On the adjacency matrix of a threshold graph

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Abstract

A threshold graph on \( n \) vertices is coded by a binary string of length \( n - 1 \). We obtain a formula for the inertia of (the adjacency matrix of) a threshold graph in terms of the code of the graph. It is shown that the number of negative eigenvalues of the adjacency matrix of a threshold graph is the number of ones in the code, whereas the nullity is given by the number of zeros in the code that are preceded by either a zero or a blank. A formula for the determinant of the adjacency matrix of a generalized threshold graph and the inverse, when it exists, of the adjacency matrix of a threshold graph are obtained. Results for antiregular graphs follow as special cases.

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1 Introduction

The graphs we consider are simple, that is, without loops or parallel edges. For basic terminology and definitions we refer to [1],[5].

Let \( G \) be a connected graph with vertex set \( V(G) = \{1, \ldots, n\} \) and edge set \( E(G) \). The adjacency matrix \( A(G) \), or simply \( A \), is the \( n \times n \) matrix with \((i, j)\)-element equal to 1 if vertices \( i \) and \( j \) are adjacent, and equal to 0 otherwise.

A threshold graph is a graph with no induced subgraph isomorphic to the path on 4 vertices, the cycle on 4 vertices, or to two disjoint copies of \( K_2 \), the complete graph on 2 vertices. Threshold graphs admit several equivalent definitions, in particular, a recursive definition based on a binary code will be relevant to this paper, and will be described later. We refer to the definitive [2] for further information concerning threshold graphs.

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An antiregular graph is a graph with at most two vertices of equal degree [3], [4]. These graphs enjoy several nice properties. There is a unique connected antiregular graph on \(n\) vertices, up to isomorphism. It can be shown that antiregular graphs are threshold graphs.

We introduce some notation. Let \(\alpha_1 \cdots \alpha_{n-1}\) be an \((n - 1)\)-tuple of real numbers. We define a generalized threshold graph on \(n\) vertices as follows. The graph is defined recursively. We start with a single vertex and label it as 1. We then add vertex 2 and make it adjacent to 1 by an edge of weight \(\alpha_1\), if \(\alpha_1\) is nonzero. If \(\alpha_1 = 0\), then 1 and 2 are not adjacent. We then add vertex 3 and make it adjacent to 1 and 2 by edges with weight \(\alpha_2\), if \(\alpha_2\) is nonzero. The process is continued. Having constructed the graph on vertices 1, \ldots, \(k\), we add vertex \(k + 1\) and make it adjacent to 1, \ldots, \(k\) by edges of weight \(\alpha_{k-1}\) if \(\alpha_{k-1} \neq 0, k = 2, 3, \ldots, n - 1\). We denote the resulting graph on \(n\) vertices by \(G[\alpha_1 \cdots \alpha_{n-1}]\). Note that if each \(\alpha_i\) is either 0 or 1, then the resulting graph is a threshold graph. Hence we refer to \(G[\alpha_1 \cdots \alpha_{n-1}]\) as a generalized threshold graph.

If \(\alpha_1 \cdots \alpha_{n-1}\) are alternately 0 and 1 (where \(\alpha_1\) is either 0 or 1) then the resulting graph is an antiregular graph. Furthermore, if \(\alpha_{n-1} = 1\) (respectively, 0) then the graph is the unique connected (respectively, disconnected) antiregular graph on \(n\) vertices. If \(\alpha_1 \cdots \alpha_{n-1}\) are alternately zero and nonzero (where \(\alpha_1\) is either zero or nonzero), then we refer to \(G[\alpha_1 \cdots \alpha_{n-1}]\) as a generalized antiregular graph.

