Domains of Attraction of Invariant Distributions of the Infinite Atlas Model

Amarjit Budhiraja
University of North Carolina at Chapel Hill
Joint work with Sayan Banerjee

A Conference on Probability & Stochastic Processes
In honor of Prof. Rajeeva Karandikar.

Atlas model with $d+1$ Brownian particles.

- Consider $d+1$ Brownian particles in $\mathbb{R}$ where the lowest particle gets a constant drift of $+1$.
- Describes a particle system with *topological interactions* [Carinci, A. De Masi, C. Giardina, and E. Presutti (2016)] referred to as the Atlas model.
- Arise in stochastic portfolio theory. [Fernholz(2002), Fernholz and Karatzas(2009)].
- Connections with interacting particle systems (e.g. simple exclusion process) [Karatzas, Pal and Shklonikov (2016)]. ..
- ...also with nonlinear diffusions and McKean-Vlasov equations [Dembo, Shklonikov, Varadhan, Zeitouni (2016)]...
- .. and Aldous' Up the River stochastic control problem [Aldous 2002].

Finite Dimensional Atlas Model: Some Known Results

- Strong existence and pathwise uniqueness [Ichiba, Karatzas, Shkolnikov (2013)].

- Denote the ranked particle system as

$$Y_0(t) \le Y_1(t) \le \cdots Y_d(t)$$

  and the gap process

$$Z_i(t) \doteq Y_i(t) - Y_{i-1}(t),\ 1 \le i \le d.$$

  Then $Z = (Z_i)_{1 \le i \le m}$ describes a reflected Brownian motion in $\mathbb{R}_+^d$ with oblique reflections at the boundary.

Finite Dimensional Atlas Model: Some Known Results

- The process Z is ergodic. The unique invariant probability measure is given as
$$\pi^{(d)} := \bigotimes_{i=1}^{d} \mathrm{Exp}(2(1 - i/(d+1))).$$
[Harrison and Williams (1987a, 1987b), Pal and Pitman(2008).]

- Geometric ergodicity [B. and Lee (2007)], explicit dimension dependent rates [Banerjee and B. (2020), Banerjee and Brown (2020)] .

Infinite Atlas Model.

We now consider an infinite system of particles, i.e.

$$d = \infty.$$

Infinite Atlas Model: $d = \infty$.

- Denoting the un-ranked particle state processes as $U_i(t)$ and the ranked as $Y_i(t)$, system SDE:

$$dU_i(s) = \mathbf{1}(U_i(s) = Y_0(s))ds + dW_i(s), \; s \geq 0, i \in \mathbb{N}_0$$

  where $\{W_i\}_{i \in \mathbb{N}_0}$ are mutually independent Brownian motions

- Wellposedness. Suppose $\sum_{i=0}^{\infty} e^{-\alpha U_i(0)^2} < \infty$, for all $\alpha > 0$. Then weak existence and uniqueness holds and the ranked process is well defined[Sarantsev(2017)].

- SDE for ranked system:

$$dY_i(t) = \mathbf{1}(i = 0)dt + dB_i(t) - \frac{1}{2}dL_{i+1}(t) + \frac{1}{2}dL_i(t), \; t \geq 0, i \in \mathbb{N}_0.$$

- $L_0(\cdot) \equiv 0$. For $i \in \mathbb{N}$, $L_i(\cdot)$ is the collision local time of $(i-1)$-th and i-th particles. B_i are independent BM.

Infinite Atlas Model: The Gap process

- Consider the $\mathbb{R}_+^\infty$-valued process $\mathbf{Z}(\cdot) = (Z_1(\cdot), Z_2(\cdot), \dots)$ defined by

$$Z_i(\cdot) := Y_i(\cdot) - Y_{i-1}(\cdot), \; i \in \mathbb{N}.$$

- One invariant distribution: $\pi_0 \doteq \otimes_{i=1}^\infty \text{Exp}(2)$. [Pal and Pitman (2008). [Recall for d-dim. Atlas, unique inv. dist. is $\pi^{(d)} := \bigotimes_{i=1}^d \text{Exp}(2(1 - i/(d+1)))$].

- PP conjectured this is the unique invariant probability measure.

- This conjecture was shown to be false in [Sarantsev and Tsai (2017)]. Showed that these are all stationary distributions:

$$\pi_a := \bigotimes_{i=1}^\infty \text{Exp}(2 + ia), \; a \geq 0.$$

- Question of interest: Local stability structure – Domains of attraction.

