

Random analytic functions from OPUC

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Pérez-Virág (2005): $f(z) = g_0 + g_1 z + g_2 z^2 + \dots$ g_k iid $N_{\mathbb{C}}(0, 1)$ density $\frac{1}{\pi} e^{-|z|^2}$ on \mathbb{C}

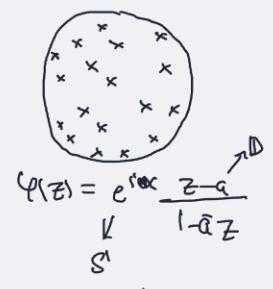
↪ random analytic function on $\mathbb{D} = \{ |z| < 1 \}$;

$Z_f = f^{-1}\{0\}$ - random discrete subset of \mathbb{D}
conformally invariant in distribution

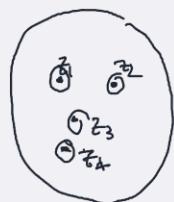
Z_f is a determinantal point process on \mathbb{D} with kernel $K(z, \omega) = \frac{1}{\pi(1-z\bar{\omega})^2}$ (Bergman)

Means $P\{Z_f \cap \mathbb{D}(z_1, \varepsilon) = \phi, 1 \leq i \leq m\} -$

$$\frac{1}{(\pi \varepsilon^2)^m} \rightarrow \det(K(z_i, z_j))_{1 \leq i, j \leq m}$$



$$\Psi \circ Z_f \stackrel{d}{=} Z_f + \Psi$$



$$K_\alpha(z, \omega) = \frac{\alpha}{\pi} \frac{(1-|z|^2)^{\frac{\alpha-1}{2}} (1-|\omega|^2)^{\frac{\alpha-1}{2}}}{(1-z\bar{\omega})^{\alpha+1}}$$

(2010) Let $\alpha \in \{1, 2, 3, \dots\}$
 $f_\alpha(z) = g_0 + g_1 z + g_2 z^2 + \dots$ $\begin{cases} \alpha \times \alpha \text{ matrix} \\ \text{analytic function} \end{cases}$
 G_k are iid $\alpha \times \alpha$ matrices whose entries are iid $N_{\mathbb{C}}(0, 1)$

$$K_\alpha(z, \omega) = \frac{\alpha}{\pi} \frac{(1-|z|^2)^{\frac{\alpha-1}{2}} (1-|\omega|^2)^{\frac{\alpha-1}{2}}}{(1-z\bar{\omega})^{\alpha+1}}$$

DPP with kernel K_α is conformally invariant

Let $Z_{f_\alpha} = \text{Zero set of } \det(f_\alpha(\cdot))$. Then $Z_{f_\alpha} \sim \text{DPP}(K_\alpha)$

Proof used a result from RMT on truncated unitary matrices.

Zyczkowski-Sommers: $U \sim \text{Haar}(U_{mn}) \rightarrow \text{delete one row one column} \rightarrow V \sim \text{Haar}(V_{(m-1) \times (n-1)})$

Eigenvalues of V have density $\frac{1}{2\pi} \prod_{i,j} |\lambda_i - \lambda_j|^2$ on \mathbb{D}

\equiv DPP with kernel $K_n(z, \omega) = ((1+2z\bar{\omega} + 3z\bar{\omega}^2 + \dots + (n-1)z^{n-1}\bar{\omega}^{n-1}) / \pi)$



$$K(z, \omega) = \frac{1}{\pi(1-z\bar{\omega})^2}$$

Hanningsky (1950s): Let $\Omega_n = \{(c_0, \dots, c_{n-1}) \in \mathbb{C}^n \mid z^n + \sum_{k=0}^{n-1} c_k z^k \text{ has all roots in } \mathbb{D}\}$

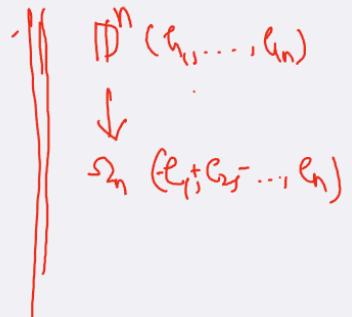
↳ bounded open set in \mathbb{C}^n

Suppose $\underline{c} \sim \text{Unif}(\Omega_n)$. Then the zeros of

$$P(z) = z^n + c_{n-1}z^{n-1} + \dots + c_1z + c_0$$

have joint density

$$\frac{1}{\Omega_n} \prod_{i < j} (c_i - c_j)^2$$



Can we show

\times $P \rightarrow \sum_{k=0}^{\infty} c_k z^k$
in appropriate sense

$$\text{Eq: } \delta_n(c_0, c_1, \dots) \xrightarrow{d} (c_0, c_1, \dots)$$

DPP with Bergman kernel

New realization: A new way of generating P

Fix $\alpha_0, \alpha_1, \dots$, independent rv where $\alpha_k \sim \frac{1}{\pi} (1 - |z|^2)^{-k}$ on \mathbb{D}

