Scaling limit for a perturbed Howard model

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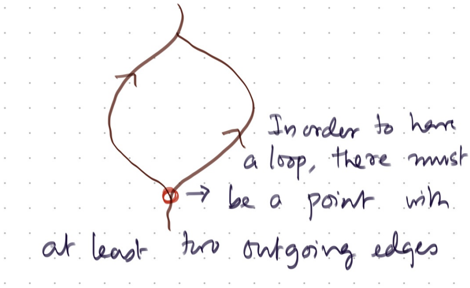
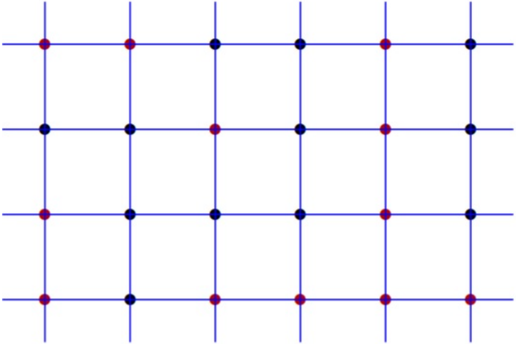
joint work with Rahul Roy and Anish Sarkar

Drainage networks

Drainage networks are random directed forests constructed on a *random* set of vertices such that each vertex has exactly one outgoing edge (which ensures that the generated random graph is **cycle free**).

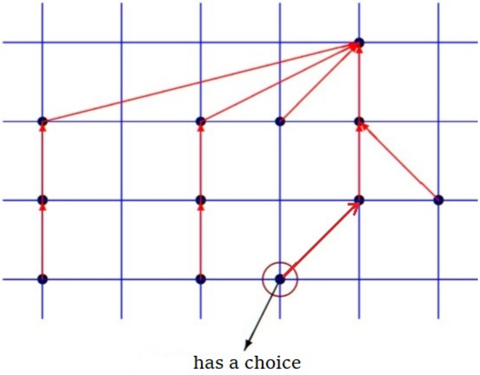


Howard's model: a drainage network model



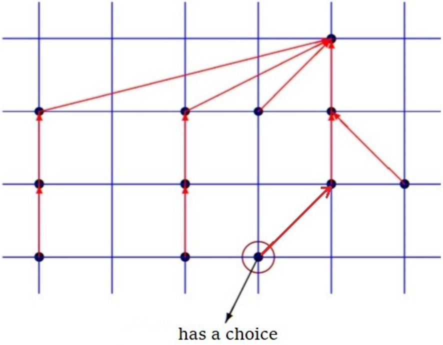
- Each lattice point is **open** with probability $p \in (0, 1)$ independent of everything else. **Open** vertices act as source vertices (for fluid flow).

Howard's model: a drainage network model



- An **open** vertex connects to the nearest open vertex at the next (upper) level.
- If there are choices available, then one is chosen uniformly independent of everything else.

Howard's model: a drainage network model



This gives us the random graph $G = (V, E)$ which is of our interest.

Some properties of Howard's model

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- Each (open) vertex has exactly **one outgoing** edge and hence, G is **cycle-free** a.s.
- Each component must be an infinite directed tree. The question is how many are they??

Some properties of Howard's model

Theorem ((Gangyopadhyay, Roy, Sarkar 05))

The random graph G is ***connected****.*

Howard's model: a drainage network model

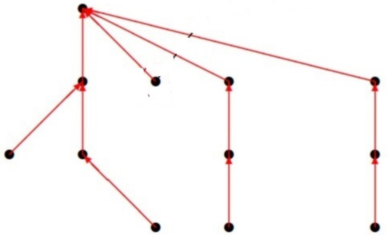


Figure : Fella river network in Northern Italy.
From Maritan *et al*, Phy. Rev. E 1996

- Thus, Howard's model gives a *random tributary* on $\mathbb{Z}^2$.

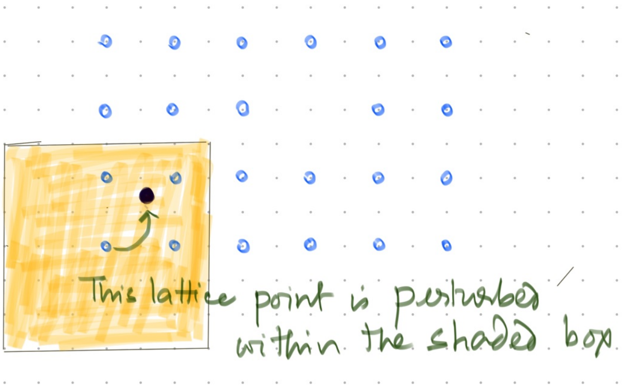
Variants of drainage network models

- Similar questions were asked for other drainage network models where construction of edges involve more dependencies.