Domains of Attraction (DoA): Recent Results

- For a suitable $\nu \in \mathcal{P}(\mathbb{R}_+^\infty)$, denote by $\hat{\nu}_t$ the probability law of $\mathbf{Z}(t)$, when $P \circ \mathbf{Z}(0)^{-1} = \nu$.
- Let $\nu_t = \frac{1}{t} \int_0^t \hat{\nu}_s ds$.
- Definition.
  - We say ν is in the strong DoA of π_a if $\hat{\nu}_t \to \pi_a$ as $t \to \infty$.
  - We say ν is in the weak DoA of π_a if $\nu_t \to \pi_a$ as $t \to \infty$.
- Sarantsev (2017): If $\nu \overset{d}{\geq} \pi_0$ then $\nu \in \text{sDoA}(\pi_0)$.

Domains of Attraction (DoA): Recent Results

- Dembo, Jara, Olla (2019): Let $\mathbf{Z}(0) = (Z_j(0))_{j\in\mathbb{N}}$ distributed as ν a.s. satisfy, for some $\beta \in [1,2)$, and eventually non-decreasing sequence $\{\theta(m) : m \geq 1\}$ with $\inf_m\{\theta(m-1)/\theta(m)\} > 0$: A.s.

$$\limsup_{m\to\infty} \frac{1}{m^\beta \theta(m)} \sum_{j=1}^{m} (\log Z_j(0))_- < \infty, \tag{1}$$

$$\liminf_{m\to\infty} \frac{1}{m^{\beta^2/(1+\beta)}\theta(m)} \sum_{j=1}^{m} Z_j(0) = \infty, \tag{2}$$

$$\limsup_{m\to\infty} \frac{1}{m^\beta \theta(m)} \sum_{j=1}^{m} Z_j(0) < \infty \tag{3}$$

with the additional requirement that $\theta(m) \geq \log m, m \geq 1$, if $\beta = 1$. Then $\nu \in \mathrm{sDoA}(\pi_0)$.

Some Remarks

- The condition $\nu \overset{d}{\geq} \pi_0$ of Sarantsev says the particles are not 'too closely packed'.
- DJO's third condition says that particles are not too spread out. E.g. $Z_j(0) \sim e^{j^2}$ does not satisfy DJO condition but satisfies S's condition.
- Condition (1) of DJO requires all gaps to be strictly positive.
- Proofs exploit reversibility and Dirichlet forms and relative entropy estimates.
- No results available for $a \neq 0$.

Some Challenges for $a \neq 0$.

- DJO approximate the infinite Atlas model by the finite Atlas model with $d+1$ particles. Main steps are:
  - Using coupling argue that, on any $[0, T]$, there is a $d_T \in \mathbb{N}$ s.t. the d_T-dimensional Atlas model stays uniformly close to the lowest $d_T + 1$ particles in the infinite Atlas model.
  - Argue that the law of the first k gaps, as $t \to \infty$ and $d \to \infty$ simultaneously, suitably, converges to the k-marginal of π_0.
- Crucial fact: The *unique* stationary measure of the finite Atlas model with $d+1$ particles, converges to π_0 as $d \to \infty$.
- The finite Atlas model has a unique stationary distribution. So a f.d. approximation approach is designed to select π_0. It cannot help for proving convergence to π_a for $a \neq 0$.!
- One may consider other types of f.d. approximations to the infinite Atlas model. However not clear how to couple them with the the infinite Atlas model.

Main Results.

- A coupling of two probability measures μ_1 and μ_2 on spaces S_1 and S_2 are two random variables X_1, X_2, on a common probability space, with values in S_1 and S_2 resp., s.t. $P \circ (X_i)^{-1} = \mu_i$, $i = 1, 2$.

- Theorem [Banerjee and B. 2022] Fix $a > 0$. Suppose $\nu \in \mathcal{P}(\mathbb{R}_+^\infty)$ satisfies the following: There exists a coupling $(\mathbf{U}, \mathbf{V}_a)$ of ν and π_a such that, almost surely,

$$\limsup_{d\to\infty} \frac{\log\log d}{\log d} \sum_{i=1}^{d} |V_{a,i} - U_i| = 0, \text{ and } \limsup_{d\to\infty} \frac{U_d}{dV_{a,d}} < \infty. \qquad (4)$$

Then $\nu \in \mathrm{wDoA}(\pi_a)$.