Set $\tilde{\Phi}_0(z) = 1$

$$\tilde{\Phi}_1(z) = z \tilde{\Phi}_0(z) + \alpha_1 \tilde{\Phi}_0^*(z)$$

$$\tilde{\Phi}_n(z) = z \tilde{\Phi}_{n-1}(z) + \alpha_n \tilde{\Phi}_{n-1}^*(z)$$

$$q(z) = \beta_0 z^n + \dots + \beta_n z + \beta_0$$

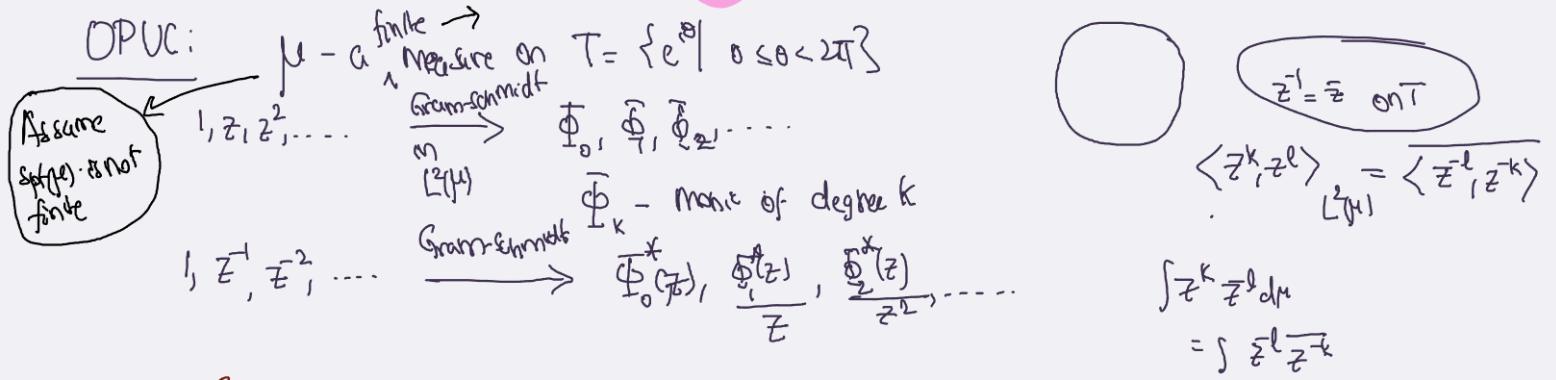
$$\sqrt{n} q(z) = z^n \overline{q(\sqrt{z})} \\ = \bar{\beta}_0 z^n + \dots + \bar{\beta}_n z + \bar{\beta}_0$$

CLAIM1: Then $\tilde{\Phi}_n \xrightarrow{d} P$

CLAIM2: Let $R_n(z) = \frac{\tilde{\Phi}_n(z)}{\tilde{\Phi}_n^*(z)}$. Then R_n has the same roots as $\tilde{\Phi}_n$ in \mathbb{D}

$$\text{and } \sqrt{n} R_n(z) \rightarrow q + q_1 z + q_2 z^2 + \dots$$

where $q_k \xrightarrow{d} N(0, 1)$



Szegő recursion: $\exists \alpha_0, \alpha_1, \dots \in \mathbb{P} \text{ s.t.}$

$$\left(\begin{array}{l} \Phi_{k+1}(z) = z\Phi_k(z) + \alpha_k \Phi_k^*(z) \\ \perp \Phi_0^*, \dots, \Phi_k^* \end{array} \right) \text{ for } k \geq 0.$$

$$\Phi_{k+1}^*(z) = \alpha_k z \Phi_k^*(z) + \Phi_k^*(z)$$

$(\alpha_k)_{k \geq 0}$ are called Verblunsky coefficients

Durbin-Levinson algorithm in time series

Connection to time series: $\mu \leftrightarrow (x_n)_{n \geq 0} \text{ s.t. } E[x_n \bar{x}_m] = \hat{\mu}(n-m)$

$\hat{\mu}_n = \text{Partial Covariance of } x_{n+1} \text{ and } x_0 \text{ eliminating } x_1, \dots, x_n$

key calculation: Suppose $Q(z) = z^m + b_{m-1}z^{m-1} + \dots + b_1z + b_0$

$$P(z) = z^{m+1} + q_{m-1}z^m + \dots + q_1z + q_0$$

$$\text{and } R(z) = zQ(z) + \alpha Q^*(z)$$

$$a_m = b_{m-1} + \alpha \bar{b}_0$$

$$c_{m-1} = b_{m-2} + \alpha \bar{b}_1$$

⋮

$$a_1 = b_0 + \alpha \bar{b}_{m-1}$$

$$a_0 = \alpha$$

$$\left| \frac{\partial (a_0, \dots, a_m)}{\partial (b_0, \dots, b_{m-1}, \alpha)} \right| = (1 - |\alpha|^2)^{m-1}$$

Repeating this leads to

$$Q(z) = z^m + a_{m-1}z^{m-1} + \dots + a_1z + a_0$$

$$\left| \frac{\partial (a_{m-1}, \dots, a_0)}{\partial (b_0, \dots, b_{m-1})} \right| = \prod_{k=0}^{m-1} (1 - |\alpha_k|^2)^{k+1}$$

From this the CLAIM follows.