We now describe the results of this paper. Recall that the inertia of the symmetric \(n \times n\) matrix \(A\) is the triple \((n_+ (A), n_0 (A), n_- (A)) = (n_+, n_0, n_-)\), where \(n_+, n_0\) and \(n_-\) are respectively the number of eigenvalues of \(A\) that are positive, zero and negative. By the inertia of a graph we mean the inertia of its adjacency matrix. It is well-known (see, for example, [3],[4]) that if \(G\) is an antiregular graph on \(n\) vertices, then the inertia of \(G\) is given by \((n_2 / 2, 0, n_2 / 2)\) if \(n\) is even, and by \((n_- - 1 / 2, 1, n_- - 1 / 2)\) if \(n\) is odd.

In Section 2 we obtain the inertia of a threshold graph. It is shown that if \(G\) is a connected threshold graph with the adjacency matrix \(A\), then \(n_- (A)\) is the number of ones in the code, whereas \(n_0(A)\), or the nullity of \(A\) is given by the number of zeros in the code that are preceded by either a zero or a blank. We remark that some partial results concerning \(n_- (A)\) and an equivalent formula for \(n_0 (A)\) are proved in [4]. Results for the inertia of an antiregular graph mentioned earlier follow as special cases from the results on threshold graphs.

In Section 3 we obtain a formula for the determinant and the inverse, when it exists, of the adjacency matrix of a threshold graph.

2 Inertia of a threshold graph

We begin by showing that the adjacency matrix of a generalized threshold graph may be reduced to a certain tridiagonal matrix by row and column operations.
Theorem 1 Let $A$ be the adjacency matrix of $G[α_1 \cdots α_{n-1}]$, where $α_1, \ldots, α_{n-1}$ are real numbers. Then there exists an $n \times n$ matrix $P$ with $\det P = 1$ such that

$$PAP' = \begin{pmatrix}
-2α_1 & α_1 & 0 & 0 & \cdots & 0 \\
α_1 & -2α_2 & α_2 & 0 & \cdots & 0 \\
0 & α_2 & -2α_3 & α_3 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & α_{n-2} & -2α_{n-1} & α_{n-1} \\
0 & \cdots & \cdots & 0 & α_{n-1} & 0
\end{pmatrix}.$$ \hspace{1cm} (1)

Proof: Note that

$$A = \begin{pmatrix}
0 & α_1 & α_2 & \cdots & \cdots & α_{n-1} \\
α_1 & 0 & α_2 & \cdots & \cdots & α_{n-1} \\
α_2 & α_2 & 0 & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
α_{n-1} & α_{n-1} & \cdots & \cdots & α_{n-1} & 0
\end{pmatrix}.$$

Replace the first row (column) of $A$ by the first row (column) minus the second row (column). The resulting matrix is

$$B = \begin{pmatrix}
-2α_1 & α_1 & 0 & \cdots & 0 \\
α_1 & 0 & A(1|1) \\
\vdots \\
0
\end{pmatrix},$$

where $A(1|1)$ is the submatrix of $A$ obtained by deleting the first row and column. Note that if $Q$ is the matrix obtained by replacing the first row of $I_n$, the identity matrix of order $n$, by the first row minus the second row, then $QAQ' = B$. Clearly, $\det Q = 1$. We may assume, as an induction assumption, that there exists an $n \times n$ matrix $R$ with determinant 1 such that

$$RA(1|1)R' = \begin{pmatrix}
-2α_2 & α_2 & 0 & 0 & \cdots & 0 \\
α_2 & -2α_3 & α_3 & 0 & \cdots & 0 \\
0 & α_3 & -2α_4 & α_3 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & α_{n-2} & -2α_{n-1} & α_{n-1} \\
0 & \cdots & \cdots & 0 & α_{n-1} & 0
\end{pmatrix}.$$

Let $S = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$. The result is proved by setting $P = S^{-1}Q$. \hfill \blacksquare
As consequences of Theorem 1, we obtain a formula for the inertia of a threshold graph and a generalized antiregular graph. We first prove a preliminary result.

