Recent developments and open problems in the theory of permanents[1]

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1 Introduction

If A is an $n \times n$ complex matrix, then the permanent of A, denoted $perA$, is defined as

$$perA = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where S_n is the symmetric group of degree n. As an example,

$$per \begin{bmatrix} 1 & 5 & 2 \\ 3 & 2 & 4 \\ 2 & 1 & 5 \end{bmatrix} = 10 + 40 + 6 + 8 + 4 + 75 = 143.$$

Thus the definition of the permanent is similar to that of the determinant except for the sign associated with each term in the summation. This minor difference in the definition makes the two functions quite unlike each other. Perhaps the permanent cannot compete with its cousin, the determinant, in terms of the depth of theory and the breadth of applications, but it is safe to say that the permanent also exhibits both these characteristics in ample measure, a fact that has not received enough attention.

The permanent has a rich structure when restricted to certain classes of matrices, particularly, matrices of zeros and ones, (entrywise) nonnegative matrices and positive semidefinite matrices. Furthermore, there is a certain similarity of its properties over the class of nonnegative matrices and the class of positive semidefinite matrices, which is not yet fully understood. In this article we describe some properties of permanents over these three classes, putting

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emphasis on recent developments and open problems. The article is by no means an extensive survey of permanents, rather it is biased towards topics in which I have been interested over the years. The interested reader can however follow the references at the end that point to several sources dealing with various aspects of the theory of permanents.

2 Matrices of zeros and ones

Let A be an $n \times n$ matrix. If $\sigma \in S_n$, then the set $\{a_{1\sigma(1)}, \ldots, a_{n\sigma(n)}\}$ is called a diagonal of A corresponding to the permutation σ. The product $\pi_{i=1}^{n} a_{i\sigma(i)}$ is called a diagonal product. A diagonal is positive if the corresponding diagonal product is positive. Note that the permanent of a $0-1$ matrix equals the number of positive diagonals of A.

The concept of a positive diagonal is clearly related to that of a system of distinct representatives in a family of sets, as well as to perfect matching in a bipartite graph. This is explained as follows.

If A is a $0-1$ $n \times n$ matrix then let $A_i = \{j : a_{ij} = 1\}, i = 1, \ldots, n$. We say that the set $\{x_1, \ldots, x_n\}$ is a system of distinct representatives of the family $\{A_1, \ldots, A_n\}$ if $x_i \in A_i, i = 1, \ldots, n$. The well-known theorem due to P.Hall asserts that the family $\{A_1, \ldots, A_n\}$ admits a system of distinct representatives if and only if the union of any k members of the family contain at least k elements, $k = 1, \ldots, n$.

Given a $0-1$ $n \times n$ matrix we may naturally associate a bipartite graph G with A. The partite sets X and Y are the index sets of rows and columns respectively. There is an edge from the i-th vertex of X to the j-th vertex of Y if and only if $a_{ij} = 1$. Then by the König-Egervary Theorem G has a perfect matching if and only if for any $S \subset X$, the neighbour set of S has at least $|S|$ elements. This statement as well as Hall's Theorem are equivalent to the next result.

Theorem 1 *[Frobenius-König Theorem] Let A be a $0-1$ $n \times n$ matrix. Then per$A$ is zero if and only if A has a zero submatrix of order $r \times s$ such that $r + s = n + 1$.*

Another result equivalent to Theorem 1 is the Marriage Theorem of Halmos and Vaughan. As these equivalences show, the permanent function on the set of $0-1$ matrices is intimately connected with several classical combinatorial problems. However, Theorem 1 also admits extensions of a different type, making it evident that the result is important in classical matrix theory as well. If A is a $0-1$ matrix then the term rank of A is the maximum k such that A has a $k \times k$ submatrix with positive permanent. Although the term rank appears to be a purely combinatorial concept, related to the matching number of a bipartite graph, it is also related to the classical rank as we indicate next.

We say that a matrix over a field is of zero type if each of its rows is a linear combination of the remaining rows and each of its columns is a linear combination of the remaining columns. The following result is obtained in [5].

Theorem 2 *Let A be an $n \times n$ matrix over a field F. Then A is singular if and only if A has a zero type submatrix B of order $r \times s$ such that $r + s \geq n + rank(B)$.*

A matrix is said to be generic if its nonzero elements are algebraically independent indeterminates. The term rank of a generic matrix coincides with its rank (over the field generated by its nonzero elements). If Theorem 2 is applied to a generic matrix then we recover Theorem 1. For related results and extensions to bimatroids, see [23].

