in the quadrilaterals \(AFIE, BDIF\) and \(CEID\) respectively, prove that

\[
\frac{r_1}{r-r_1} + \frac{r_2}{r-r_2} + \frac{r_3}{r-r_3} = \frac{r_1 r_2 r_3}{(r-r_1)(r-r_2)(r-r_3)}
\]

Problem 113 [Loney]

Given the product \(p\) of the sines of the angles of a triangle and the product \(q\) of the cosines, show that the tangents of the angles are the roots of the equation

\[
qx^3 - px^2 + (1 + q)x - p = 0
\]
Problem 114

The altitude of a right triangle drawn to the hypotenuse is equal to \( h \). Prove that the vertices of the acute angles of the triangle and the projections of the foot of the altitude on the legs all lie on the same circle. Determine the length of the chord cut by this circle on the line containing the altitude and the segments of the chord into which it is divided by the hypotenuse.

Problem 115

Four villages are situated at the vertices of a square of side 2 Km. The villages are connected by roads so that each village is joined to any other. Is it possible for the total length of the roads to be less than 5.5 Km?

Problem 116

Prove that if the lengths of the internal angle bisectors of a triangle are less than 1, then its area is less than \( \frac{\sqrt{3}}{3} \).

Problem 117

Given a convex quadrilateral ABCD circumscribed about a circle of diameter 1. Inside ABCD, there is a point M such that

\[
|MA|^2 + |MB|^2 + |MC|^2 + |MD|^2 = 2.
\]

Find the area of ABCD.

Problem 118

The circle inscribed in a triangle ABC divides the median BM into three equal parts. Find the ratio \(|BC| : |CA| : |AB|\).

Problem 119

Prove that if the centres of the squares constructed externally on the sides of a given triangle serve as the vertices of the triangle whose area is twice the area of the given triangle, then the centres of the squares constructed internally on the sides of the triangle lie on a straight line.

Problem 120

Prove that the median drawn to the largest side of a triangle forms with the sides enclosing this median angles each of which is not less than half the smallest angle of the triangle.
Problem 121

Three squares $BCDE$, $ACFG$ and $BAHK$ are constructed externally on the sides $BC$, $CA$ and $AB$ of a triangle $ABC$. Let $FCDQ$ and $EBKP$ be parallelograms. Prove that the triangle $APQ$ is a right-angled isosceles triangle.

Problem 122

Three points are given in a plane. Through these points three lines are drawn forming a regular triangle. Find the locus of centres of those triangles.

Problem 123

Drawn in an inscribed polygon are non-intersecting diagonals separating the polygon into triangles. Prove that the sum of the radii of the circles inscribed in those triangles is independent of the way the diagonals are drawn.

Problem 124

A polygon is circumscribed about a circle. Let $l$ be an arbitrary line touching the circle and coinciding with no side of the polygon. Prove that the ratio of the product of the distances from the vertices of the polygon to the line $l$ to the product of the distances from the points of tangency of the sides of the polygon with the circle to $l$ is independent of the position of the line $l$.

Problem 125 [Loney]

If $2\phi_1$, $2\phi_2$, $2\phi_3$ are the angles subtended by the circle escribed to the side $a$ of a triangle at the centres of the inscribed circle and the other two escribed circles, prove that

$$\sin(\phi_1) \sin(\phi_2) \sin(\phi_3) = \frac{r_1^2}{16R^2}$$

Problem 126

If from any point in the plane of a regular polygon perpendiculars are drawn on the sides, show that the sum of the squares of these perpendiculars is equal to the sum of the squares on the lines joining the feet of the perpendiculars with the centre of the polygon.
Problem 127 [Loney]

The three medians of a triangle $ABC$ make angles $\alpha$, $\beta$, $\gamma$ with each other. Prove that

$$\cot \alpha + \cot \beta + \cot \gamma + \cot A + \cot B + \cot C = 0$$

Problem 128 [Loney]

A railway curve, in the shape of a quadrant of a circle, has $n$ telegraph posts at its ends and at equal distances along the curve. A man stationed at a point on one of the extreme radii produced sees the $p$th and $q$th posts from the end nearest him in a straight line. Show that the radius of the curve is

$$a \cos(p+q)\phi \over 2\sin(p\phi)\sin(q\phi)$$

where $\phi = \pi \over 4(n-1)$, and $a$ is the distance from the man to the nearest end of the curve.