Variants of drainage network models

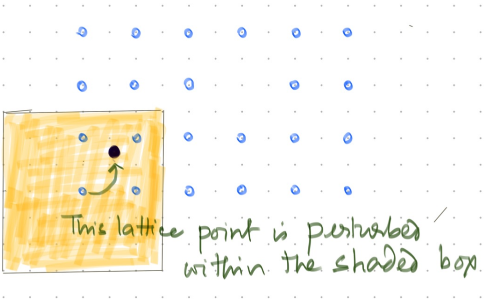
- Similar questions were asked for other drainage network models where construction of edges involve more dependencies.
- Typically, for all these models random sets of vertices enjoy independence over disjoint regions.
- What happens if don't have the assumption of independence over disjoint regions?

A drainage network model on perturbed lattice



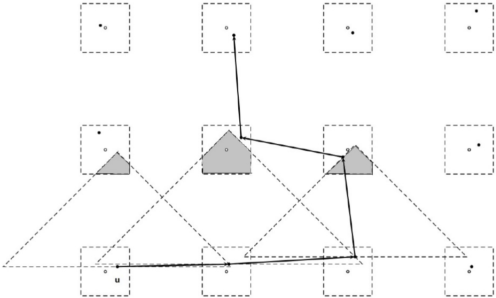
- For each lattice point, consider a compact box centred at that point and an independent (random) perturbation vector uniformly distributed over that box.
- Lattice points are independently perturbed using these i.i.d. perturbation vectors.

A drainage network model on perturbed lattice



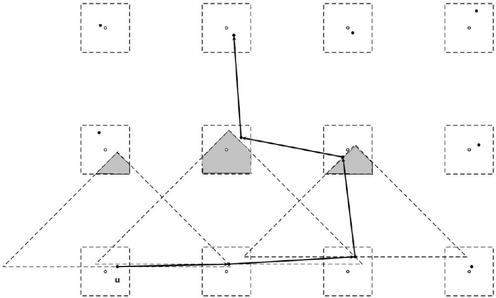
- Consider the point process of randomly perturbed lattice points and construct a drainage network model as follows: each vertex connects to the nearest vertex (in ℓ_1 norm) with higher y co ordinate.

A drainage network model on perturbed lattice



- Note that the boxes can overlap and the generated point process does not have independence over disjoint region.

A drainage network model on perturbed lattice



- Note that the boxes can overlap and the generated point process does not have independence over disjoint region.
- In fact, the generated point process is not even 'tolerant' (in the sense of Ghosh and Peres [17]).

A perturbed Howard's model

Theorem (Ghosh, S. (20))

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Under diffusive scaling with suitable normalization constants, the ℓ_1 DSF converges to the Brownian web.

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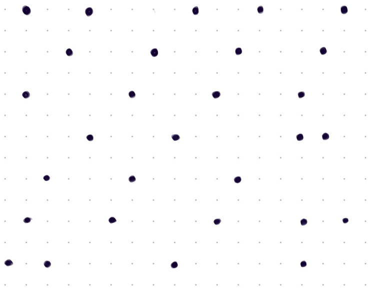
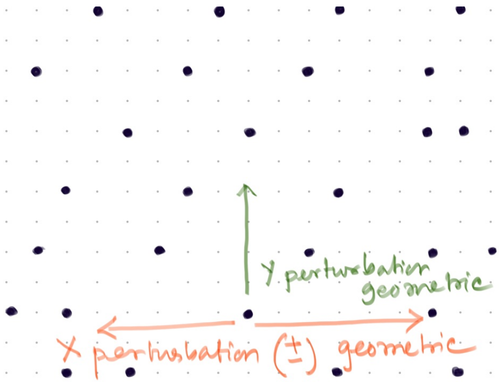


- Each lattice point is independently open (closed) with probability p $(1-p)$ where $p \in (0,1)$. We consider the random set of open vertices only.

A drainage network model on perturbed lattice



Each open vertex is perturbed as follows:

- Y coordinate is perturbed according to i.i.d. geometric;
- X coordinate is perturbed according to $(\pm)$ i.i.d. geometric;

A perturbed Howard's model

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A perturbed Howard's model

- We consider the resultant set of *perturbed open* vertices and construct Howard's network on this.
- The resultant point process does not have independence over disjoint regions (and not even a tolerant point process too).

A perturbed Howard's model

Theorem (Roy, S. , Sarkar (22))

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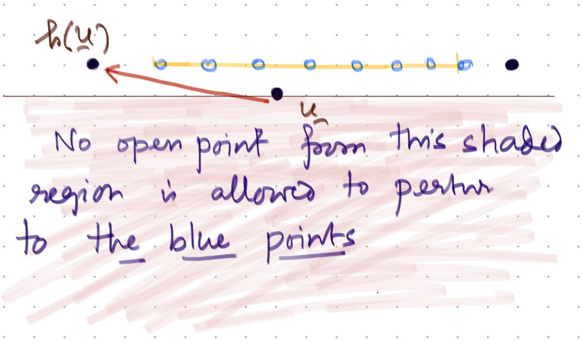
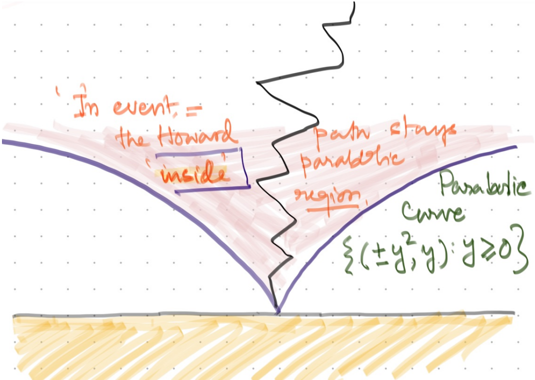
Under diffusive scaling with suitable normalization constants, the perturbed Howard model converges to the Brownian web.