It was conjectured by Minc in 1963 that if A is a $0-1$ matrix with row sums $r_1, \ldots, r_n$, then

$$per A \leq \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}.$$

The conjecture was proved by Brègman in 1973 and Schrijver gave an easier proof in 1978. Recently, Soules[31,32] has obtained extensions of the bound to nonnegative matrices, using a different proof technique.

3 Nonnegative matrices

The class of entrywise nonnegative matrices, which properly includes the class of $0-1$ matrices, is another class on which the permanent is well-behaved. We will write $A \geq 0$ to indicate that each element of A is nonnegative. Note that when $A \geq 0$, there are no cancelations in the expression for the permanent and it provides the sum of all the diagonal products of the matrix. In the context of graph theory it can be interpreted as the sum of the weights of all perfect matchings in the associated bipartite graph.

The $n \times n$ matrix A is said to be doubly stochastic if $A \geq 0$ and each row and column sum of A is 1. The set of $n \times n$ doubly stochastic matrices is a compact, convex set, and we denote it by Ω_n.

A permutation matrix is a matrix obtained from the identity matrix by permuting its rows and columns. Clearly a permutation matrix is doubly stochastic. A celebrated theorem of Birkhoff and von Neumann asserts that the extreme points of Ω_n are precisely the $n!$ permutation matrices of order n. Thus a matrix is doubly stochastic if and only if it can be expressed as a convex combination of permutation matrices. As a simple consequence of the Birkhoff-von Neumann Theorem we can conclude that the permanent of a doubly stochastic matrix must be positive. Therefore it is natural to enquire about the minimum of the permanent over Ω_n. Van der Waerden conjectured in 1928 that the minimum of the permanent over Ω_n equals $\frac{n!}{n^n}$ and is attained uniquely at the matrix J_n, which is the $n \times n$ matrix with each entry equal to $\frac{1}{n}$.

Marcus and Newman[21] obtained some important partial results towards the solution of the van der Waerden conjecture. In particular they showed that if $A \in \Omega_n$ is a permanent minimizer, then all permanental cofactors of A must exceed or equal $perA$. From this result it can be seen that if $A \in \Omega_n$ is a permanent minimizer such that each entry of A is positive, then $A = J_n$.

After the Marcus and Newman paper, attempts towards the solution of the van der Waerden conjecture again picked up in the mid sixties and seventies and it gave a significant impetus to work in the area of combinatorial matrix theory.

The breakthrough came around 1981 when Egorychev and Falikman independently proved the van der Waerden conjecture. The main tools in the proof due to Egorychev were the Marcus-Newman result and the Alexandroff inequality which we now describe.

If A is an $n \times n$ matrix, then let a_i denote the i-th column of $A, i = 1, 2, \ldots, n$. Let A be an $n \times n$ positive matrix and consider the bilnear form

$$per(a_1, \ldots, a_{n-2}, x, y), x, y \in R^n. \tag{1}$$

It turns out that the bilnear form (1) is indefinite with exactly one positive eigenvalue. As a consequence one obtains the Alexandroff inequality which asserts that if A is a nonnegative $n \times n$ matrix and if $x, y \in R^n$ where x has positive coordinates, then

$$(perA)^2 \geq per(a_1, \ldots, a_{n-2}, x, x)per(a_1, \ldots, a_{n-2}, y, y). \tag{2}$$

The Alexandroff inequality was in fact proved for the mixed discriminant, which we will discuss in the next section.

The Birkhoff-von Neumann Theorem has been extensively studied and generalized. For a recent extension in the area of quantum probability, see [24].

Let A be an $n \times n$ $0-1$ matrix with k ones in each row and column. The permanent of such a matrix is of special interest since it counts the number of perfect matchings in a regular bipartite graph. Note that $\frac{1}{k}A \in \Omega_n$ and hence we do get a bound for $perA$ by the Egorychev-Falikman proof of the van der Waerden bound. Let $\Omega_{k,n}$ be the set of matrices in Ω_n with each entry 0 or $\frac{1}{k}$. The following bound was conjectured by Schrijver and Valiant in 1980, and proved by Schrijver [27]. If A is an $n \times n$ matrix in $\Omega_{k,n}$ with each entry 0 or $1/k$, then

$$\min\{perA : A \in \Omega_{k,n}\} \geq (\frac{k-1}{k})^{(k-1)n}.$$

For any k and n, let $p(k, n)$ be the number of perfect matchings in any k-regular bipartite graph with $2n$ vertices. The van der Waerden bound implies that

$$inf_{k \in N} \frac{p(k, n)}{k^n} = \frac{n!}{n^n}.$$

Note that the Schrijver-Valiant bound implies that

$$p(k, n) \geq \frac{(k-1)^{(k-1)}}{k^{(k-2)}}.$$

Moreover, Schrijver[27] has shown that

$$inf_{n \in N} p(k, n)^{1/n} = \frac{(k-1)^{(k-1)}}{k^{(k-2)}}.$$

Thus both bounds are best possible in different asymptotic directions.