Problem 129

Let $D$ be an arbitrary point on the side $BC$ of a triangle $ABC$. Let $E$ and $F$ be points on the sides $AC$ and $AB$ such that $DE$ is parallel to $AB$ and $DF$ is parallel to $AC$. A circle passing through $D$, $E$ and $F$ intersects for the second time $BC$, $CA$ and $AB$ at points $D_1$, $E_1$ and $F_1$, respectively. Let $M$ and $N$ denote the intersection points of $DE$ and $F_1D_1$, $DF$ and $D_1E_1$, respectively. Prove that $M$ and $N$ lie on the symedian emanating from the vertex $A$. If $D$ coincides with the foot of the symedian, then the circle passing through $D$, $E$ and $F$ touches the side $BC$. (This circle is called Tucker’s Circle.)

Problem 130

Let $ABCD$ be a cyclic quadrilateral. The diagonal $AC$ is equal to $a$ and forms angles $\alpha$ and $\beta$ with the sides $AB$ and $AD$, respectively. Prove that the magnitude of the area of the quadrilateral lies between $a^2 \sin(\alpha+\beta) \sin \beta$ and $a^2 \sin(\alpha+\beta) \sin \alpha \over 2\sin \beta$ and $a^2 \sin \alpha \over 2\sin \alpha$.

Problem 131

A triangle has sides of lengths $a$, $b$, $c$ and respective altitudes of lengths $h_a$, $h_b$, $h_c$. If $a \geq b \geq c$ show that $a + h_a \geq b + h_b \geq c + h_c$. 

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Problem 132 [Crux]

Given a right-angled triangle $ABC$ with $\angle BAC = 90^\circ$. Let $I$ be the incentre and let $D$ and $E$ be the intersections of $BI$ and $CI$ with $AC$ and $AB$ respectively. Prove that

$$\frac{|BI|^2 + |ID|^2}{|IC|^2 + |IE|^2} = \frac{|AB|^2}{|AC|^2}$$

Problem 133 [Hobson]

Straight lines whose lengths are successively proportional to $1, 2, 3, \cdots, n$ form a rectilineal figure whose exterior angles are each equal to $\frac{2\pi}{n}$; if a polygon be formed by joining the extremities of the first and last lines, show that its area is

$$\frac{n(n+1)(2n+1)}{24} \cot\left(\frac{\pi}{n}\right) + \frac{n}{16} \cot\left(\frac{\pi}{n}\right) \csc^2\left(\frac{\pi}{n}\right)$$

Problem 134

An arc $AB$ of a circle is divided into three equal parts by the points $C$ and $D$ ($C$ is nearest to $A$). When rotated about the point $A$ through an angle of $\frac{\pi}{3}$, the points $B, C$ and $D$ go into points $B_1, C_1$ and $D_1$. $F$ is the point of intersection of the straight lines $AB_1$ and $DC_1$; $E$ is a point on the bisector of the angle $B_1BA$ such that $|BD| = |DE|$. Prove that the triangle $CEF$ is regular (Finlay’s theorem).

Problem 135

In a triangle $ABC$, a point $D$ is taken on the side $AC$. Let $O_1$ be the centre of the circle touching the line segments $AD, BD$ and the circle circumscribed about the triangle $ABC$ and let $O_2$ be the centre of the circle touching the line segments $CD, BD$ and the circumscribed circle. Prove that the line $O_1O_2$ passes through the centre $O$ of the circle inscribed in the triangle $ABC$ and $|O_1O| : |OO_2| = \tan^2(\phi/2)$, where $\phi = \angle BDA$ (Thebault’s theorem).

Problem 136

Prove the following statement. If there is an $n$-gon inscribed in a circle $\alpha$ and circumscribed about another circle $\beta$, then there are infinitely many $n$-gons inscribed in the circle $\alpha$ and circumscribed about the circle $\beta$ and any point of the circle can be taken as one of the vertices of such an $n$-gon (Poncelet’s theorem).
Problem 137 [Loney]

A point is taken in the plane of a regular polygon of $n$ sides at a distance $c$ from the centre and on the line joining the centre to a vertex, and the radius of the inscribed circle is $r$. Show that the product of the distances of the point from the sides of the polygon is

$$\frac{c^n}{2^{n-2}} \cos^2\left(\frac{n}{2} \cos^{-1} \frac{r}{c}\right) \text{ if } c > r$$

$$\frac{c^n}{2^{n-2}} \cosh^2\left(\frac{n}{2} \cosh^{-1} \frac{r}{c}\right) \text{ if } c < r$$

Problem 138 [Loney]