Challenges for studying perturbed Howard model

The Howard path $\{h^n(\mathbf{0}) : n \geq 1\}$ is not a random walk.
It is not even a Markov process.



Renewal structure for perturbed Howard model: ‘In’ event

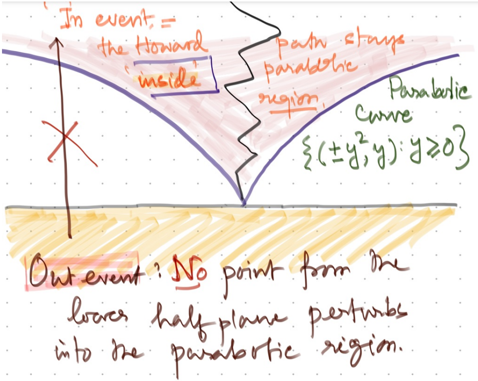


Definition

(‘In’ event) Let $\nabla(\mathbf{x}) \subset \mathbb{R}^2$ denote the parabolic region centred at $\mathbf{x}$. ‘In’ event occurs at step n if we have

$$h^{n+j}(\mathbf{u}) \in \nabla(h^n(\mathbf{u})) \text{ for all } j \geq 1.$$

Renewal structure for perturbed Howard model: ‘Out’ event

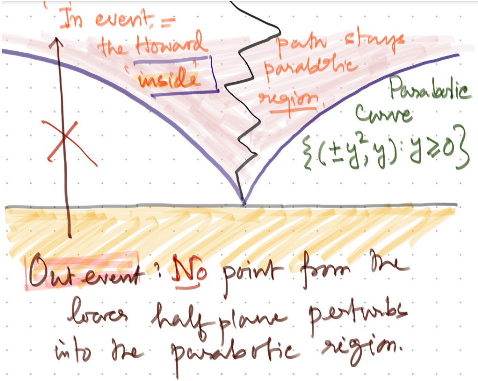


Definition

(‘Out’ event) ‘Out’ event occurs at step n if we have

$$\text{perturbation of } \mathbf{w} \notin \nabla(h^n(\mathbf{u})) \text{ for all } \mathbf{w}(2) \leq h^n(\mathbf{u})(2).$$

Renewal structure for perturbed Howard model



Definition

(Renewal event) Renewal event occurs at the n-th step if both 'In' event and 'Out' event occurs at $h^n(\mathbf{u})$.

- The marginal process $\{h^n(\mathbf{0}) : n \geq 1\}$ observed at renewal steps gives a random walk with i.i.d. increments.

Renewal structure for perturbed Howard model

- The marginal process $\{h^n(\mathbf{0}) : n \geq 1\}$ observed at renewal steps gives a random walk with i.i.d. increments.
- The (random) time gap between any two successive renewal steps has moments of all orders.
- The renewal step for joint process $\{h^n(\mathbf{u}), h^n(\mathbf{v}) : n \geq 1\}$ starting from two different points is joint occurrence for renewal steps for each of the marginal processes.

Coalescing time tail decay for the perturbed Howard model

Theorem

(Roy, S. , Sarkar (22))
Let $\tau = \tau(x, y)$ *denote the* *coalescing time* *for the joint Howard paths* $\{(h^n(x, 0), h^n(y, 0)) : n \in \mathbb{N}\}$ *starting from* $(x, 0)$ *and* $(y, 0)$. *Then* τ *is a.s. finite and there exists* $C_0 > 0$ *such that*

$$\mathbb{P}(\tau(x, y) > n) < C_0 |x - y| / \sqrt{n} \text{ for all } n \in \mathbb{N}.$$

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Away form the origin the difference between the two paths observed at renewal steps behaves like symmetric random walk on an *event with high probability*.

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- Under diffusively scaling, individual path converges to Brownian motion.
- In the scaling limit, we should have independent coalescing Brownian motions starting from everywhere on $\mathbb{R}^2$. This justifies Brownian web as the scaling limit for the perturbed Howard's model.

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Future extensions

- What happens the lattice points are perturbed using an i.i.d. collection of Gaussian random vectors?
- Jointly with Subhroshekhar Ghosh and Kartick Adhikari, we are studying a model where only y coordinates lattice points are perturbed using an i.i.d. collection of $(\pm)$ geometric random variables. [Challenge: *future* may affect *past*]
- Can we extend this technique for general point processes where covariance between disjoint regions decays fast in terms of (Hausdorff) distance between these regions?

Thank you