Recently, Gurvits[14] has given a unified proof of the van der Waerden conjecture and the Schrijver-Valiant conjecture using hyperbolic polynomials.

According to the van der Waerden bound the permanent achieves its minimum over Ω_n at the matrix with each entry $\frac{1}{n}$. However finding the minimum of the permanent on certain subsets has also been considered. In this context the following folklore conjecture appears to be very notorious.

Conjecture 1 The permanent achieves its minimum over the set of $n \times n$ doubly stochastic matrices with zeros on the diagonal uniquely at the $n \times n$ matrix with each diagonal entry zero and each off-diagonal entry $\frac{1}{n-1}$.

4 Mixed discriminants

If $A^k = (a^k_{ij})$ are $n \times n$ matrices, $k = 1, 2, \dots, n$, then their mixed discriminant, denoted by $D(A^1, \dots, A^n)$, is defined as

$$D(A^1, \dots, A^n) = \frac{1}{n!} \sum_{\sigma \in S_n} \begin{vmatrix} a_{11}^{\sigma(1)} & \cdots & a_{1n}^{\sigma(n)} \\ \vdots & \ddots & \vdots \\ a_{n1}^{\sigma(1)} & \cdots & a_{nn}^{\sigma(n)} \end{vmatrix}, \tag{3}$$

where S_n denotes, as usual, the set of permutations of $1, 2, \dots, n$. (Throughout this section, A^k should not be confused with the k-th power of A.) Thus, if $A = (a_{ij}), B = (b_{ij})$ are 2×2 matrices, then

$$D(A, B) = \frac{1}{2}(a_{11}b_{22} - a_{21}b_{12} - a_{12}b_{21} + a_{22}b_{11}).$$

We now indicate that the mixed descriminant provides a generalization of both the determinant and the permanent. If $A^k = A, k = 1, 2, \dots, n$, then clearly, $D(A^1, \dots, A^n) = detA$. Also, if each A_k is a diagonal matrix,

$$A^k = \begin{bmatrix} a_{11}^k & & \\ & \ddots & \\ & & a_{nn}^k \end{bmatrix},$$

then $D(A^1, \dots, A^n)$ equals $\frac{1}{n!}\text{per}B$ where $B = (b_{ij}) = (a^j_{ii})$.

We now consider some properties of mixed discriminants of positive semidefinite matrices.

Let $A^k, k = 1, 2, \dots, n$, be positive semidefinite $n \times n$ matrices and suppose $A^k = X_k {X_k}^T$ for each k. Then it can be proved that

$$D(A^1, \dots, A^n) = \frac{1}{n!} \sum (det(x_1, \dots, x_n))^2,$$

where the sum is over all choices $\{x_1, \ldots, x_n\}$ such that x_k is a column of $X_k, k = 1, 2 \ldots, n$. As an immediate consequence we conclude that the mixed discriminant of positive semidefinite matrices is nonnegative. When each A^k is a diagonal positive semidefinite matrix then the statement merely reduces to the fact that the permanent of a nonnegative matrix is nonnegative.

It is natural to enquire about the positivity of D $(A^1, \ldots, A^n)$ when each A^k is positive semidefinite. Here one can prove the following, using Rado's generalization of Hall's theorem. Let $A^k, k = 1, 2, \ldots, n$, be $n \times n$ positive semidefinite matrices. Then $D(A^1, \ldots, A^n) > 0$ if and only if for any $T \subset \{1, 2, \ldots, n\}$, the rank of $\left(\sum_{i \in T} A^i\right)$ is at least $|T|$.