An infinite straight line is divided by an infinite number of points into portions each of length $a$. Prove that the sum of the fourth powers of the reciprocals of the distances of a point $O$ on the line from all the points of division is

$$\frac{\pi^4}{3n^4} (3 \csc^4 \frac{\pi b}{a} - 2 \csc^2 \frac{\pi b}{a})$$

Problem 139 [Loney]

If $\rho_1, \rho_2, \cdots, \rho_n$ be the distances of the vertices of a regular polygon of $n$ sides from any point $P$ in its plane, prove that

$$\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \cdots + \frac{1}{\rho_n^2} = \frac{n}{r^2 - a^2 - 2nr \cos \theta + a^2n}$$

where $a$ is the radius of the circumcircle of the polygon, $r$ is the distance of $P$ from its centre $O$ and $\theta$ is the angle that $OP$ makes with the radius to any angular point of the polygon.

Problem 140

Given an angle with vertex $A$ and a circle inscribed in it. An arbitrary straight line touching the given circle intersects the sides of the angle at points $B$ and $C$. Prove that the circle circumscribed about the triangle $ABC$ touches the circle inscribed in the given angle.

Problem 141

Let $ABCDEF$ be an inscribed hexagon in which $|AB| = |CD| = |EF| = R$, where $R$ is the radius of the circumscribed circle, $O$ its centre. Prove that the points of pairwise intersections of the circles circumscribed about the triangles $BOC$, $DOE$, $FOA$, distinct from $O$, serve as the vertices of an equilateral triangle with side $R$. 

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Problem 142

The diagonals of an inscribed quadrilateral are mutually perpendicular. Prove that the midpoints of its sides and the feet of the perpendiculars dropped from the point of intersection of the diagonals on the sides lie on a circle. Find the radius of that circle if the radius of the given circle is $R$ and the distance from its centre to the point of intersection of the diagonals of the quadrilateral is $d$.

Problem 143

Prove that if a quadrilateral is both inscribed in a circle and circumscribed about a circle of radius $r$, the distance between the centres of those circles being $d$, then the relationship

$$\frac{1}{(R+d)^2} + \frac{1}{(R-d)^2} = \frac{1}{r^2}$$

is true.

Problem 144

Let $ABCD$ be a convex quadrilateral. Consider four circles each of which touches three sides of this quadrilateral.

(a) Prove that the centres of these circles lie on one circle.

(b) Let $r_1, r_2, r_3$ and $r_4$ denote the radii of these circles ($r_1$ does not touch the side $DC$, $r_2$ the side $DA$, $r_3$ the side $AB$ and $r_4$ the side $BC$). Prove that

$$\frac{|AB|}{r_1} + \frac{|CD|}{r_3} = \frac{|BC|}{r_2} + \frac{|AD|}{r_4}$$

Problem 145

The sides of a square is equal to $a$ and the products of the distances from the opposite vertices to a line $l$ are equal to each other. Find the distance from the centre of the square to the line $l$ if it is known that neither of the sides of the square is parallel to $l$.

Problem 146

Find the angles of a triangle if the distance between the centre of the circumcircle and the intersection point of the altitudes is one-half the length of the largest side and equals the smallest side.
Problem 147

Prove that for the perpendiculars dropped from the points $A_1$, $B_1$ and $C_1$ on the sides $BC$, $CA$ and $AB$ of a triangle $ABC$ to intersect at the same point, it is necessary and sufficient that

$$|A_1B|^2 - |BC_1|^2 + |C_1A|^2 - |AB_1|^2 + |B_1C|^2 - |CA_1|^2 = 0.$$ 

Problem 148

Each of the sides of a convex quadrilateral is divided into $(2n + 1)$ equal parts. The division points on the opposite sides are joined correspondingly. Prove that the area of the central quadrilateral amounts to $1/(2n + 1)^2$ of the area of the entire quadrilateral.

Problem 149

A straight line intersects the sides $AB$, $BC$ and the extension of the side $AC$ of a triangle $ABC$ at points $D$, $E$ and $F$, respectively. Prove that the midpoints of the line segments $DC$, $AE$ and $BF$ lie on a straight line ($Gaussian$ line).

Problem 150

Given two intersecting circles. Find the locus of centres of rectangles with vertices lying on these circles.

Problem 151

An equilateral triangle is inscribed in a circle. Find the locus of intersection points of the altitudes of all possible triangles inscribed in the circle if two sides of the triangles are parallel to those of the given one.

Problem 152

Given two circles touching each other internally at a point $A$. A tangent to the smaller circle intersects the larger one at points $B$ and $C$. Find the locus of centres of circles inscribed in triangles $ABC$.