**Lemma 2** Let \( n \geq 2 \) be a positive integer and let
\[
T_n = \begin{pmatrix}
-2 & 1 & 0 & \cdots & 0 \\
1 & -2 & 1 & \cdots & 0 \\
0 & 1 & -2 & \cdots & 0 \\
\vdots & & & & 1 \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}.
\]
Then \( \det T_n = (-1)^{n-1}(n-1) \). Furthermore, the inertia of \( T_n \) is \((1,0,n-1)\).

**Proof:** We prove the result by induction on \( n \), the cases \( n = 2,3 \) being easy. Assume the result to be true for \( T_k, 2 \leq k \leq n-1 \). A simple Laplace expansion shows that
\[
\det T_n = -2 \det T_{n-1} - \det T_{n-2} = (-2)(-1)^{n-2}(n-2) - (-1)^{n-3}(n-3) = (-1)^{n-1}(n-1).
\]
It follows by the Cauchy interlacing inequalities that the inertia of \( T_n \) is \((1,0,n-1)\). This completes the proof.

**Theorem 3** Let \( G \) be a connected threshold graph on \( n \) vertices with the code \( \alpha_1 \cdots \alpha_{n-1} \) where each \( \alpha_i \) is 0 or 1 and \( \alpha_{n-1} = 1 \). Let \( A \) be the adjacency matrix of \( G \). Then \( n_-(A) \) equals the number of ones in the code, while \( n_0(A) \) equals the number of zeros in the code that are preceded by a zero or a blank (a zero is preceded by a blank if it is the first element of the code).

**Proof:** Let the code \( \alpha_1 \cdots \alpha_{n-1} \) be given by
\[
\begin{array}{c}
0 \cdots 0 1 \\
t_1 \\
1 \cdots 1 \\
s_1 \\
1 \cdots 0 1 \\
t_2 \\
\vdots \\
s_2 \\
1 \cdots 0 \\
t_k \\
\vdots \\
s_k \end{array}
\]
where \( t_1 + \cdots + t_k + s_1 + \cdots + s_k = n-1 \). Since \( A \) and \( PAP' \) have the same inertia for a nonsingular \( P \), by Theorem 1, \( A \) has the same inertia as the matrix on the right side of (1). Let \( O_m \) be the \( m \times m \) null matrix and let \( T_n \) be the \( n \times n \) matrix defined as in Lemma 2. It can be seen that the matrix on the right side of (1) is the direct sum of \( O_{t_1}, T_{s_1+1}, O_{t_2-1}, T_{s_2+1}, \ldots, O_{t_k-1} \) and \( T_{s_k+1} \). By Lemma 2, \( T_{s_i+1} \) has \( s_i \) negative eigenvalues, \( i = 1, \ldots, k \), and therefore \( A \) has \( s_1 + \cdots + s_k \) negative eigenvalues. Note that \( s_1 + \cdots + s_k \) is the number of ones in the code. The zero eigenvalues of \( A \) come only from \( O_{t_1}, O_{t_2-1}, \ldots, O_{t_k-1} \) and their total number is \( t_1 + (t_2 - 1) + \cdots + (t_k - 1) \), which is precisely the number of zeros in the code that are preceded by a zero or a blank. This completes the proof.
Theorem 4 Let $G$ be a connected generalized antiregular graph on $n$ vertices with the code $\alpha_1 \cdots \alpha_{n-1}$. Let $A$ be the adjacency matrix of $G$. If $n$ is even, then $n_+(A) = n_-(A) = \frac{n}{2}$, and if $n$ is odd, then $n_+(A) = n_-(A) = \frac{n-1}{2}$.