Let $\mathcal{D}_n$ denote the set of all $n-$tuples $\mathbf{A} = (A^1, A^2, \ldots, A^n)$ of $n \times n$ positive semidefinite matrices satisfying trace $A^i = 1, i = 1, 2, \ldots, n; \sum_{i=1}^{n} A^i = I$. Then by the process of identifying a nonnegative $n \times n$ matrix with an $n-$tuple of diagonal matrices described in the preceding discussion, $\mathcal{D}_n$ can be viewed as a generalization of the class of $n \times n$ doubly stochastic matrices. The permanent function on Ω_n, the polytope of $n \times n$ doubly stochastic matrices, is generalized to the mixed discriminant over $\mathcal{D}_n$. It can be shown [2] that if $\mathbf{A} = (A^1, \ldots, A^n) \in \mathcal{D}_n$, then $D(A^1, A^2, \ldots, A^n) > 0$.

It was conjectured in [2] that the mixed discriminant achieves its minimum over $\mathcal{D}_n$ precisely at $(A^1, \ldots, A^n)$ where each A^i is the diagonal matrix

$$\begin{bmatrix} \frac{1}{n} & & \\ & \ddots & \\ & & \frac{1}{n} \end{bmatrix}.$$

The conjecture has been recently proved by Gurvits[13].

Another open problem posed in [2] is to characterize the extreme points of $\mathcal{D}_n$.

5 Positive definite matrices

When one deals with the concept of positivity in the context of matrices then one must keep in mind the two notions of positivity, the notion of a positive operator, which corresponds to a positive semidefinite matrix and the notion of an entrywise positive matrix. There are curious similarities regarding the properties of the two classes. Olga Taussy Todd proposed the problem of explaining this similarity and it is commonly known as the Taussy unification problem, see [26].

If A is an $n \times n$ complex hermitian matrix, then A is said to be positive definite if $x^*Ax > 0$ for any nonzero x, and positive semidefinite if $x^*Ax \geq 0$ for any x.

The permanent exhibits interesting properties on the class of positive definite matrices as well. As an example, if A is a nonnegative matrix then $perA$ is clearly nonnegative. It turns out that if A is positive semidefinite, then $perA \geq 0$. This fact is not obvious from the definition of permanent, since the sum in the definition may contain both positive as well as negative terms.

If A and B are both positive semidefinite such that $A - B$ is also positive semidefinite, then it is known that $perA \geq perB$. Now if A is a positive semidefinite matrix which is also doubly stochastic, then it is not difficult to show that $A - J_n$ is positive semidefinite. Then $perA \geq perJ_n$ and thus the van der Waerden conjecture is verified for positive semidefinite matrices. Later we will give an example of an open problem which appears difficult when the matrix is positive semidefinite but is easy for nonnegative matrices.

We now introduce some notation. If A, B are matrices of order $m \times n$ and $p \times q$ respectively, then the Kronecker product of A, B is denoted by $A \otimes B$. Thus $A \otimes B$ is the $mp \times nq$ matrix given by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

The main property of Kronecker product is that if A, B, C, D are matrices such that AC and BD are defined, then

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

It can be proved using the preceding property that if A, B are positive semidefinite, then $A \otimes B$ is positive semidefinite.

Let us assume that the elements of S_n, the permutation group of degree n, have been ordered in some way. This ordering will be assumed fixed in the subsequent discussion. Let A be an $n \times n$ matrix. The Schur power of A, denoted by $\pi(A)$, is the $n! \times n!$ matrix whose rows as well as columns are indexed by S_n and whose $(\sigma, \tau)-$entry is $\prod_{i=1}^n a_{\sigma(i)\tau(i)}$ if $\sigma, \tau \in S_n$.

As an illustration, suppose the elements of S_3 are ordered as 123, 132, 213, 231, 312, 321, and let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}.$$

Then

$$\pi(A) = \begin{bmatrix} aek & afh & bdk & bfg & cdh & ceg \\ afh & aek & bfg & bdk & ceg & cdh \\ bdk & cdh & aek & ceg & afh & bfg \\ cdh & bdk & ceg & aek & bfg & afh \\ bfg & ceg & afh & cdh & aek & bdk \\ ceg & bfg & cdh & afh & bdk & aek \end{bmatrix}.$$

We make some simple observations about $\pi(A)$. The diagonal entries of $\pi(A)$ are all equal to $a_{11} \cdots a_{nn}$ where A is $n \times n$. The sum of the entries in any row or column of $\pi(A)$ equals $perA$. In particular, $perA$ is an eigenvalue of $\pi(A)$.

If A is $n \times n$, then after a permutation of the rows and an identical permutation of the columns, $\pi(A)$ can be viewed as a principal submatrix of $\otimes^n A = A \otimes A \otimes \cdots \otimes A$, taken n times. If A is positive semidefinite, then $\otimes^n A$ is positive semidefinite and hence so is $\pi(A)$. Since $perA$ is an eigenvalue of $\pi(A)$, we immediately have a proof of the fact that if A is positive semidefinite, then $perA \geq 0$.