Problem 153 [Loney]

Two circles, the sum of whose radii is $a$, are placed in the same plane with their centres at a distance $2a$ and an endless string is fully stretched so as partly to surround the circles and to cross between them. Show that the length of the string is $(4\pi + 2\sqrt{3})a$. 

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Problem 154 [Loney]

If $p$, $q$, $r$ are the perpendiculars from the vertices of a triangle upon any straight line meeting the sides externally in $D$, $E$, $F$, prove that

$$a^2(p-q)(p-r) + b^2(q-r)(q-p) + c^2(r-p)(r-q) = 4\Delta^2.$$

Problem 155 [Loney]

A regular polygon is inscribed in a circle; show that the arithmetic mean of the squares of the distances of its corners from any point (not necessarily in its plane) is equal to the arithmetic mean of the sum of the squares of the longest and shortest distances of the point from the circle.

Problem 156

In the cyclic quadrilateral $ABCD$, the diagonal $AC$ bisects the angle $DAB$. The side $AD$ is extended beyond $D$ to a point $E$. Show that $CE = CA$ if and only if $DE = AB$.

Problem 157 [BMO]

Let $G$ be a convex quadrilateral. Show that there is a point $X$ in the plane of $G$ with the property that every straight line through $X$ divides $G$ into two regions of equal area if and only if $G$ is a parallelogram.

Problem 158

Given a triangle $ABC$ and a point $M$. A straight line passing through the point $M$ intersects the lines $AB$, $BC$ and $CA$ at points $C_1$, $A_1$ and $B_1$, respectively. The lines $AM$, $BM$ and $CM$ intersect the circle circumscribed about the triangle $ABC$ at points $A_2$, $B_2$ and $C_2$, respectively. Prove that the lines $A_1A_2$, $B_1B_2$ and $C_1C_2$ intersect at a point situated on the circle circumscribed about the triangle $ABC$.

Problem 159 [AMM]

Let $P$ be a point in the interior of triangle $ABC$ and let $r_1$, $r_2$, $r_3$ denote the distances from $P$ to the sides of the triangle with lengths $a_1$, $a_2$, $a_3$, respectively. Let $R$ be the circumradius of $ABC$ and let $0 < a < 1$ be a real number. Let $b = 2a/(1-a)$. Prove that

$$r_1^a + r_2^a + r_3^a \leq \frac{1}{(2R)^a}(a_1^b + a_2^b + a_3^b)^{1-a}.$$
Problem 160 [AMM]

Let \( K \) be the circumcentre and \( G \) the centroid of a triangle with side lengths \( a, b, c \) and area \( \Delta \).

(a) Show that the distance \( d \) from \( K \) to \( G \) satisfies
\[
12\Delta d = a^2b^2c^2 - (b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)
\]

(b) Show that \( d(< \frac{abc}{12\Delta}, = \frac{abc}{12\Delta}, > \frac{abc}{12\Delta}) \) when the triangle is respectively acute, right-angled, obtuse.

Problem 161 [AMM]

Let \( ABC \) be an acute-triangle and let \( P \) be a point in its interior. Denote by \( a, b, c \) the lengths of the triangle’s sides, by \( d_a, d_b, d_c \) the distances from \( P \) to the triangle’s sides, and by \( R_a, R_b, R_c \) the distances from \( P \) to the vertices \( A, B, C \) respectively. Show that
\[
d_a^2 + d_b^2 + d_c^2 \geq R_a^2\sin^2(A/2) + R_b^2\sin^2(B/2) + R_c^2\sin^2(C/2) \geq (d_a + d_b + d_c)^2 / 3
\]

Problem 162 [Loney]

\( A_1A_2 \cdots A_n \) is a regular polygon of \( n \) sides which is inscribed in a circle, whose radius is \( a \) and whose centre is \( O \); prove that the product of the distances of its angular points from a straight line at right angles to \( OA \) and at a distance \( b(> a) \) from the centre is
\[
b^n[\cos^n(\frac{1}{2}\sin^{-1}\frac{a}{b}) - \sin^n(\frac{1}{2}\sin^{-1}\frac{a}{b})]^2
\]

Problem 163 [Loney]

