First Passage Percolation on Hyperbolic groups

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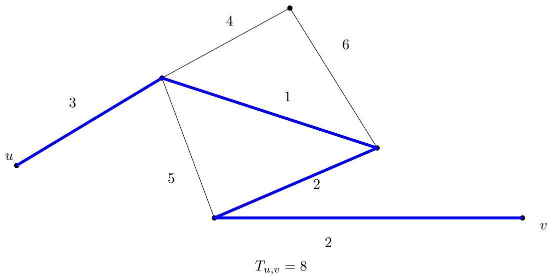
Best Wishes to RLK

First passage percolation on a graph

- Random perturbation of the graph distance.
- Assign i.i.d. positive lengths to the edges.
- First passage time $T_{u,v}$ between two vertices u and v is the distance between them in the perturbed metric: i.e., the length of the shortest path between the two.



Basic objects or interest

- Asymptotics of the first passage time between two vertices when the graph distance becomes large.
- Asymptotics of the metric ball $B_v(t)$ with centre v and large radius t.
- Existence and geometry of finite/ semi-infinite or bi-infinite geodesics.

Cayley graph of finitely generated groups

- A natural choice for the background geometry-locally finite translation invariant infinite graphs.
- Cayley graph of finitely generated groups.

Definition

Consider a group G with a finite symmetric generating set S. Then the corresponding Cayley graph Γ has elements of G as its vertex set and g_1 and g_2 are connected by an edge if $g^{-1}g_2 \in S$.

Examples

- Consider the group $\mathbb{Z}^d$ with generators $\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$- the Cayley graph is the standard Euclidean lattice.
- Finitely generated free groups with standard generators (and their inverses): the Calyley graph is a regular tree.

FPP on $\mathbb{Z}^d$

- Hammersley-Welsh (1965)- model for fluid flow through inhomogeneous media, Eden (1961)- growth of bacterial colony.
- First order behaviour by subadditivity.

  - Let $T_{m,n}$ denote the passage time from me_1 to ne_1.
  - $T_{0,m+n} \leq T_{0,m} + T_{m.m+n}$.
  - $\mathbb{E}T_{0,m+n} \leq \mathbb{E}T_{0,m} + \mathbb{E}T_{0,n}$.
  - $n^{-1}\mathbb{E}T_n \to \mu = \mu(e_1) \in (0, \infty)$ under mild conditions on the passage time variables.

- Same is true in all other directions.
- Can be upgraded to almost sure convergence by subadditive ergodic theorem.

FPP on $\mathbb{Z}^d$

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Shape theorem

- One can patch the limits in each directions to get a limit shape.
- Under mild conditions

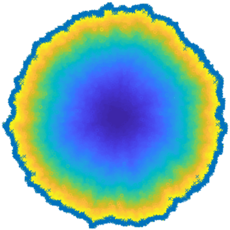
$$t^{-1}\tilde{B}_0(t) \to \mathcal{B}$$

  almost surely as $t \to \infty$.
- $\tilde{B}$ is the "filled" metric ball.
- Limit shape $\mathcal{B}$ is a compact convex subset of $\mathbb{R}^d$.

Predictions for FPP on $\mathbb{Z}^d$

Under mild conditions

- Limit shape $\mathcal{B}$ is strictly convex with uniformly curved boundary.
- $\mathrm{Var}(T_{0,n}) \approx n^{2\chi}$ for some $\chi < 1/2$.
- For $d = 2$, $\chi = 1/3$– KPZ universality class.
- Typical deviations of the geodesics from the straight line $\approx n^{\xi}$ where $\chi = 2\xi - 1$.



Prediction for infinite geodesics (at least for $d = 2$)

Under mild conditions

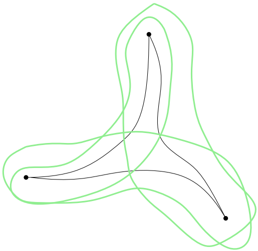
- Almost surely, every semi-infinite geodesic has a limiting direction.
- For any given direction, almost surely there exists a unique infinite geodesic starting from each vertex going in this direction.
- For a given direction, and two distinct starting points, the semi-infinite geodesics almost surely coalesce.
- Almost surely, there does not exist any bi-infinite geodesic.

Gromov hyperbolic groups

Definition

- A geodesic metric space is called δ-hyperbolic if it satisfies the "thin triangle condition"– the third side of each triangle is contained in the δ-neighbourhood of the fiirst two sides.
- A finitely generated group is called Gromov hyperbolic if the Calyley graph with respect to some symmetric generating set is δ-hyperbolic for some finite δ.

- Trees are 0-hyperbolic.
- $SL_2(\mathbb{Z})$, or other groups acting properly discontinuously on a tree.



Boundary and Patterson-Sullivan measure

- To parametrize directions, one can define a boundary ∂G of hyperbolic groups.
- Points on the boundary can be thought of as equivalence classes of word geodesic rays.
- Cantor set like object.
- One can define a natural measure ν on ∂G that is obtained as a weak limit of appropriately defined measures on the boundaries of n-balls– called Patterson-Sullivan measure.
- These are usually defined for given base points, we shall always think of the identity element as the base point.