Proof: First let $n = 2m$ be even. Then $\alpha_2 = \alpha_4 = \cdots = \alpha_{2m-2} = 0$, whereas the remaining $\alpha_i$’s are nonzero. The matrix on the right side of (1) is the direct sum of

$$
\left(\begin{array}{rr}
-2\alpha_1 & \alpha_1 \\
\alpha_1 & 0 \\
\end{array}\right),
\left(\begin{array}{rr}
-2\alpha_3 & \alpha_3 \\
\alpha_3 & 0 \\
\end{array}\right), \ldots, 
\left(\begin{array}{rr}
-2\alpha_{n-1} & \alpha_{n-1} \\
\alpha_{n-1} & 0 \\
\end{array}\right),
$$

Since $\left(\begin{array}{rr}
-2\alpha_i & \alpha_i \\
\alpha_i & 0 \\
\end{array}\right)$ has negative determinant, it has one positive and one negative eigenvalue, $i = 1, 3, \ldots, n-1$. Hence by Lemma 2, $A$ has $m$ positive and $m$ negative eigenvalues. The proof is similar when $n$ is odd.

As remarked earlier, Theorem 4 is well-known in the case of antiregular graphs, see [3],[4]. An equivalent description of the nullity of a threshold graph ($n_0(A)$ in the notation of Theorem 3) as well as some partial results concerning the inertia of a threshold graph are given in [4].

3 Determinant and inverse

Theorem 5 Let $G$ be a connected threshold graph on $n$ vertices with the code

$$
\begin{array}{cccccccc}
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\
t_1 & s_1 & t_2 & s_2 & t_3 & s_3 & \ldots & t_k & s_k \\
\end{array}
$$

where $t_1 + \cdots + t_k + s_1 + \cdots + s_k = n - 1$. Let $A$ be the adjacency matrix of $G$. Then det $A = 0$ if $t_1 > 0$ or if $t_i \geq 2$ for some $i \in \{2, \ldots, k\}$. If $t_1 = 0$ and $t_i = 1, i = 2, \ldots, k$, then det $A = (-1)^{s_1+\cdots+s_k} \prod_{i=1}^{k} s_i$.

Proof: If $t_1 > 0$ or if $t_i \geq 2$ for some $i \in \{2, \ldots, k\}$, then by Theorem 3, $A$ has a zero eigenvalue and det $A = 0$. So we assume that $t_1 = 0$ and $t_i = 1, i = 2, \ldots, k$. The result will be proved by induction on $n$. Let the code

$$
\begin{array}{cccccccc}
1 & \cdots & 1 & 0 & 1 & \cdots & 0 & 1 \\
s_1 & \ldots & s_2 & \ldots & s_k \\
\end{array}
$$

be denoted as $\alpha_1 \cdots \alpha_{n-1}$. By Theorem 1, det $A$ equals the determinant of the matrix on the right side of (1).

Let $G_1$ and $G_{12}$ denote the graphs obtained from $G$ by deleting vertex 1 and vertices 1, 2 respectively and let $A_1$ and $A_{12}$ be the corresponding adjacency matrices. A simple determinant expansion shows that

$$
\text{det } A = -2\alpha_1 \text{ det } A_1 - \alpha_1^2 \text{ det } A_{12}.
$$

(2)
We consider cases:

**Case (i):** \( \alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 1. \)

By the induction assumption and (2), \( \det A = -2(0) - (-1)^{s_1 + \cdots + s_k} \prod_{i=2}^k s_i. \) Since \( s_1 = 1, \)
\[ \det A = (-1)^{s_1 + \cdots + s_k} \prod_{i=2}^k s_i. \]

**Case (ii):** \( \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 0. \)

By the induction assumption and (2), \( \det A = -2(-1) \prod_{i=2}^k s_i - 0. \) Since \( s_1 = 2, \)
\[ \det A = (-1)^{s_1 + \cdots + s_k} \prod_{i=2}^k s_i. \]

**Case (iii):** \( \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1. \)

By the induction assumption and (2),
\[ \det A = -2(-1)^{s_1} + (-1)^{s_1} \prod_{i=1}^k s_i. \]

and the proof is complete.

The next result follows readily from Theorem 5.