The radii of an infinite series of concentric circles are \( a, \frac{a}{2}, \frac{a}{3}, \cdots \). From a point at a distance \( c(> a) \) from their common centre a tangent is drawn to each circle. Prove that
\[
\sin(\theta_1)\sin(\theta_2)\sin(\theta_3)\cdots = \sqrt{\frac{\pi}{na}\sin\frac{\pi a}{c}}
\]

where \( \theta_1, \theta_2, \theta_3, \cdots \) are the angles that the tangents subtend at the common centre.
Problem 164 [Crux]

Construct equilateral triangles $A'B'C'$, $B'C'A'$, $C'A'B'$ exterior to triangle $ABC$ and take points $P$, $Q$, $R$ on $AA'$, $BB'$, $CC'$, respectively, such that

$$\frac{AP}{AA'} + \frac{BQ}{BB'} + \frac{CR}{CC'} = 1.$$  

Prove that $\triangle PQR$ is equilateral.

Problem 165 [Crux]

Given a triangle $ABC$, we take variable points $P$ on segment $AB$ and $Q$ on segment $AC$. $CP$ meets $BQ$ in $T$. Where should $P$ and $Q$ be located so that area of $\triangle PQT$ is maximized?

Problem 166 [Crux]

Let $ABC$ be a triangle and $A_1$, $B_1$, $C_1$ the common points of the inscribed circle with the sides $BC$, $CA$, $AB$, respectively. We denote the length of the arc $B_1C_1$ (not containing $A_1$) of the incircle by $S_a$, and similarly define $S_b$ and $S_c$. Prove that

$$\frac{a}{S_a} + \frac{b}{S_b} + \frac{c}{S_c} \geq \frac{9\sqrt{3}}{\pi}.$$  

Problem 167 [AMM]

A cevian of a triangle is a line segment that joins a vertex to the line containing the opposite side. An equicevian point of a triangle $ABC$ is a point $P$ (not necessarily inside the triangle) such that the cevians on the lines $AP$, $BP$ and $CP$ have equal length. Let $SBC$ be an equilateral triangle and let $A$ be chosen in the interior of $SBC$ on the altitude dropped from $S$.

(a) Show that $ABC$ has two equicevian points.
(b) Show that the common length of the cevians through either of the equicevian points is constant, independent of the choice of $A$.
(c) Show that the equicevian points divide the cevian through $A$ in a constant ratio, independent of the choice of $A$.
(d) Find the locus of the equicevian points as $A$ varies.
(e) Let $S'$ be the reflection of $S$ in the line $BC$. Show that (a), (b) and (c) hold if $A$ moves on any ellipse with $S$ and $S'$ as its foci. Find the locus of the equicevian points as $A$ varies on the ellipse.
Problem 168 [Crux]

Let $ABCD$ be a trapezoid with $AD$ parallel to $BC$. $M$, $N$, $P$, $Q$, $O$ are the midpoints of $AB$, $CD$, $AC$, $BD$, $MN$, respectively. Circles $m$, $n$, $p$, $q$ all pass through $O$ and are tangent to $AB$ at $M$, to $CD$ at $N$, to $AC$ at $P$, and to $BD$ at $Q$, respectively. Prove that the centres of $m$, $n$, $p$, $q$ are collinear.

Problem 169 [AMM]

Let $ABC$ be an equilateral triangle inscribed in a circle with radius 1 unit. Suppose $P$ is a point inside the triangle. Prove that $|PA||PB||PC| \leq \frac{32}{27}$. Generalize the result to a regular polygon of $n$ sides. (Erdos)

Problem 170

Given two circles. Find the locus of points $M$ such that the ratio of the lengths of the tangents drawn from $M$ to the given circles is a constant $k$.

Problem 171

In a quadrilateral $ABCD$, $P$ is the intersection point of $BC$ and $AD$, $Q$ that of $CA$ and $BD$ and $R$ that of $AB$ and $CD$. Prove that the intersection points of $BC$ and $QR$, $CA$ and $RP$, $AB$ and $PQ$ are collinear.

Problem 172

Given two squares whose sides are respectively parallel. Determine the locus of points $M$ such that for any point $P$ of the first square there is a point $Q$ of the second one such that the triangle $MPQ$ is equilateral. Let the side of the first square be $a$ and that of the second square be $b$. For what relationship between $a$ and $b$ is the desired locus non-empty?

