THE GREATEST PRIME DIVISOR OF A PRODUCT OF TERMS IN AN ARITHMETIC PROGRESSION

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1. Introduction

Let $d \geq 1, k \geq 2, n \geq 1$ and $y \geq 1$ be integers with $\gcd(n, d) = 1$. We write

$$\Delta = \Delta(n, d, k) = n(n+d)\cdots(n+(k-1)d).$$

For an integer $\nu > 1$, we denote by $\omega(\nu)$ and $P(\nu)$ the number of distinct prime divisors of ν and the greatest prime factor of ν, respectively, and we put $\omega(1) = 0$, $P(1) = 1$. Further we write $\pi_d(\nu)$ for the number of primes $\leq \nu$ coprime to d and we put $\pi(\nu) = \pi_1(\nu)$. Let $W(\Delta)$ be the number of terms in Δ divisible by a prime $> k$.

Let $d = 1$. A well known theorem of Sylvester [15] states that

$$P(\Delta) > k \text{ if } n > k. \tag{1}$$

We observe that $P(\Delta(1, 1, k)) \leq k$ and therefore, the assumption $n > k$ in (1) cannot be removed. For $n > k$, Moser [10] sharpened (1) to $P(\Delta) > \frac{11}{10}k$ and Hanson [4] to $P(\Delta) > 1.5k$ unless $(n, k) = (3, 2), (8, 2), (6, 5)$. Further Laishram and Shorey [5] proved that $P(\Delta) > 1.95k$ with $n > k$ except for an explicitly given finite set of pairs (n, k). We refer to [5] for a precise formulation of the above result. We observe that $P(\Delta(k+1, 1, k)) \leq 2k$ and therefore 1.95 cannot be replaced by 2 in the preceding result. Further it has been proved in [5] that $P(\Delta) > 2k$ for $n > \max(k + 13, \frac{279}{262}k)$.

Now we consider (1) when $d > 1$. Let $d = 2$. If $n > k$, then (2) follows from Laishram and Shorey [7, Theorem 1]. Let $n \leq k$. Then we observe that $P(\Delta(n, 2, k)) \leq 2k$ implies $P(\Delta(n + k, 1, k)) \leq 2k$. Therefore the case $d = 2$ when considering $P(\Delta(n, 2, k)) > 2k$ reduces to considering $P(\Delta(n + k, 1, k)) > 2k$ discussed above in the case $d = 1$. Therefore we may suppose that $d > 2$. Sylvester [15] proved that

$$P(\Delta) > k \text{ if } n \geq d + k.$$

Langevin [8] sharpened it to $P(\Delta) > k$ if $n > k$. Shorey and Tijdeman [14] improved it to $P(\Delta) > k$ for $k \geq 3$ unless $(n, d, k) = (2, 7, 3)$. The case $k = 2$ is clear since $P(\Delta(n, d, 2)) = 2$ if and only if $n = 1, d = 2^r - 1$ with $r > 1$. We prove

Theorem 1. *Let $d > 2$ and $k \geq 3$. Then*

$$P(\Delta) = P(n(n+d)\cdots(n+(k-1)d)) > 2k \tag{2}$$

unless (n, d, k) *is given by*

$$\begin{aligned} k = 3,\ & n = 1, d = 4, 7; \\ & n = 2, d = 3, 7, 23, 79; \\ & n = 3, d = 61;\ n = 4, d = 23; \\ & n = 5, d = 11;\ n = 18, d = 7; \\ k = 4,\ & n = 1, d = 3, 13;\ n = 3, d = 11; \\ k = 10,\ & n = 1, d = 3. \end{aligned}$$

It is necessary to exclude the exceptions stated in Theorem 1. Lower bounds for $P(\Delta)$ have been useful at several places. For example, see [12].

The computations in this paper were carried out using MATHEMATICA and MAGMA. We thank Professor R. Tijdeman for his useful remarks and for pointing out an exception $(9, 247)$ in (6).

