On the adjacency matrix of a threshold graph

R.B. Bapat*

Indian Statistical Institute,

Delhi Centre, 7 S.J.S.S. Marg,

New Delhi 110 016, India

rbb@isid.ac.in

Abstract

A threshold graph on n vertices is coded by a binary string of length $n-1$. We obtain a formula for the inertia of (the adjacency matrix of) a threshold graph in terms of the code of the graph. It is shown that the number of negative eigenvalues of the adjacency matrix of a threshold graph is the number of ones in the code, whereas the nullity is given by the number of zeros in the code that are preceded by either a zero or a blank. A formula for the determinant of the adjacency matrix of a generalized threshold graph and the inverse, when it exists, of the adjacency matrix of a threshold graph are obtained. Results for antiregular graphs follow as special cases.

AMS Classification: 05C50

Keywords: threshold graph, antiregular graph, adjacency matrix, inertia

1 Introduction

The graphs we consider are simple, that is, without loops or parallel edges. For basic terminology and definitions we refer to [1],[5].

Let G be a connected graph with vertex set $V(G) = \{1, \dots, n\}$ and edge set $E(G)$. The adjacency matrix $A(G)$, or simply A, is the $n \times n$ matrix with (i, j)-element equal to 1 if vertices i and j are adjacent, and equal to 0 otherwise.

A threshold graph is a graph with no induced subgraph isomorphic to the path on 4 vertices, the cycle on 4 vertices, or to two disjoint copies of K_2, the complete graph on 2 vertices. Threshold graphs admit several equivalent definitions, in particular, a recursive definition based on a binary code will be relevant to this paper, and will be described later. We refer to the definitive [2] for further information concerning threshold graphs.

*The author acknowledges support from the JC Bose Fellowship, Department of Science and Technology, Government of India.

An antiregular graph is a graph with at most two vertices of equal degree [3], [4]. These graphs enjoy several nice properties. There is a unique connected antiregular graph on n vertices, up to isomorphism. It can be shown that antiregular graphs are threshold graphs.

We introduce some notation. Let $\alpha_1 \cdots \alpha_{n-1}$ be an $(n-1)$-tuple of real numbers. We define a generalized threshold graph on n vertices as follows. The graph is defined recursively. We start with a single vertex and label it as 1. We then add vertex 2 and make it adjacent to 1 by an edge of weight α_1, if α_1 is nonzero. If $\alpha_1 = 0$, then 1 and 2 are not adjacent. We then add vertex 3 and make it adjacent to 1 and 2 by edges with weight α_2, if α_2 is nonzero. The process is continued. Having constructed the graph on vertices $1, \ldots, k$, we add vertex $k+1$ and make it adjacent to $1, \ldots, k$ by edges of weight α_{k-1} if $\alpha_{k-1} \neq 0, k = 2, 3, \ldots, n-1$. We denote the resulting graph on n vertices by $G[\alpha_1 \cdots \alpha_{n-1}]$. Note that if each α_i is either 0 or 1, then the resulting graph is a threshold graph. Hence we refer to $G[\alpha_1 \cdots \alpha_{n-1}]$ as a generalized threshold graph.

If $\alpha_1 \cdots \alpha_{n-1}$ are alternately 0 and 1 (where α_1 is either 0 or 1) then the resulting graph is an antiregular graph. Furthermore, if $\alpha_{n-1} = 1$ (respectively, 0,) then the graph is the unique connected (respectively, disconnected) antiregular graph on n vertices. If $\alpha_1 \cdots \alpha_{n-1}$ are alternately zero and nonzero (where α_1 is either zero or nonzero), then we refer to $G[\alpha_1 \cdots \alpha_{n-1}]$ as a generalized antiregular graph.

We now describe the results of this paper. Recall that the inertia of the symmetric $n \times n$ matrix A is the triple $(n_+(A), n_0(A), n_-(A)) = (n_+, n_0, n_-)$, where n_+, n_0 and n_- are respectively the number of eigenvalues of A that are positive, zero and negative. By the inertia of a graph we mean the inertia of its adjacency matrix. It is well-known (see, for example, [3],[4]) that if G is an antiregular graph on n vertices, then the inertia of G is given by $(\frac{n}{2}, 0, \frac{n}{2})$ if n is even, and by $(\frac{n-1}{2}, 1, \frac{n-1}{2})$ if n is odd.