Problem 173 [AMM]

Let $C_1C_2 \ldots C_n$ be a regular $n$-gon and let $C_{n+1} = C_1$. Let $O$ be the inscribed circle. For $1 \leq k \leq n$, let $T_k$ be the point at which $O$ is tangent to $C_kC_{k+1}$. Let $X$ be a point on the open arc $(T_{n-1}T_n)$ and let $Y$ be a point other than $X$ on $O$. For $1 \leq i \leq n$, let $B_i$ be the second point at which the line $XC_i$ meets $O$ and let $p_i = |XB_i||XC_i|$. Let $M_i$ be the mid-point of chord $T_iT_{i+1}$ and let $N_i$ be the second point, other than $Y$, at which $YM_i$ meets $O$. Let $q_i = |YM_i||YN_i|$. Prove that $\sum_{i=1}^{n} q_i = (\sum_{i=1}^{n} p_i) - p_n$. 

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Problem 174 [AMM]

Let \(ABC\) be an acute triangle, with semi-perimeter \(p\) and with inscribed and circumscribed circles of radius \(r\) and \(R\), respectively.

(a) Show that \(ABC\) has a median of length at most \(p/\sqrt{3}\).
(b) Show that \(ABC\) has a median of length at most \(R + r\).
(c) Show that \(ABC\) has an altitude of length at least \(R + r\).

Problem 175 [RMO, India]

Let \(ABC\) be an acute-angled triangle and \(CD\) be the altitude through \(C\). If \(AB = 8\) and \(CD = 6\) find the distance between the mid-points of \(AD\) and \(BC\).

Problem 176 [RMO, India]

Let \(ABCD\) be a rectangle with \(AB = a\) and \(BC = b\). Suppose \(r_1\) is the radius of the circle passing through \(A\) and \(B\) and touching \(CD\); and similarly \(r_2\) is the radius of the circle passing through \(B\) and \(C\) and touching \(AD\). Show that

\[ r_1 + r_2 \geq \frac{5}{8}(a + b) \]

Problem 177 [INMO]

Two circles \(C_1\) and \(C_2\) intersect at two distinct points \(P\) and \(Q\) in a plane. Let a line passing through \(P\) meet the circles \(C_1\) and \(C_2\) in \(A\) and \(B\) respectively. Let \(Y\) be the mid-point of \(AB\) and \(QY\) meet the circles \(C_1\) and \(C_2\) in \(X\) and \(Z\) respectively. Show that \(Y\) is also the mid-point of \(XZ\).

Problem 178 [INMO]

In a triangle \(ABC\) angle \(A\) is twice angle \(B\). Show that \(a^2 = b(b + c)\).

Problem 179 [INMO]

The diagonals \(AC\) and \(BD\) of a cyclic quadrilateral \(ABCD\) intersect at \(P\). Let \(O\) be the circumcentre of triangle \(APB\) and \(H\) be the orthocentre of triangle \(CPD\). Show that the points \(H, P, O\) are collinear.
Problem 180 [INMO]

Let $ABC$ be a triangle in a plane $\Sigma$. Find the set of all points $P$ (distinct from $A, B, C$) in the plane $\Sigma$ such that the circumcircles of triangles $ABP$, $BCP$ and $CAP$ have the same radii.

Problem 181 [INMO]

Let $ABC$ be a triangle right-angled at $A$ and $S$ be its circumcircle. Let $S_1$ be the circle touching the lines $AB$ and $AC$ and the circle $S$ internally. Further let $S_2$ be the circle touching the lines $AB$ and $AC$ and the circle $S$ externally. If $r_1$ and $r_2$ be the radii of the circles $S_1$ and $S_2$ respectively, show that

$$r_1r_2 = 4\text{Area}(ABC)$$

Problem 182 [INMO]

Show that there exists a convex hexagon in the plane such that

(a) all its interior angles are equal.
(b) all its sides are 1,2,3,4,5,6 in some order.

Problem 183 [INMO]

Let $G$ be the centroid of a triangle $ABC$ in which the angle $C$ is obtuse and $AD$ and $CF$ be the medians from $A$ and $C$ respectively onto the sides $BC$ and $AB$. If the four points $B, D, G$ and $F$ are concyclic, show that

$$\frac{AC}{BC} > \sqrt{2}$$

If further $P$ is a point on the line $BG$ extended such that $AGCP$ is a parallelogram, show that the triangle $ABC$ and $GAP$ are similar.

Problem 184 [INMO]

A circle passes through a vertex $C$ of a rectangle $ABCD$ and touches its sides $AB$ and $AD$ at $M$ and $N$ respectively. If the distance from $C$ to the line segment $MN$ is equal to 5 units, find the area of the rectangle $ABCD$. 