- Let $\mathbf{W} \sim \pi_0$, then for $a, a' > 0$, $V_{a,i} = \frac{2}{2+ia} W_i$, $V_{a',i} = \frac{2}{2+ia'} W_i$, $i \in \mathbb{N}$ defines a coupling of π_a and $\pi_{a'}$. Clearly, $\pi_{a'} \notin \mathrm{wDoA}(\pi_a)$. Also,

$$\sum_{i=1}^{d} |V_{a,i} - V_{a',i}| \sim O(|a - a'| \log d) \text{ and } \limsup_{d\to\infty} \frac{V_{a',d}}{dV_{a,d}} = 0.$$

Conjecture. $(\log d)^{-1} \sum_{i=1}^{d} |V_{a,i} - U_i| \to 0$ is necessary.

Remarks.

- Corollary. Fix $a > 0$. Let

$$\nu \doteq \bigotimes_{d=1}^{\infty} \mathrm{Exp}(2 + da + \lambda_d),$$

where

$$-a < \liminf_{d\to\infty} \frac{\lambda_d}{d} \le \limsup_{d\to\infty} \frac{\lambda_d}{d} < \infty.$$

Also assume

$$\limsup_{d\to\infty} \frac{\log\log d}{\log d} \sum_{i=1}^{d} \frac{|\lambda_i|}{i^2} = 0. \tag{5}$$

Then $\nu \in \mathrm{wDoA}(\pi_a)$.

- Need for the second condition: For $a' > 0$, $\lambda_d = da'$ satisfies the first condition but $\nu \notin \mathrm{wDoA}(\pi_a)$.

- If $|\lambda_d| = o(d/\log\log d)$, then both conditions hold.

The Case $a = 0$.

- Theorem (Banerjee and B. (2022)) Suppose $\nu \in \mathcal{P}(\mathbb{R}_+^\infty)$ satisfies the following: there exists a coupling $(\mathbf{U}, \mathbf{V})$ of ν and π_0 s.t., a.s.,

$$\liminf_{d \to \infty} \frac{1}{\sqrt{d}(\log d)} \sum_{i=1}^{d} U_i \wedge V_i = \infty. \tag{6}$$

  Then $\nu \in \mathrm{wDoA}(\pi_0)$.

- Note that if $\nu \overset{d}{\geq} \pi_0$ then (6) holds.
- The condition (6) does not impose any upper bound constraints on gaps...
- ... nor does it require gaps to be strictly positive.
- But gives wDoA instead of sDoA.

Conjecture: Under (6) $\nu \in \mathrm{sDoA}(\pi_0)$.

Case $a = 0$.

- Corollary. Let $\mathbf{U} \sim \nu$. Suppose

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}(\log d)} \sum_{i=1}^{d} (U_i \wedge 1) = \infty, \text{ a.s.}$$

  Then $\nu \in \mathrm{wDoA}(\pi_0)$.

- Corollary. Let $\mathbf{U} \sim \nu$. Suppose $\pi_0 \overset{d}{\geq} \nu$ and, a.s.,

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}(\log d)} \sum_{i=1}^{d} U_i = \infty.$$

  Then $\nu \in \mathrm{wDoA}(\pi_0)$.

Case $a = 0$.

- Corollary. Let $\mathbf{U} \sim \nu$. Suppose $U_j = \lambda_j \Theta_j$, $j \in \mathbb{N}$, where $\{\Theta_j\}$ are iid and non-negative satisfying $\mathbb{P}(\Theta_1 > 0) > 0$, and$\{\lambda_j\}$ satisfy one of the following:

  (a) $\liminf_{j\to\infty} \lambda_j > 0$,

  (b) $\limsup_{j\to\infty} \lambda_j < \infty$ and

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}(\log d)} \sum_{i=1}^{d} \lambda_i = \infty.$$

  Then $\nu \in \mathrm{wDoA}(\pi_0)$.

Remarks.

DJO: Suppose $U_j = \lambda_j \Theta_j$, Θ_j are iid, and

- $E\Theta_1 < \infty$, $E \log(\Theta_1)_- < \infty$.

Then $\nu \in sDoA(\pi_0)$ if one of the following hold:

(i) for some $c \in [1, \infty)$, $\lambda_j \in [c^{-1}, c]$ for all $j \in \mathbb{N}$.

(ii) $\Theta_1 \sim Exp(1)$ and one of the following holds:

  (a) $\lambda_d \uparrow \infty$ and $\limsup_{d\to\infty} \frac{1}{d^\beta} \sum_{i=1}^d \lambda_i < \infty$ for some $\beta < 2$.

  (b) $\lambda_d \downarrow 0$ and $\frac{1}{\sqrt{d} \log d} \sum_{i=1}^d \lambda_i \to \infty$, as $d \to \infty$.