In Section 2 we obtain the inertia of a threshold graph. It is shown that if G is a connected threshold graph with the adjacency matrix A, then $n_-(A)$ is the number of ones in the code, whereas $n_0(A)$, or the nullity of A is given by the number of zeros in the code that are preceded by either a zero or a blank. We remark that some partial results concerning $n_-(A)$ and an equivalent formula for $n_0(A)$ are proved in [4]. Results for the inertia of an antiregular graph mentioned earlier follow as special cases from the results on threshold graphs.

In Section 3 we obtain a formula for the determinant and the inverse, when it exists, of the adjacency matrix of a threshold graph.

2 Inertia of a threshold graph

We begin by showing that the adjacency matrix of a generalized threshold graph may be reduced to a certain tridiagonal matrix by row and column operations.

Theorem 1 *Let A be the adjacency matrix of $G[\alpha_1 \cdots \alpha_{n-1}]$, where $\alpha_1, \ldots, \alpha_{n-1}$ are real numbers. Then there exists an $n \times n$ matrix P with $\det P = 1$ such that*

$$PAP' = \begin{pmatrix} -2\alpha_1 & \alpha_1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & -2\alpha_2 & \alpha_2 & 0 & \cdots & 0 \\ 0 & \alpha_2 & -2\alpha_3 & \alpha_3 & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_{n-2} & -2\alpha_{n-1} & \alpha_{n-1} \\ 0 & \cdots & \cdots & 0 & \alpha_{n-1} & 0 \end{pmatrix}. \tag{1}$$

Proof: Note that

$$A = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \cdots & \cdots & \alpha_{n-1} \\ \alpha_1 & 0 & \alpha_2 & \cdots & \cdots & \alpha_{n-1} \\ \alpha_2 & \alpha_2 & 0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \alpha_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & \alpha_{n-1} & \cdots & \cdots & \alpha_{n-1} & 0 \end{pmatrix}.$$

Replace the first row (column) of A by the first row (column) minus the second row (column). The resulting matrix is

$$B = \begin{pmatrix} -2\alpha_1 & \alpha_1 & 0 & \cdots & 0 \\ \alpha_1 & & & & \\ 0 & & A(1|1) & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix},$$

where $A(1|1)$ is the submatrix of A obtained by deleting the first row and column. Note that if Q is the matrix obtained by replacing the first row of I_n, the identity matrix of order n, by the first row minus the second row, then $QAQ' = B$. Clearly, $\det Q = 1$. We may assume, as an induction assumption, that there exists an $n \times n$ matrix R with determinant 1 such that

$$RA(1|1)R' = \begin{pmatrix} -2\alpha_2 & \alpha_2 & 0 & 0 & \cdots & 0 \\ \alpha_2 & -2\alpha_3 & \alpha_3 & 0 & \cdots & 0 \\ 0 & \alpha_3 & -2\alpha_4 & \alpha_3 & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_{n-2} & -2\alpha_{n-1} & \alpha_{n-1} \\ 0 & \cdots & \cdots & 0 & \alpha_{n-1} & 0 \end{pmatrix}.$$

Let $S = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$. The result is proved by setting $P = S^{-1}Q$. ■

As consequences of Theorem 1, we obtain a formula for the inertia of a threshold graph and a generalized antiregular graph. We first prove a preliminary result.

Lemma 2 *Let $n \geq 2$ be a positive integer and let*

$$T_n = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & & & & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then $\det T_n = (-1)^{n-1}(n-1)$. *Furthermore, the inertia of T_n is* $(1, 0, n-1)$.