2. Lemmas

We begin with

Lemma 1. *It suffices to prove Theorem 1 for* k *such that* $2k - 1$ *is prime.*

Proof. Let (n, d, k) be as in Theorem 1. Let k_1 and k_2 be such that $k_1 < k < k_2$ and $2k_1 - 1, 2k_2 - 1$ are consecutive primes. Assume that (2) holds at (n, d, k_1). Then

$$P(n(n+d)\cdots(n+(k-1)d) \geq P(n\cdots(n+(k_1-1)d)) > 2k_1$$

implying $P(\Delta(n, d, k)) \geq 2k_2 - 1 > 2k$. Thus (2) holds at (n, d, k).

Therefore (2) is valid except possibly for those triples (n, d, k) with (n, d, k_1) as one of the exceptions in Theorem 1. We check the validity of (2) at those (n, d, k). For instance, let $k = 11$. Then $k_1 = 10$. We see that $(1, 3, 10)$ is the only exception in Theorem 1. We check that (2) holds at $(1, 3, 11)$. $\square$

For a proof of the following result, we refer to de Weger [16, Theorem 5.2]. It is a particular case of Catalan equation which has been solved completely by Mihăilescu [9].

Lemma 2. *Let* $a, b \in \{2, 3, 5\}$ *and* $a < b$*. Then the solutions of*

$$a^x - b^y = \pm 1 \text{ in integers } x > 0, y > 0$$

are given by

$$(a^x, b^y) \in \{(2^2, 3), (2, 3), (2^3, 3^2), (2^2, 5)\}.$$

The next result is due to Nagell [11], see [1].

Lemma 3. *Let $a, b, c \in \{2, 3, 5\}$ and $a < b$. Then the solutions of*

$$a^x + b^y = c^z \text{ in integers } x > 0, y > 0, z > 0$$

are given by

$$\begin{aligned}(a^x, b^y, c^z) \in \{&(2,3,5), (2^4, 3^2, 5^2), (2, 5^2, 3^3),\\ &(2^2, 5, 3^2), (3, 5, 2^3), (3^3, 5, 2^5), (3, 5^3, 2^7)\}.\end{aligned}$$

We shall also need some more equations given by the following. See also de Weger [16, Theorem 5.5].

Lemma 4. *Let $\delta \in \{1, -1\}$. The solutions of*

$$\begin{aligned}&(i) \quad 2^x - 3^y 5^z = \delta\\ &(ii) \; 3^x - 2^y 5^z = \delta\\ &(iii) \; 5^x - 2^y 3^z = \delta\end{aligned}$$

in integers $x > 0, y > 0, z > 0$ are given by

$$(x, y, z, \delta) = \begin{cases}(4,1,1,1) & \text{for } (i);\\ (4,4,1,1), (2,1,1,-1) & \text{for } (ii);\\ (2,3,1,1), (1,1,1,-1) & \text{for } (iii),\end{cases}$$

respectively.

Proof. (i) Let $\delta = 1$. By $2^x \equiv 1 (\text{mod } 5)$, we get $4|x$. This implies $2^{\frac{x}{2}} - 1 = 3^y, 2^{\frac{x}{2}} + 1 = 5^z$ and the assertion follows from Lemma 2. Let $\delta = -1$. Then $2^x \equiv -1 (\text{mod } 5)$ and $2^x \equiv -1 (\text{mod } 3)$ implying $2|x$ and $2 \nmid x$, respectively. This is a contradiction.