It is true that $detA$ is also an eigenvalue of $\pi(A)$. To see this, define a vector ϵ of order $n!$ as follows. Index the elements of ϵ by S_n. If $\tau \in S_n$, then set $\epsilon(\tau) = 1$ if τ is even and -1 if τ is odd. Then for any $\sigma \in S_n$,

$$\begin{aligned} \sum_{\tau \in S_n} \epsilon(\tau) \prod_{i=1}^n a_{\sigma(i)\tau(i)} &= \sum_{\tau \in S_n} \epsilon(\tau) \prod_{i=1}^n a_{i\tau o\sigma^{-1}(i)} \\ &= \sum_{\rho \in S_n} \epsilon(\rho\sigma) \prod_{i=1}^n a_{i\rho(i)} \\ &= \epsilon(\sigma) \sum_{\rho \in S_n} \epsilon(\rho) \prod_{i=1}^n a_{i\rho(i)} \\ &= \epsilon(\sigma) detA. \end{aligned}$$

Thus $\pi(A)\epsilon = (detA)\epsilon$ and hence $detA$ is an eigenvalue of $\pi(A)$ with ϵ as the corresponding eigenvector.

Thus $perA$ and $detA$ are both eigenvalues of $\pi(A)$ and this explains the similarity of certain properties of the permanent and the determinant, restricted to the class of positive semidefinite matrices.

A remarkable result due to Schur asserts that if A is positive semidefinite, then $detA$ is in fact the smallest eigenvalue of $\pi(A)$. Recall the extremal characterization of the eigenvalues of a hermitian matrix. If B is hermitian with the least eigenvalue λ_n, then λ_n is the minimum of x^*Bx taken over unit vectors x. Therefore Schur's result provides a rich source of inequalities for the determinant of a positive definite matrix since we can make a judicious choice of x and get an inequality. For example, if x has all coordinates zero except one, then Schur's Theorem reduces to the well-known Hadamard Inequality, that the determinant of a positive semidefinite matrix is bounded above by the product of the main diagonal elements. Another example is the following.

Let A be an $n \times n$ positive semidefinite matrix and let G be a subgroup of S_n. Then

$$detA \leq \sum_{\sigma \in G} \prod_{i=1}^{n} a_{i\sigma(i)}. \tag{4}$$

The expression appearing on the right hand side of (4) is an example of an immanant of A. Note that in (4), if G is the subgroup consisting of the identity permutation only, then we get the Hadamard Inequality.

One of the most important outstanding open problems at present is to decide whether an anlogue of Schur's result holds for the permanent. More precisely, the problem is formulated as a conjecture due to Soules as follows:

Conjecture 2 If A is positive semidefinite, then $perA$ is the largest eigenvalue of $\pi(A)$.

The conjecture has been proved for matrices of order at most 3 , see [1]. For some further ideas, see [29,30]. We remark in passing that Conjecture 2 is easily verified if A is assumed to be entrywise nonnegative as well. For in this case $\pi(A)$ is a nonnegative matrix and the vector of all ones is an eigenvector of $\pi(A)$ corresponding to $perA$. It follows by the Perron-Frobenius theorem that $perA$ must be the spectral radius, and hence the maximal eigenvalue, of $\pi(A)$.

There are several conjectures weaker that Conjecture 2 which have also received considerable attention. We discuss some of these below. The next is the so called “permanent- on-top conjecture” or the “permanental dominance conjecture” which asserts that any immanant of a positive semidefinite matrix is dominated by the permanent, see [17,22]. More precisely, the conjecture is the following.

Conjecture 3 Let G be a subgroup of S_n and let χ be a complex character on G. If A is an $n \times n$ positive semidefinite matrix, then

$$perA \geq \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

The Schur product of two $n \times n$ matrices A and B, denoted $A \circ B$, is simply their entrywise product. If A and B are $n \times n$ positive semidefinite matrices then $A \circ B$ is also positive semidefinite. Oppenheim's inequality asserts that if A and B are $n \times n$ positive semidefinite matrices then $det(A \circ B) \geq (detA) b_{11} \cdots b_{nn}$. Note that the inequality reduces to the Hadamard inequality when B is the identity matrix. The next conjecture, which can be shown to be weaker than Conjecture 2, has been proposed in [1].