Proof: We prove the result by induction on n, the cases $n = 2, 3$ being easy. Assume the result to be true for $T_k, 2 \leq k \leq n-1$. A simple Laplace expansion shows that

$$\begin{aligned} \det T_n &= -2 \det T_{n-1} - \det T_{n-2} \\ &= (-2)(-1)^{n-2}(n-2) - (-1)^{n-3}(n-3) \\ &= (-1)^{n-1}(n-1). \end{aligned}$$

It follows by the Cauchy interlacing inequalities that the inertia of T_n is $(1, 0, n-1)$. This completes the proof. ■

Theorem 3 *Let G be a connected threshold graph on n vertices with the code $\alpha_1 \cdots \alpha_{n-1}$ where each α_i is 0 or 1 and $\alpha_{n-1} = 1$. Let A be the adjacency matrix of G. Then $n_-(A)$ equals the number of ones in the code, while $n_0(A)$ equals the number of zeros in the code that are preceded by a zero or a blank (a zero is preceded by a blank if it is the first element of the code).*

Proof: Let the code $\alpha_1 \cdots \alpha_{n-1}$ be given by

$$\underbrace{0 \cdots 0}_{t_1}\underbrace{1 \cdots 1}_{s_1}\underbrace{0 \cdots 0}_{t_2}\underbrace{1 \cdots 1}_{s_2} \cdots \underbrace{0 \cdots 0}_{t_k}\underbrace{1 \cdots 1}_{s_k},$$

where $t_1 + \cdots + t_k + s_1 + \cdots + s_k = n-1$. Since A and PAP' have the same inertia for a nonsingular P, by Theorem 1, A has the same inertia as the matrix on the right side of (1). Let $\mathcal{O}_m$ be the $m \times m$ null matrix and let T_n be the $n \times n$ matrix defined as in Lemma 2. It can be seen that the matrix on the right side of (1) is the direct sum of $\mathcal{O}_{t_1}, T_{s_1+1}, \mathcal{O}_{t_2-1}, T_{s_2+1}, \cdots, \mathcal{O}_{t_k-1}$ and T_{s_k+1}. By Lemma 2, T_{s_i+1} has s_i negative eigenvalues, $i = 1, \ldots, k$, and therefore A has $s_1 + \cdots + s_k$ negative eigenvalues. Note that $s_1 + \cdots + s_k$ is the number of ones in the code. The zero eigenvalues of A come only from $\mathcal{O}_{t_1}, \mathcal{O}_{t_2-1}, \cdots, \mathcal{O}_{t_k-1}$ and their total number is $t_1 + (t_2 - 1) + \cdots + (t_k - 1)$, which is precisely the number of zeros in the code that are preceded by a zero or a blank. This completes the proof. ■

Theorem 4 *Let G be a connected generalized antiregular graph on n vertices with the code $\alpha_1 \cdots \alpha_{n-1}$. Let A be the adjacency matrix of G. If n is even, then $n_+(A) = n_-(A) = \frac{n}{2}$, and if n is odd, then $n_+(A) = n_-(A) = \frac{n-1}{2}$.*

Proof: First let $n = 2m$ be even. Then $\alpha_2 = \alpha_4 = \cdots = \alpha_{2m-2} = 0$, whereas the remaining α_i's are nonzero. The matrix on the right side of (1) is the direct sum of

$$\begin{pmatrix} -2\alpha_1 & \alpha_1 \\ \alpha_1 & 0 \end{pmatrix}, \begin{pmatrix} -2\alpha_3 & \alpha_3 \\ \alpha_3 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} -2\alpha_{n-1} & \alpha_{n-1} \\ \alpha_{n-1} & 0 \end{pmatrix}.$$

Since $\begin{pmatrix} -2\alpha_i & \alpha_i \\ \alpha_i & 0 \end{pmatrix}$ has negative determinant, it has one positive and one negative eigenvalue, $i = 1, 3, \ldots, n-1$. Hence by Lemma 2, A has m positive and m negative eigenvalues. The proof is similar when n is odd. ■

As remarked earlier, Theorem 4 is well-known in the case of antiregular graphs, see [3],[4]. An equivalent description of the nullity of a threshold graph ($n_0(A)$ in the notation of Theorem 3) as well as some partial results concerning the inertia of a threshold graph are given in [4].