**Corollary 6** Let \( G \) be the connected antiregular graph on \( n = 2m \) vertices, and let \( A \) be the adjacency matrix of \( G. \) Then \( \det A = (-1)^m. \)

We now turn to the inverse of the adjacency matrix of a threshold graph. Let \( s_1, \ldots, s_k \) be positive integers with \( s_1 + \cdots + s_k + k = n, \) and consider the threshold graph \( G \) on \( n \) vertices with the code
\[ \overbrace{1 \cdots 1}^{s_1} \overbrace{0 \cdots 0}^{s_2} \overbrace{1 \cdots 1}^{s_k}. \]

Let \( X_1 \) be the \((s_1 + 2) \times (s_1 + 2)\) matrix given by
\[ X_1 = \begin{pmatrix}
\frac{1}{s_1} & -1 & \frac{1}{s_1} & \cdots & \frac{1}{s_1} & -1 \\
\frac{1}{s_1} & \frac{1}{s_1} & -1 & \cdots & \frac{1}{s_1} & -1 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\frac{1}{s_1} & \cdots & \frac{1}{s_1} & -1 & -\frac{1}{s_1} \\
-\frac{1}{s_1} & \cdots & -\frac{1}{s_1} & -\frac{1}{s_1} & \frac{1}{s_1}
\end{pmatrix}. \]

For \( r = 2, \ldots, k - 1, \) define the \((s_r + 2) \times (s_r + 2)\) matrix
Finally, define the \((s_k + 1) \times (s_k + 1)\) matrix
\[
X_k = \begin{pmatrix}
\frac{1}{s_k} & \frac{1}{s_k} & \cdots & \frac{1}{s_k} & -\frac{1}{s_k} \\
\frac{1}{s_k} & \frac{1}{s_k} - 1 & \cdots & \frac{1}{s_k} & -\frac{1}{s_k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{s_k} & \cdots & \frac{1}{s_k} & -1 & -\frac{1}{s_k} \\
-\frac{1}{s_k} & \cdots & -\frac{1}{s_k} & \frac{1}{s_k} & \frac{1}{s_k}
\end{pmatrix}.
\]

For \(r = 0, 1, \ldots, k - 2\), let \(C_r\) be the \(n \times n\) matrix whose principal submatrix indexed by the rows and the columns \(s_1 + \cdots + s_r + r + 1, \ldots, s_1 + \cdots + s_{r+1} + r + 2\) equals \(X_{r+1}\) and with its remaining entries equal to zero. Let \(C_{k-1}\) be the \(n \times n\) matrix whose principal submatrix indexed by the rows and the columns \(s_1 + \cdots + s_{k-1} + k, \ldots, s_1 + \cdots + s_k + k\) equals \(X_k\) and with its remaining entries equal to zero. With this notation we have the following result.

**Theorem 7** Let \(s_1, \ldots, s_k\) be positive integers with \(s_1 + \cdots + s_k + k = n\), and let \(G\) be the threshold graph on \(n\) vertices with the code

\[
\begin{pmatrix}
1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\
\end{pmatrix}.
\]

If \(A\) is the adjacency matrix of \(G\), then \(A\) is nonsingular, and \(A^{-1} = C_0 + \cdots + C_{k-1}\).

**Proof:** By Theorem 3, \(A\) does not have an eigenvalue equal to zero and hence \(A\) is nonsingular. Let \(J_m\) denote the \(m \times m\) matrix of all ones, and let \(1\) be the column vector of all ones of appropriate order. We let \(J_{p \times q}\) denote the \(p \times q\) matrix of all ones. The boldface \(0\) will denote the matrix of all zeros, whose size will be clear from the context. We have

\[
X_1 = \begin{pmatrix}
\frac{1}{s_1} J_{s_1 + 1} - I_{s_1 + 1} & -\frac{1}{s_1} 1 \\
-\frac{1}{s_1} 1' & \frac{1}{s_1}
\end{pmatrix}.
\]

For \(r = 2, \ldots, k - 1\), we may write

\[
X_r = \frac{1}{s_r} \begin{pmatrix}
1 & \cdots & 1' \\
1 & \cdots & 1' \\
\vdots & \ddots & \vdots \\
-1 & \cdots & -1
\end{pmatrix}.
\]