Conjecture 4 If A and B are $n \times n$ positive semidefinite matrices then

$$per(A \circ B) \leq (perA) b_{11} \cdots b_{nn}.$$

In this context it may be mentioned that the Hadamard inequality for the permanent (which is a special case of Conjecture 4) has been proved by Marcus[20]. For the relevance of Conjecture 4 in some topics in mathematical physics, see [9].

If A is an $n \times n$ matrix, then let $A(i,j)$ denote the submatrix of A obtained by deleting its i-th row and j-th column. The next conjecture, which is also weaker than Conjecture 2, was proposed in [1].

Conjecture 5 Let A be an $n \times n$ positive semidefinite matrix. Then $perA$ is the largest eigenvalue of the $n \times n$ matrix with its (i,j)-entry equal to $a_{ij} perA(i,j), i, j, = 1, \ldots, n$.

If $k \leq n$, then let $G_{k,n}$ denote the set of all strictly increasing functions from $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$, ordered lexicographically. If A is an $n \times n$ matrix and $\alpha, \beta \in G_{k,n}$, then $A[\alpha, \beta]$ will denote the $k \times k$ submatrix of A whose rows and columns are indexed by α, β respectively, while the $(n-k) \times (n-k)$ submatrix of A whose rows and columns are indexed by α^c, β^c will be denoted $A(\alpha, \beta)$. Let $\mathcal{C}_k(A)$ be $\binom{n}{k} \times \binom{n}{k}$ matrix indexed by $G_{k,n}$ with $(\mathcal{C}_k(A))_{\alpha,\beta} = per(A(\alpha, \beta)) per(A[\alpha, \beta])$. The following generalization of Conjecture 5 has recently been proposed by Pate[25] who has also obtained some partial results.

Conjecture 6 If A is an $n \times n$ positive semidefinite matrix, then $perA$ is the largest eigenvalue of $\mathcal{C}_k(A), k = 1, \ldots, n-1$.

We remark that Conjecture 6 reduces to Conjecture 5 when $k = n-1$.

6 A q-analogue of the permanent

If $\sigma \in S_n$, then an *inversion* of σ is a pair (i, j) such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$. As an example, the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 2 & 1 & 5 \end{pmatrix}$$

in S_6 has 9 inversions. If $\sigma \in S_n$ then let $\ell(\sigma)$ denote the number of inversions of σ. The identity permutation has zero inversions. The maximum number of inversions in S_n is $\frac{n(n-1)}{2}$, attained at the permutation $n, n-1, \ldots, 2, 1$. Note that $\ell(\sigma)$ is even (odd) if σ is an even (odd) permutation.

If A is an $n \times n$ matrix and q a real number, then we define the $q-$*permanent* of A, denoted by $\mathrm{per}_q(A)$ as

$$\mathrm{per}_q(A) = \sum_{\sigma \in S_n} q^{\ell(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Observe that $\mathrm{per}_{-1}(A) = detA$, $\mathrm{per}_0(A) = \prod_{i=1}^{n} a_{ii}$ and $\mathrm{per}_1(A) = \mathrm{per}\, A$. Here we have made the usual convention that $0^0 = 1$. The $q-$permanent thus provides a parametric generalization of both the determinant and the permanent. The $q-$permanent appears to be a function with a very rich structure but at the same time it does not lend itself to manipulations very easily.

If A is a positive semidefinite matrix and $-1 \leq q \leq 1$, then $\mathrm{per}_q(A) \geq 0$. This result has been proved by Bożejko and Speicher [8] in connection with a problem in Mathematical Physics dealing with parametric generalizations of Brownian motion. A proof based on conditionally negative definite matrices has been given in [6, Chapter 4].

The following monotonicity property of the $q-$permanent has been conjectured in [3].

Conjecture 7 If A is positive semidefinite, then $\mathrm{per}_q(A)$ as a function of q is monotonically increasing in $[-1, 1]$.

Note that Conjecture 7 can be motivated by the known fact that

$$perA \geq \prod_{i=1}^{n} a_{ii} \geq detA$$

for a positive semidefinite matrix A. Conjecture 7 has been verified for $n \leq 3$ and their is overwhelming numerical evidence in its favour.

A different generalization of the permanent and the determinant has been considered by Vere-Jones [33], which is as follows. If A is an $n \times n$ matrix and if α is a real number then let

$$det_\alpha A = \sum_{\sigma \in S_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $\nu(\sigma)$ is the number of cycles in σ. Note that when $\alpha = -1$, then $det_\alpha A = detA$, while if $\alpha = 1$, then $det_\alpha A = perA$. This function arises in connection with some random point processes; see [28], where the following conjecture is posed.