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Problem 185 [RMO, India]

In a quadrilateral $ABCD$, it is given that $AB$ is parallel to $CD$ and the diagonals $AC$ and $BD$ are perpendicular to each other. Show that

(a) $AD \cdot BC \geq AB \cdot CD$
(b) $AD + BC \geq AB + CD$

Problem 186 [RMO, India]

In the triangle $ABC$, the incircle touches the sides $BC$, $CA$ and $AB$ respectively at $D$, $E$ and $F$. If the radius of the incircle is 4 units and if $BD$, $CE$ and $AF$ are consecutive integers, find the sides of the triangle $ABC$.

Problem 187 [RMO, India]

Let $ABCD$ be a square and $M$, $N$ points on sides $AB$, $BC$, respectively, such that $\angle MDN = 45^\circ$. If $R$ is the midpoint of $MN$ show that $RP = RQ$ where $P$, $Q$ are the points of intersection of $AC$ with the lines $MD$ and $ND$.

Problem 188 [RMO, India]

Let $AC$ and $BD$ be two chords of a circle with centre $O$ such that they intersect at right angles inside the circle at the point $M$. Suppose $K$ and $L$ are the mid-points of the chord $AB$ and $CD$ respectively. Prove that $OKML$ is a parallelogram.

Problem 189 [AMM]

Let $P$ be a convex $n$-gon inscribed in a circle $O$ and let $\Delta$ be a triangulation of $P$ without new vertices. Compute the sum of the squares of distances from the centre $O$ to the incentres of the triangles of $\Delta$ and show that this sum is independent of $\Delta$.

Problem 190 [AMM]

Let $T_1$ and $T_2$ be triangles such that for $i \in 1, 2$, triangle $T_i$ has circumradius $R_i$, inradius $r_i$ and side lengths $a_i$, $b_i$ and $c_i$. Show that

$$8R_1R_2 + 4r_1r_2 \geq a_1a_2 + b_1b_2 + c_1c_2 \geq 36r_1r_2$$

and determine when equality holds.
Problem 191 [AMM]

Let $ABC$ be an acute triangle. $T$ the mid-point of arc $BC$ of the circle circumscribing $ABC$. Let $G$ and $K$ be the projections of $A$ and $T$ respectively on $BC$, let $H$ and $L$ be the projections of $B$ and $C$ on $AT$ and let $E$ be the mid-point of $AB$. Prove that:

(a) $KH \parallel AC$, $GL \parallel BT$, $GH \parallel TC$, $LK \parallel AB$.
(b) $G$, $H$, $K$ and $L$ are concyclic.
(c) The centre of the circle through $G$, $H$ and $K$ lies on the Euler circle of $ABC$.

Problem 192 [AMM]

A trapezoid $ABRS$ with $AB \parallel RS$ is inscribed in a non-circular ellipse $E$ with axes of symmetry $a$ and $b$. The points $A$ and $B$ are reflected through $a$ to points $P$ and $Q$ on $E$.

(a) Show that $P$, $Q$, $R$ and $S$ are concyclic.
(b) Show that if the line $PQ$ intersects the line $RS$ at $T$, then the angle bisector of $\angle PTR$ is parallel to $a$.

Problem 193 [RMO, India]

$ABCD$ is a cyclic quadrilateral with $AC \perp BD$; $AC$ meets $BD$ at $E$. Prove that $EA^2 + EB^2 + EC^2 + ED^2 = 4R^2$.

Problem 194 [RMO, India]

$ABCD$ is a cyclic quadrilateral; $x$, $y$, $z$ are the distances of $A$ from the lines $BD$, $BC$, $CD$ respectively. Prove that

$$\frac{BD}{x} = \frac{BC}{y} + \frac{CD}{z}$$

Problem 195 [RMO, India]

$ABCD$ is a quadrilateral and $P$, $Q$ are mid-points of $CD$, $AB$ respectively. Let $AP$, $DQ$ meet at $X$ and $BP$, $CQ$ meet at $Y$. Prove that

$$\text{area}(ADX) + \text{area}(BCY) = \text{area}(PXQY).$$
Problem 196 [RMO, India]

The cyclic octagon $ABCDEFGH$ has sides $a, a, a, b, b, b$ respectively. Find the radius of the circle that circumscribes $ABCDEFGH$ in terms of $a$ and $b$.