3 Determinant and inverse

Theorem 5 *Let G be a connected threshold graph on n vertices with the code*

$$\underbrace{0\cdots0}_{t_1}\underbrace{1\cdots1}_{s_1}\underbrace{0\cdots0}_{t_2}\underbrace{1\cdots1}_{s_2}\cdots\underbrace{0\cdots0}_{t_k}\underbrace{1\cdots1}_{s_k},$$

where $t_1 + \cdots + t_k + s_1 + \cdots + s_k = n-1$. Let A be the adjacency matrix of G. Then $\det A = 0$ if $t_1 > 0$ or if $t_i \geq 2$ for some $i \in \{2, \ldots, k\}$. If $t_1 = 0$ and $t_i = 1, i = 2, \ldots, k$, then $\det A = (-1)^{s_1 + \cdots + s_k} \prod_{i=1}^{k} s_i$.

Proof: If $t_1 > 0$ or if $t_i \geq 2$ for some $i \in \{2, \ldots, k\}$, then by Theorem 3, A has a zero eigenvalue and $\det A = 0$. So we assume that $t_1 = 0$ and $t_i = 1, i = 2, \ldots, k$. The result will be proved by induction on n. Let the code

$$\underbrace{1\cdots1}_{s_1}0\underbrace{1\cdots1}_{s_2}0\cdots0\underbrace{1\cdots1}_{s_k}$$

be denoted as $\alpha_1 \cdots \alpha_{n-1}$. By Theorem 1, $\det A$ equals the determinant of the matrix on the right side of (1).

Let G_1 and G_{12} denote the graphs obtained from G by deleting vertex 1 and vertices $1, 2$ respectively and let A_1 and A_{12} be the corresponding adjacency matrices. A simple determinant expansion shows that

$$\det A = -2\alpha_1 \det A_1 - \alpha_1^2 \det A_{12}. \tag{2}$$

We consider cases:

Case (i): $\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 1$.

By the induction assumption and (2), $\det A = -2(0) - (-1)^{s_2+\cdots+s_k} \prod_{i=2}^k s_i$. Since $s_1 = 1$, $\det A = (-1)^{s_1+\cdots+s_k} \prod_{i=1}^k s_i$.

Case (ii): $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 0$.

By the induction assumption and (2), $\det A = -2(-1)^{1+s_2+\cdots+s_k} \prod_{i=2}^k s_i - 0$. Since $s_1 = 2$, $\det A = (-1)^{s_1+\cdots+s_k} \prod_{i=1}^k s_i$.

Case (iii): $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1$.

By the induction assumption and (2),

$$\begin{aligned}
\det A &= -2(-1)^{(s_1-1)+s_2+\cdots+s_k}(s_1-1)s_2\cdots s_k \\
&- (-1)^{(s_1-2)+s_2+\cdots+s_k}(s_1-2)s_2\cdots s_k \\
&= (-1)^{s_1+\cdots+s_k} s_2\cdots s_k(2s_1 - 2 - s_1 + 2) \\
&= (-1)^{s_1+\cdots+s_k} \prod_{i=1}^k s_i
\end{aligned}$$

and the proof is complete. ■

The next result follows readily from Theorem 5.

Corollary 6 *Let G be the connected antiregular graph on $n = 2m$ vertices, and let A be the adjacency matrix of G. Then* $\det A = (-1)^m$.

We now turn to the inverse of the adjacency matrix of a threshold graph. Let $s_1, \ldots, s_k$ be positive integers with $s_1 + \cdots + s_k + k = n$, and consider the threshold graph G on n vertices with the code

$$\underbrace{1\cdots 1}_{s_1}0\underbrace{1\cdots 1}_{s_2}0\cdots 0\underbrace{1\cdots 1}_{s_k}.$$

Let X_1 be the $(s_1+2)\times(s_1+2)$ matrix given by

$$X_1 = \begin{pmatrix}
\frac{1}{s_1}-1 & \frac{1}{s_1} & \cdots & \frac{1}{s_1} & -\frac{1}{s_1} \\
\frac{1}{s_1} & \frac{1}{s_1}-1 & \cdots & \frac{1}{s_1} & -\frac{1}{s_1} \\
\vdots & & \ddots & & \vdots \\
\frac{1}{s_1} & \cdots & & \frac{1}{s_1}-1 & -\frac{1}{s_1} \\
-\frac{1}{s_1} & \cdots & & -\frac{1}{s_1} & \frac{1}{s_1}
\end{pmatrix}.$$