Conjecture 8 If A is positive semidefinite $n \times n$ matrix and if $0 \leq \alpha \leq 2$, then $det_\alpha A \geq 0$.

For some related stochastic processes involving the permanent, see [12,15].

7 Computation of the permanent

The determinant can be evaluated efficiently using Gaussian elimination. The computation of the permanent is however much more complicated. In the last two decades many contributions in the area of computational complexity have been made towards exact or approximate computation of the permanent. A classical result of Valiant asserts that the problem of computing the permanent is $\sharp P$-complete, which basically means that there is almost no possibility of finding a polynomial time algorithm for computing the permanent. At the same time the possibility of computing the permanent within arbitrarily small relative error in polynomial time is not ruled out.

Given a $0-1$ matrix A, form a random matrix B by assigning $\pm$ signs independently at random to the elements of A. Then $(detB)^2$ is an unbiased estimator of $perA$. In general the variance of the estimator may be very large. Karmarkar et al [18] replaced the ± 1 entries of B by randomly choosing complex roots of unity and later Barvinok[7] used random quaternions with the aim of reducing the variance. This idea was naturally extended by employing Clifford algebras, see [11]. A polynomial-time approximation algorithm for the permanent of a nonnegative matrix is given in [16].

8 Symmetric function means and permanents

We conclude by recalling a conjecture posed in [4]. First we introduce some notation. Let $x = (x_1, \ldots, x_n)$ be a positive vector. We denote by $e_{r,n}(x)$ the r-th elementary symmetric function in $x_1, \ldots, x_n$. Thus

$$e_{r,n}(x) = \sum_{i,<\ldots<i_r} x_{i_1} \ldots x_{i_r}.$$

We set $e_{0,n}(x) = 1$.

Several inequalities are available in the literature for ratios of elementary symmetric functions, also known as symmetric function means. Let

$$M_{r,n}(x) = \frac{e_{r,n}(x)}{e_{r-1,n}(x)}, r = 1, 2, \ldots, n.$$

A well-known result of Marcus and Lopes [19] asserts that for any two positive vectors x, y;

$$M_{r,n}(x+y) \geq M_{r,n}(x) + M_{r,n}(y). \tag{5}$$

Let $c, b_1, b_2, \ldots$ be positive vectors in R^n which will be held fixed. For any positive vector x in R^n and for $1 \leq r \leq n$, define

$$S_{r,n}(x) = \frac{per[\underbrace{x, \ldots, x}_{r}, b_1, \ldots, b_{n-r}]}{per[\underbrace{x, \ldots, x}_{r-1}, b_1, \ldots, b_{n-r}, c]} \tag{6}$$

If $c = b_i = \mathbf{1}$, the vector of all ones, for all i, then

$$S_{r,n}(x) = \frac{r}{n-r+1} M_{r,n}(x)$$

and thus the function in (6) is more general than a symmetric function mean. It is thus natural to conjecture that a generalization of (5) holds; more precisely, the following was posed in [4].

Conjecture 9 For any positive vectors x, y;

$$S_{r,n}(x+y) \geq S_{r,n}(x) + S_{r,n}(y). \tag{7}$$

The case $r = 1$ of (7) is trivial. The case $r = 2$ which is closely related to the Alexandroff inequality, is proved in [4].

For a recent survey concerning permanents we refer to [10] where further references can be found.

REFERENCES

1. Bapat, R. B. ; Sunder, V. S. An extremal property of the permanent and the determinant. Linear Algebra Appl. 76 (1986), 153–163.

2. Bapat, R. B. Mixed discriminants of positive semidefinite matrices. Linear Algebra Appl. 126 (1989), 107–124.

3. Bapat, R. B. Interpolating the determinantal and permanental Hadamard inequality. Linear and Multilinear Algebra 32 (1992), no. 3-4, 335–337.

4. Bapat, R. B. Symmetric function means and permanents. Linear Algebra Appl. 182 (1993), 101–108.

5. Bapat, R. B. König's theorem and bimatroids. Proceedings of the 3rd ILAS Conference (Pensacola, FL, 1993). Linear Algebra Appl. 212/213 (1994), 353–365.

6. Bapat, R. B. ; Raghavan, T. E. S. Nonnegative matrices and applications. Encyclopedia of Mathematics and its Applications, 64. Cambridge University Press, Cambridge, 1997. xiv+336 pp. ISBN: 0-521-57167-7

7. Barvinok, Alexander. Polynomial time algorithms to approximate permanents and mixed discriminants within a simply exponential factor. Random Structures Algorithms 14 (1999), no. 1, 29–61.