Problem 197 [AMM]

Prove that in an acute triangle with angles $A, B$ and $C$

$$\frac{(1-\cos A)(1-\cos B)(1-\cos C)}{\cos A \cos B \cos C} \geq \frac{8(\tan A+\tan B+\tan C)^3}{27(\tan A+\tan B)(\tan B+\tan C)}$$

Problem 198 [Mathscope, Vietnam]

In a triangle $ABC$, denote by $l_a, l_b, l_c$ the internal angle bisectors, $m_a, m_b, m_c$ the medians and $h_a, h_b, h_c$ the altitudes to the sides $a, b, c$ of the triangle. Prove that

$$\frac{m_a}{l_b+h_b} + \frac{m_b}{h_c+l_c} + \frac{m_c}{l_a+h_a} \geq \frac{3}{2}$$

Problem 199 [Mathscope, Vietnam]

Let $AM, BN, CP$ be the medians of triangle $ABC$. Prove that if the radius of the incircles of triangles $BCN, CAP$ and $ABM$ are equal in length, then $ABC$ is an equilateral triangle.

Problem 200 [Mathscope, Vietnam]

Given a triangle with incentre $I$, let $l$ be a variable line passing through $I$. Let $l$ intersect the ray $CB$, sides $AC, AB$ at $M, N, P$ respectively. Prove that the value of

$$\frac{AB}{PA \cdot PB} + \frac{AC}{NA \cdot NC} - \frac{BC}{MB \cdot MC}$$

is independent of the choice of $l$.

Problem 201 [Mathscope, Vietnam]

Let $I$ be the incentre of triangle $ABC$ and let $m_a, m_b, m_c$ be the lengths of the medians from vertices $A, B$ and $C$, respectively. Prove that

$$\frac{IA^2}{m_a^2} + \frac{IB^2}{m_b^2} + \frac{IC^2}{m_c^2} \leq \frac{3}{4}$$
Problem 202 [Mathscope, Vietnam]

Let $R$ and $r$ be the circumradius and inradius of triangle $ABC$; the incircle touches the sides of the triangle at three points which form a triangle of perimeter $p$. Suppose that $q$ is the perimeter of triangle $ABC$. Prove that
\[
\frac{r}{R} \leq \frac{p}{q} \leq \frac{1}{2}
\]

Problem 203 [AMM]

Let $a$, $b$ and $c$ be the lengths of the sides of a triangle and let $R$ and $r$ be the circumradius and inradius of that triangle, respectively. Show that
\[
R^2r \geq \exp\left(\frac{(a-b)^2}{2a^2} + \frac{(b-c)^2}{2b^2} + \frac{(c-a)^2}{2c^2}\right)
\]

Problem 204 [AMM]

Consider an acute triangle with sides of lengths $a$, $b$ and $c$ and with an inradius of $r$ and circumradius of $R$. Show that
\[
r \leq \frac{\sqrt{2(2a^2-(b-c)^2)(2b^2-(c-a)^2)(2c^2-(a-b)^2)}}{(a+b)(b+c)(c+a)}.
\]

Problem 205 [AMM]

Let $a$, $b$ and $c$ be the lengths of the sides of a triangle, and let $R$ and $r$ denote the circumradius and inradius of the triangle. Show that
\[
\frac{R}{2r} \geq \left(\frac{4a^2}{4a^2-(b-c)^2} \frac{4b^2}{4b^2-(c-a)^2} \frac{4c^2}{4c^2-(a-b)^2}\right)^2.
\]

Problem 206 [AMM]

Let $ABC$ be a triangle with sides $a$, $b$ and $c$ all different, and corresponding angles $\alpha$, $\beta$ and $\gamma$. Show that
(a) $(a + b) \cot(\beta + \frac{\gamma}{2}) + (b + c) \cot(\gamma + \frac{\alpha}{2}) + (c + a) \cot(\alpha + \frac{\beta}{2}) = 0$.
(b) $(a - b) \tan(\beta + \frac{\gamma}{2}) + (b - c) \tan(\gamma + \frac{\alpha}{2}) + (c - a) \tan(\alpha + \frac{\beta}{2}) = 4(R + r)$.

Problem 207 [AMM]

Let $r$, $R$ and $s$ be the radii of the incircle, circumcircle and semi-perimeter of a triangle. Prove that
\[
\sqrt[n]{\frac{4}{3}} \leq \sqrt{\frac{r^2+4Rr}{3}} \leq \frac{s}{3}
\]

Problem 208 [AMM]

Let $a$, $b$ and $c$ be the lengths of the sides of a nondegenerate triangle, let $p = (1/2)(a+b+c)$, and let $r$ and $R$ be the inradius and circumradius of the triangle, respectively. Show that
\[
\frac{a}{2} \left(\frac{4r-R}{R}\right) \leq \sqrt{(p-b)(p-c)} \leq \frac{a}{2}.
\]