Earlier works

- The interest in studying FPP on general background geometries is rather recent.
- Initiated primarily by Itai Benjamini and co-authors.
- Certain questions turn out to be easier to answer for hyperbolic groups, although answers might be different to what is expected in the Euclidean case.
- Benjamini-Tessera showed bigoedesics exist for FPP on hyperbolic groups.
- Variance is predicted to grow linearly.

Our set-up and results

- Consider a finitely generated Gromov hyperbolic group G.
- Fix a symmetric generating set S and consider the corresponding Cayley graph Γ.
- Let 1 denote the vertex corresponding to the identity element.
- Let ∂G denote the boundary and let ν denote the Patterson-Sullivan measure on it.
- Consider FPP on Γ with i.i.d. positive passage times with a non-degenerate continuous distribution and sufficiently light tails.

Results: metric growth

- Let us consider a word geodesic $\gamma = \{1 = x_0, x_1, \dots, x_n, \dots\}$ converging to $\xi \in \partial G$.

Question

Does $\frac{\mathbb{E}T_{1,x_n}}{n}$ converge?

- What about the previous sub-additivity argument?
- Works if there are points $g, g^2, g^3, \dots$ on γ.
- That is, it works if ξ is a *polar* direction, i.e., $\xi = g^{\pm\infty}$ for some $g \in G$.

Results: metric growth

Theorem (B., Mj)

For ν-a.e. $\xi \in \partial G$, there exists $v(\xi)$ such that

$$\frac{\mathbb{E}T_{1,x_n}}{n} \to v(\xi).$$

Further, $v(\xi)$ is constant ν-a.e..

- Can be upgraded to convergence in probability.
- One can find directions along which the convergence does not hold.

Results: Fluctuations

- What about the fluctuations of T_{1,x_n}?

Theorem (B., Mj)

For each $\xi \in \partial G$ *and each word geodesic ray* $\gamma = \{1 = x_0, x_1, \ldots, x_n, \ldots\}$ *converging to* ξ

$$C_1 \leq \frac{\mathrm{Var}(T_{1,x_n})}{n} \leq C_2.$$

- The upper bound comes from the standard Poincare inequality argument, essentially same as Kesten's proof in the Euclidean case.
- The lower bound is the one that requires the hyperbolic structure.

Results: geodesics

Definition

For a given environment ω of passage times, consider a geodesic ray $\sigma = \{x_0, x_1, \ldots\}$ (in the FPP metric). We say σ accumulates on $\xi \in \partial G$ if there is a subsequence x_{n_k} converging to ξ; σ is said to have direction ξ if it is the only accumulation point.

Theorem (B., Mj)

For almost every ω, each geodesic ray has a direction. Further, given $o \in G$ and $\xi \in \partial G$ almost surely there exists a unique geodesic ray starting at o in direction ξ.

Results: geodesics

- What about coalescence of geodesics?

Theorem (B., Mj)

For $\xi \in \partial G$ and $o_1, o_2 \in G$ almost surely the geodesic rays from o_1 and o_2 in the direction ξ coalesce.

Some ideas from the proof

Automatic structure

- Hyperbolic groups admit an automatic structure– Cannon (1984).
- There exists a finite state automaton parametrizing a geodesic combing of the Calyley graph.
- Associated to the automaton there is a topological Markov chain.
- One can roughly think that the Patterson-Sullivan measure on ∂G can be lifted to a Markov chain on the space of words and almost every geodesic ray eventually enters one of the irreducible components – Calegari-Fujiwara (2010).

Existence of the time constant

- Using the standard theory of ergodic Markov chains, one can show that for ν-a.e., the proportion of occurrence of finite words along the ray converges.
- One can then approximate, for $m, n \gg 1$, $T_{1,x_{mn}}$ by

$$T_{1,x_n} + T_{x_n,x_{2n}} + \cdots + T_{x_{(m-1)n},x_{mn}}.$$

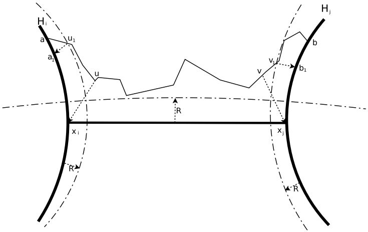
- The proof is then completed by translation invariance and SLLN.

Exponential divergence of hyperplanes

Definition

For a word geodesic ray $\gamma = \{1, x_1, x_2, \ldots\}$ define the hyperplane H_i through x_i as the set of all points whose nearest neighbour projection onto γ is x_i.

- Coarsely well-defined.
- The hyperplanes diverge exponentially, i.e., if $j - i$ is sufficiently large any path from H_i to H_j that stays at least distance R from γ has word length larger than $(j-i)e^{\alpha R}$.



Proof of coalescence

- Each geodesic must cross all hyperplanes from some index onwards.
- Exponential divergence of hyperplanes imply that it is extremely likely for the geodesics to come close to γ while crossing between two hyperplanes.
- This can be used to show that the geodesics meet between two hyperplanes with uniformly positive probability.
- The proof is completed by Borel-Cantelli Lemma and uniqueness of geodesics.

Linearity of variance

- Consider the variance decomposition given by the Doob martingale where the filtration successively exposes the region between consecutive hyperplanes.
- A similar argument as above shows that the contribution to the variance coming from each term is uniformly bounded below.
- This shows that the variance grows at least linearly.
- Similar arguments have previously been used for FPP/LPP across thin cylinders.

Thank You

Questions?