The sufficient conditions in the corollary are substantially weaker:

$\mathbb{P}(\Theta_1 > 0) > 0$, and $\{\lambda_j\}$ satisfy one of the following:

(a) $\liminf_{j\to\infty} \lambda_j > 0$,

(b) $\limsup_{j\to\infty} \lambda_j < \infty$ and

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}(\log d)} \sum_{i=1}^d \lambda_i = \infty.$$

Proof Ingredients.

- Synchronous coupling of infinite Atlas model starting from two different initial conditions.
- Strong existence uniqueness results lacking for the infinite Atlas model.
- Approach: Strong approximative versions [Sarantsev(2017)]. Basic idea:
  - Given an initial ordered vector $\mathbf{x}$ and a sequence of Brownian motions $\{B_i\}_{i\in\mathbb{N}_0}$ construct for each m a finite ordered Atlas model with $m+1$ particles with initial condition $\mathbf{x}|_m = (x_0, \ldots, x_m)$ and BM $(B_0, \ldots B_m)$ as unique strong solutions of the SDE.
  - These converge a.s. as $m \to \infty$ in $C([0,\infty) : \mathbb{R}^\infty)$ to a limit process $\mathbf{X}$ that has the same law as the ordered particles in the infinite Atlas model.
  - Such strong approximative versions can be synchronously coupled.

Proof Sketch for $a > 0$ case.

- Sandwich the initial gap vectors $\mathbf{U} \sim \nu$ and $\mathbf{V} \sim \pi_a$ between $\mathbf{U} \wedge \mathbf{V}$ and $\mathbf{U} \vee \mathbf{V}$.

- Using monotonicity properties bound $\mathbf{Z}^{\mathbf{U}}(\cdot)$ from above and below by the coupled processes $\mathbf{Z}^{\mathbf{U}\vee\mathbf{V}}(\cdot)$ and $\mathbf{Z}^{\mathbf{U}\wedge\mathbf{V}}(\cdot)$ respectively.

- Consider measures

$$\mu_t^{\max}(F) := \frac{1}{t}\int_0^t \mathbb{P}\left(\mathbf{Z}^{\mathbf{U}\vee\mathbf{V}}(s) \in F\right) ds, \;\; F \in \mathcal{B}\left(\mathbb{R}_+^\infty\right),$$
$$\mu_t^{\min}(F) := \frac{1}{t}\int_0^t \mathbb{P}\left(\mathbf{Z}^{\mathbf{U}\wedge\mathbf{V}}(s) \in F\right) ds, \;\; F \in \mathcal{B}\left(\mathbb{R}_+^\infty\right).$$

- Suffices to show, for all k, the marginals on first k coordinates converge:

$$\mu_t^{\max,(k)} \to \pi_a^{(k)} \text{ and } \mu_t^{\min,(k)} \to \pi_a^{(k)}.$$

Proof Sketch for $a > 0$ case.

- Introduce an intermediate process $\mathbf{Z}^d$ where the initial vector has first d entries of $\mathbf{U} \wedge \mathbf{V}$ and remaining of of $\mathbf{U} \vee \mathbf{V}$.

- A Key Stability Estimate. For $\epsilon, \delta > 0$, $k \in \mathbb{N}$,

$$\limsup_{d \to \infty} \mathbb{P}\left(\frac{1}{\log d} \int_0^{\log d} \mathbf{1}\left(|Z_k^{\mathbf{U} \vee \mathbf{V}}(s) - Z_k^d(s)| \geq \epsilon \right) ds \geq \delta \right) = 0.$$

Ingredients in the stability estimate.

- L^1 norm estimate in terms of intersection local time of bottom two particles. Suppose the gap processes $\mathbf{Z}$ and $\tilde{\mathbf{Z}}$ satisfy $\tilde{\mathbf{Z}}(0) \geq \mathbf{Z}(0)$. Then

$$\sum_{j=1}^{\infty} |\tilde{Z}_j(t) - Z_j(t)| = \sum_{j=1}^{\infty} |\tilde{Z}_j(0) - Z_j(0)| + \frac{1}{2} |\tilde{L}_1(t) - L_1(t)|.$$

- Influence of far away coordinates. A quantitative estimate for the time taken for the first k gaps between the ranked particles to 'feel the effect' of the unordered processes $U_i(\cdot)$ for $i \gg k$: There exists $\delta_0 \in (0,1)$ such that,

$$\limsup_{d \to \infty} \mathbb{P}\left(\inf_{s \in [0, \delta_0 \log d]} \inf_{i \geq d} U_i(s) \leq \frac{1}{8a} \log d \right) = 0,$$

and for any $k \in \mathbb{N}$

$$\limsup_{d \to \infty} \mathbb{P}\left(\sup_{s \in [0, \delta_0 \log d]} Y_k(s) \geq \frac{1}{8a} \log d \right) = 0.$$

Ingredients in the stability estimate.