8. Bożejko, M. and Speicher, R. An example of a generalized Brownian motion. Comm. Math. Phys. 137 (1991), 519-531.

9. Chefles, Anthony ; Andersson, Erika ; Jex, Igor. Unambiguous comparison of the states of multiple quantum systems. J. Phys. A 37 (2004), no. 29, 7315–7340.

10. Cheon, Gi-Sang ; Wanless, Ian M. An update on Minc's survey of open problems involving permanents. Linear Algebra Appl. 403 (2005), 314–342.

11. Chien, Steve ; Rasmussen, Lars ; Sinclair, Alistair. Clifford algebras and approximating the permanent. Special issue on STOC2002 (Montreal, QC). J. Comput. System Sci. 67 (2003), no. 2, 263–290.

12. Eisenbaum, Nathalie and Kaspi, Haya. On permanental processes. arXiv:math.PR/0610600v3, 10 Nov. 2006.

13. Gurvits, Leonid. The van der Waerden conjecture for mixed discriminants. Adv. Math. 200 (2006), no. 2, 435–454.

14. Gurvits, Leonid. Hyperbolic polynomials approach to Van der Waerden/Schrijver-Valiant like conjectures: sharper bounds, simpler proofs and algorithmic

applications. STOC'06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing, 417–426, ACM, New York, 2006.

15. Hough, J. Ben ; Krishnapur, Manjunath ; Peres, Yuval ; Virág, Bálint. Determinantal processes and independence. Probab. Surv. 3 (2006), 206–229 (electronic).

16. Jerrum, Mark ; Sinclair, Alistair ; Vigoda, Eric. A polynomial-time approximation algorithm for the permanent of a matrix with nonnegative entries. J. ACM 51 (2004), no. 4, 671–697 (electronic).

17. Johnson, Charles R. The permanent-on-top conjecture: a status report. Current trends in matrix theory (Auburn, Ala., 1986), 167–174, North-Holland, New York, 1987.

18. Karmarkar, N. ; Karp, R. ; Lipton, R. ; Lovász, L. ; Luby, M. A Monte Carlo algorithm for estimating the permanent. SIAM J. Comput. 22 (1993), no. 2, 284–293.

19. Marcus, M. ; Lopes, L. Inequalities for symmetric functions and Hermitian matrices. Canad. J. Math. 9 (1957), 305–312.

20. Marcus, Marvin. The permanent analogue of the Hadamard determinant theorem. Bull. Amer. Math. Soc. 69 1963 494–496.

21. Marcus, Marvin ; Newman, Morris. On the minimum of the permanent of a doubly stochastic matrix. Duke Math. J. 26 1959 61–72.

22. Merris, Russell. The permanental dominance conjecture. Current trends in matrix theory (Auburn, Ala., 1986), 213–223, North-Holland, New York, 1987.

23. Murota, K. Matrices and Matroids for Systems Analysis, Springer-Verlag 2000.

24. Parthasarathy, K. R. Extremal quantum states in coupled systems, Ann. Inst. H. Poincar Probab. Statist. 41 (2005), no. 3, 257–268.

25. Pate, Thomas H. On permanental componds. Linear Algebra Appl. to appear

26. Schneider, Hans. Olga Taussky-Todd's influence on matrix theory and matrix theorists. A discursive personal tribute. Linear and Multilinear Algebra 5 (1977/78), no. 3, 197–224.

27. Schrijver, Alexander. Counting 1-factors in regular bipartite graphs. J. Combin. Theory Ser. B 72 (1998), no. 1, 122–135.

28. Shirai, Tomoyuki ; Takahashi, Yoichiro. Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes. J. Funct. Anal. 205 (2003), no. 2, 414–463.

29. Soules, George W. Constructing symmetric nonnegative matrices. Linear and Multilinear Algebra 13 (1983), no. 3, 241–251.

30. Soules, George W. An approach to the permanental-dominance conjecture. Linear Algebra Appl. 201 (1994), 211–229.

31. Soules, George W. Extending the Minc-Brègman upper bound for the permanent. Linear and Multilinear Algebra 47 (2000), no. 1, 77–91.

32. Soules, George W. New permanental upper bounds for nonnegative matrices. Linear Multilinear Algebra 51 (2003), no. 4, 319–337.

33. Vere-Jones, D. A generalization of permanents and determinants. Linear Algebra Appl. 111 (1988), 119–124.