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Finally,

\[ X_k = \frac{1}{s_k} \begin{pmatrix} 1 & 1' \\ 1 & J_{s_k} - s_k I_{s_k} \end{pmatrix}. \]

The result is proved by verifying that \( A(C_0 + C_1 + \cdots + C_{k-1}) = I_n \). For clarity, we illustrate the argument for \( k = 3 \). The general case is similar. If \( k = 3 \), then we have

\[
A = \begin{pmatrix}
J_{s_1+1} - I_{s_1+1} & 0 & J_{(s_1+1)\times s_2} & 0 & J_{(s_1+1)\times s_3} \\
0 & 0 & 1' & 0 & 1' \\
J_{s_2\times(s_1+1)} & 1 & J_{s_2} - I_{s_2} & 0 & J_{s_2\times s_3} \\
0 & 0 & 0 & 0 & 1' \\
J_{s_3\times(s_1+1)} & 1 & J_{s_3\times s_2} & 1 & J_{s_3} - I_{s_3}
\end{pmatrix},
\]

\[
C_0 = \frac{1}{s_1} \begin{pmatrix}
J_{s_1+1} - s_1 I_{s_1+1} & -1 & 0 & 0 & 0 \\
-1' & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
C_1 = \frac{1}{s_2} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1' & -1 & 0 \\
0 & 1 & J_{s_2} - s_2 I_{s_2} & -1 & 0 \\
0 & -1 & -1' & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
C_2 = \frac{1}{s_3} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1' \\
0 & 0 & 0 & 1 & J_{s_3} - s_3 I_{s_3}
\end{pmatrix}.
\]

A routine calculation shows that

\[
AC_0 = \begin{pmatrix}
I_{s_1+1} & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0
\end{pmatrix},
\]

\[
AC_1 = \begin{pmatrix}
0 & 1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 1 & I_{s_2} & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0
\end{pmatrix}.
\]
\[
AC_2 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & I_3
\end{pmatrix}.
\]

It follows that \(AC_0 + AC_1 + AC_2 = I_n\) and hence \(A^{-1} = C_0 + C_1 + C_2\). In the general case we can similarly conclude that \(A^{-1} = C_0 + C_1 + \cdots + C_{k-1}\) and the proof is complete. \(\blacksquare\)

**Inverse of the adjacency matrix of an antiregular graph**

Define the matrices

\[
U = \begin{pmatrix}
0 & 1 & -1 \\
1 & 0 & -1 \\
-1 & -1 & 1
\end{pmatrix},
V = \begin{pmatrix}
1 & 1 & -1 \\
1 & 0 & -1 \\
-1 & -1 & 1
\end{pmatrix}
\text{ and } W = \begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}.
\]

Let \(G\) be the connected antiregular graph on \(n = 2m\) vertices. Let \(H_0\) be the \(n \times n\) matrix whose principal submatrix indexed by the rows and the columns 1, 2, 3 equals \(U\) and with its remaining entries equal to zero. For \(r = 1, \ldots, m-2\), let \(H_r\) be the \(n \times n\) matrix whose principal submatrix indexed by the rows and the columns \(2r+1, 2r+2, 2r+3\) equals \(V\) and with its remaining entries equal to zero. Let \(H_{m-1}\) be the \(n \times n\) matrix whose principal submatrix indexed by the rows and the columns \(2m-1, 2m\) equals \(V\) and with its remaining entries equal to zero. With this notation we have the following result, which follows from Theorem 7.

**Theorem 8** Let \(G\) be the connected, antiregular graph on \(n = 2m\) vertices, and let \(A\) be the adjacency matrix of \(G\). Then \(A^{-1} = \sum_{r=0}^{m-1} H_r\).

We conclude with an example. The adjacency matrix of the connected antiregular graph on 8 vertices is given by

\[
A = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]
Then

\[
A^{-1} = \begin{pmatrix}
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 2 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 2 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

thereby verifying the formula given in Theorem 8.

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