- Contraction by Analyzing Excursions. Consider coupled infinite Atlas systems $\mathbf{Z}$ and $\tilde{\mathbf{Z}}$ with $\mathbf{Z}(0) \leq \tilde{\mathbf{Z}}(0)$. Fix k and let $\Delta Z_k(t) \doteq \tilde{Z}_k(t) - Z_k(t)$.

- Fix $\epsilon > 0$ and consider stopping times

$$\sigma_1 \doteq \inf\{s \geq 0 : \Delta Z_k(s) \geq \epsilon\}$$

$$\sigma_2 \doteq \text{ first time after } \sigma_1 \text{ that } \Delta Z_k, \Delta Z_{k-1}, \ldots \Delta Z_1 \text{ have successively hit } 0.$$

- Contraction estimate.

$$\sum_{i=1}^{\infty} |\Delta Z_i(\sigma_2)| - \sum_{i=1}^{\infty} |\Delta Z_i(\sigma_1)| \leq -\epsilon/2^k.$$

Proof Sketch for $a > 0$ case.

- These estimates give

$$\limsup_{d\to\infty} \mathbb{P}\left(\frac{1}{\log d}\int_0^{\log d} \mathbf{1}\left(|Z_k^{\mathbf{U}\vee\mathbf{V}}(s) - Z_k^d(s)| \geq \epsilon\right) ds \geq \delta\right) = 0.$$

- Equivalently, with $d(t) = e^t$,

$$\limsup_{t\to\infty} \mathbb{P}\left(\frac{1}{t}\int_0^{t} \mathbf{1}\left(|Z_k^{\mathbf{U}\vee\mathbf{V}}(s) - Z_k^{d(t)}(s)| \geq \epsilon\right) ds \geq \delta\right) = 0. \qquad (7)$$

- Consider measure for the intermediate process:

$$\tilde{\mu}_t(F) := \frac{1}{t}\int_0^t \mathbb{P}\left(\mathbf{Z}^{d(t)}(s) \in F\right) ds,\ \ F \in \mathcal{B}\left(\mathbb{R}_+^\infty\right).$$

- Recall

$$\mu_t^{\max}(F) := \frac{1}{t}\int_0^t \mathbb{P}\left(\mathbf{Z}^{\mathbf{U}\vee\mathbf{V}}(s) \in F\right) ds,\ \ F \in \mathcal{B}\left(\mathbb{R}_+^\infty\right).$$

- Using monotonicity properties and (7)

$$\limsup_{t\to\infty} \tilde{\mu}_t^{(k)}((-\infty, \mathbf{z}]) \leq \limsup_{t\to\infty} \mu_t^{\max,(k)}((-\infty, \mathbf{z}]) \leq \pi_a^{(k)}((-\infty, \mathbf{z}]).$$

Proof Sketch for $a > 0$ case.

- By using the quantitative estimates on the influence of far away coordinates argue that for any $k \in \mathbb{N}, \mathbf{z} \in \mathbb{R}_+^k$,

$$\mu_t^{\min,(k)}((-\infty, \mathbf{z}]) - \tilde{\mu}_t^{(k)}((-\infty, \mathbf{z}]) \to 0 \text{ as } t \to \infty.$$

- Combining with previous estimate

$$\limsup_{t\to\infty} \mu_t^{\min,(k)}((-\infty, \mathbf{z}]) \le \pi_a^{(k)}((-\infty, \mathbf{z}]).$$

- Using monotonicity $\liminf_{t\to\infty} \mu_t^{\min,(k)}((-\infty, \mathbf{z}]) \ge \pi_a^{(k)}((-\infty, \mathbf{z}])$.
- This says $\mu_t^{\min,(k)} \to \pi_a^{(k)}$ as $t \to \infty$.
- Similarly argue $\mu_t^{\max,(k)} \to \pi_a^{(k)}$.

References.

- A. Sarantsev, Infinite systems of competing Brownian particles. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, (53), 2279–2315, (2017).

- A. Dembo, M. Jara, and S. Olla, The infinite Atlas process: Convergence to equilibrium. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, (55) 607–619, (2019).

- S. Banerjee and A. Budhiraja, Domains of Attraction of Invariant Distributions of the Infinite Atlas Model, to appear in *Ann. Prob.* (2022).