For $r = 2, \ldots, k-1$, define the $(s_r+2)\times(s_r+2)$ matrix

$$X_r = \begin{pmatrix} \frac{1}{s_r} & \frac{1}{s_r} & \cdots & \frac{1}{s_r} & -\frac{1}{s_r} \\ \frac{1}{s_r} & \frac{1}{s_r}-1 & \cdots & \frac{1}{s_r} & -\frac{1}{s_r} \\ \vdots & & \ddots & & \vdots \\ \frac{1}{s_r} & \cdots & & \frac{1}{s_r}-1 & -\frac{1}{s_r} \\ -\frac{1}{s_r} & \cdots & & -\frac{1}{s_r} & \frac{1}{s_r} \end{pmatrix}.$$

Finally, define the $(s_k+1)\times(s_k+1)$ matrix

$$X_k = \begin{pmatrix} \frac{1}{s_k} & \frac{1}{s_k} & \cdots & \frac{1}{s_k} & \frac{1}{s_k} \\ \frac{1}{s_k} & \frac{1}{s_k}-1 & \cdots & \frac{1}{s_k} & \frac{1}{s_k} \\ \vdots & & \ddots & & \vdots \\ \frac{1}{s_k} & \cdots & & \frac{1}{s_k}-1 & \frac{1}{s_k} \\ \frac{1}{s_k} & \cdots & & \frac{1}{s_k} & \frac{1}{s_k}-1 \end{pmatrix}.$$

For $r = 0, 1, \dots, k-2$, let C_r be the $n \times n$ matrix whose principal submatrix indexed by the rows and the columns $s_1+\cdots+s_r+r+1, \dots, s_1+\cdots+s_{r+1}+r+2$ equals X_{r+1} and with its remaining entries equal to zero. Let C_{k-1} be the $n\times n$ matrix whose principal submatrix indexed by the rows and the columns $s_1+\cdots+s_{k-1}+k, \dots, s_1+\cdots+s_k+k$ equals X_k and with its remaining entries equal to zero. With this notation we have the following result.

Theorem 7 *Let $s_1, \dots, s_k$ be positive integers with $s_1+\cdots+s_k+k=n$, and let G be the threshold graph on n vertices with the code*

$$\underbrace{1\cdots1}_{s_1}0\underbrace{1\cdots1}_{s_2}0\cdots0\underbrace{1\cdots1}_{s_k}.$$

If A is the adjacency matrix of G, then A is nonsingular, and $A^{-1} = C_0+\cdots+C_{k-1}$.

Proof: By Theorem 3, A does not have an eigenvalue equal to zero and hence A is nonsingular. Let J_m denote the $m\times m$ matrix of all ones, and let $\mathbf{1}$ be the column vector of all ones of appropriate order. We let $J_{p\times q}$ denote the $p\times q$ matrix of all ones. The boldface $\mathbf{0}$ will denote the matrix of all zeros, whose size will be clear from the context. We have

$$X_1 = \begin{pmatrix} \frac{1}{s_1}J_{s_1+1} - I_{s_1+1} & -\frac{1}{s_1}\mathbf{1} \\ -\frac{1}{s_1}\mathbf{1}' & \frac{1}{s_1} \end{pmatrix}.$$

For $r = 2, \dots, k-1$, we may write

$$X_r = \frac{1}{s_r}\left(\begin{array}{c|c|c} 1 & \mathbf{1}' & 1 \\ \hline \mathbf{1} & J_{s_r} - s_r I_{s_r} & -\mathbf{1} \\ \hline -1 & -\mathbf{1}' & 1 \end{array}\right).$$

Finally,

$$X_k = \frac{1}{s_k}\begin{pmatrix} 1 & \mathbf{1}' \\ \mathbf{1} & J_{s_k} - s_k I_{s_k} \end{pmatrix}.$$

The result is proved by verifying that $A(C_0 + C_1 + \cdots + C_{k-1}) = I_n$. For clarity, we illustrate the argument for $k = 3$. The general case is similar. If $k = 3$, then we have

$$A = \left(\begin{array}{c|c|c|c|c} J_{s_1+1} - I_{s_1+1} & \mathbf{0} & J_{(s_1+1)\times s_2} & \mathbf{0} & J_{(s_1+1)\times s_3} \\ \hline \mathbf{0} & 0 & \mathbf{1}' & 0 & \mathbf{1}' \\ \hline J_{s_2\times(s_1+1)} & \mathbf{1} & J_{s_2} - I_{s_2} & \mathbf{0} & J_{s_2\times s_3} \\ \hline \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{1}' \\ \hline J_{s_3\times(s_1+1)} & \mathbf{1} & J_{s_3\times s_2} & \mathbf{1} & J_{s_3} - I_{s_3} \end{array}\right),$$

$$C_0 = \frac{1}{s_1}\left(\begin{array}{c|c|c|c|c} J_{s_1+1} - s_1 I_{s_1+1} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline -\mathbf{1}' & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right),$$

$$C_1 = \frac{1}{s_2}\left(\begin{array}{c|c|c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 1 & \mathbf{1}' & -1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & J_{s_2} - s_2 I_{s_2} & -\mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & -1 & -\mathbf{1}' & 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right),$$

$$C_2 = \frac{1}{s_3}\left(\begin{array}{c|c|c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 0 & \mathbf{0} & 1 & \mathbf{1}' \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & J_{s_3} - s_3 I_{s_3} \end{array}\right).$$

A routine calculation shows that

$$AC_0 = \left(\begin{array}{c|c|c|c|c} I_{s_1+1} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right),$$

$$AC_1 = \left(\begin{array}{c|c|c|c|c} \mathbf{0} & \mathbf{1} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & 1 & \mathbf{0} & -1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & I_{s_2} & -\mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \end{array}\right),$$

$$AC_2 = \left(\begin{array}{c|c|c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & 0 & \mathbf{0} & 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & 0 & \mathbf{0} & 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & I_{s_3} \end{array}\right).$$

It follows that $AC_0 + AC_1 + AC_2 = I_n$ and hence $A^{-1} = C_0 + C_1 + C_2$. In the general case we can similarly conclude that $A^{-1} = C_0 + C_1 + \cdots + C_{k-1}$ and the proof is complete. ■

Inverse of the adjacency matrix of an antiregular graph

Define the matrices

$$U = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix} \text{ and } W = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let G be the connected antiregular graph on $n = 2m$ vertices. Let H_0 be the $n \times n$ matrix whose principal submatrix indexed by the rows and the columns $1, 2, 3$ equals U and with its remaining entries equal to zero. For $r = 1, \ldots, m-2$, let H_r be the $n \times n$ matrix whose principal submatrix indexed by the rows and the columns $2r+1, 2r+2, 2r+3$ equals V and with its remaining entries equal to zero. Let H_{m-1} be the $n \times n$ matrix whose principal submatrix indexed by the rows and the columns $2m-1, 2m$ equals V and with its remaining entries equal to zero. With this notation we have the following result, which follows from Theorem 7.

Theorem 8 *Let G be the connected, antiregular graph on $n = 2m$ vertices, and let A be the adjacency matrix of G. Then $A^{-1} = H_0 + \cdots + H_{m-1}$.*

We conclude with an example. The adjacency matrix of the connected antiregular graph on 8 vertices is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Then

$$A^{-1} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

thereby verifying the formula given in Theorem 8.

Acknowledgment I sincerely thank Arbind K. Lal and S. Sivasubramanian for a careful reading of the manuscript and for making several helpful suggestions.

References

[1] R.B. Bapat, *Graphs and Matrices*, Springer, London; Hindustan Book Agency, New Delhi, 2010.

[2] N.V.R. Mahadev and U.N. Peled, *Threshold Graphs and Related Topics*, Annals of Discrete Math., **58**, Elsevier, Amsterdam, 1995.

[3] Russell Merris, Antiregular graphs are universal for trees, *Publ. Elektrotehn. Fak. Univ. Beograd.* Ser. Mat. **14** (2003), 1–3.

[4] Irene Sciriha and Stephaie Farrugia, On the spectrum of threshold graphs, *ISRN Discrete Mathematics* (2011) doi:10.5402/2011/108509

[5] D. B. West, *Introduction to Graph Theory.* Prentice Hall, Inc., Upper Saddle River, NJ, 1996.