(ii) Let $\delta = 1$. By $3^x \equiv 1 (\text{mod } 5)$ giving $4|x$ and the assertion follows as in (i) with $\delta = 1$. Let $\delta = -1$. Let $y \geq 2$. Then $3^x \equiv -1 (\text{mod } 5)$ and $3^x \equiv -1 (\text{mod } 4)$ implying $2|x$ and $2 \nmid x$, respectively. Therefore $y = 1$ and we rewrite equation (ii) as $2 \cdot 5^z - 3^x = 1$. We may assume that $z \geq 2$ and further x is even by reading mod 4. Thus $3^x \equiv -1 (\text{mod } 25)$ giving $x \equiv 10 (\text{mod } 20)$. Then $\frac{x}{10}$ is odd and

$$1 + 9^5 \text{ divides } 1 + (9^5)^{\frac{x}{10}} = 2 \cdot 5^z,$$

a contradiction.

(iii) Let $\delta = 1$. By mod 3, we get x even and the assertion follows as in (i) with $\delta = 1$. Let $\delta = -1$. We may assume that $y = 1$ by mod 4 and $z \geq 2$. Then we derive as in (ii) with $\delta = -1$ that $\frac{x}{3}$ is odd by using mod 9 and $1 + 5^3$ divides $1 + 5^x = 2 \cdot 3^z$, a contradiction. □

Now we state a result due to Saradha, Shorey and Tijdeman [13] for $k = 6, 7$.

Lemma 5. *Let $n \geq 1, d > 2$ and $k = 6, 7$. Assume that*

$$(n, d, k) \notin \{(1,3,6), (1,3,7), (1,4,7), (2,3,7), (2,5,7)\}.$$

Then

$$\omega(\Delta) \geq \pi(k) + 2.$$

For $k \geq 9$, Laishram and Shorey [7, Theorem 1] proved the following result.

Lemma 6. *Let $n \geq 1, d > 2$ and $k \geq 9$. Assume $(n, d, k) \notin V$ where V is given by*

$$\begin{cases} n = 1,\ d = 3,\ k = 9, 10, 11, 12, 19, 22, 24, 31; \\ n = 2,\ d = 3,\ k = 12;\ n = 4,\ d = 3,\ k = 9, 10; \\ n = 2,\ d = 5,\ k = 9, 10; n = 1,\ d = 7,\ k = 10. \end{cases}$$

Then

$$W(\Delta) \geq \pi(2k) - \pi_d(k).$$

We observe that Δ is divisible by every prime $p \leq k$ with $p \nmid d$ and $\omega(\Delta) \geq W(\Delta) + \pi_d(k)$. Therefore Lemma 6 implies the following result immediately.

Corollary 1. *Let n, d and k be as in Lemma 6. Then*

$$\omega(\Delta) \geq \pi(2k).$$

We shall also need some estimates for the number of primes due to Dusart [2, p.14].

Lemma 7. *For $\nu > 1$, we have*

$$\pi(\nu) \leq \frac{\nu}{\log \nu}\left(1 + \frac{1.2762}{\log \nu}\right)$$

We write $p(d)$ for the least prime divisor of d. We shall use the following computational result.

Lemma 8. *Assume that $p(d) > k$ if $k = 6, 7$ and $p(d) > 2k$ if $k = 9, 10, 12, 15, 16$. Then (2) holds if*

$$n + d \leq N$$

where

$$N = \begin{cases} 20 \cdot 3^5 & \text{if } k = 6, 7, \\ 40 \cdot 3^6 & \text{if } k = 9, 10, \\ 360 & \text{if } \ = 12, 15, 16. \end{cases}$$

Proof. For each n with $1 \leq n \leq N$ and $P(n) \leq 2k$, we check the validity of $\max\{P(n+(k-1)d), P(n+(k-2)d), P(n+(k-3)d)\} > 2k$ whenever $d \leq N - n$ and $p(d) > k$ if $k = 6, 7$ and $p(d) > 2k$ if $k \geq 9$. If $\max\{P(n+(k-1)d), P(n+(k-2)d), P(n+(k-3)d)\} \leq 2k$, then we check the validity of $\max\{P(n+d), P(n+2d)\} > 2k$. Then we find that either $\max\{P(n+d), P(n+2d)\} > 2k$ or

(3)
$(n, d) \in \{(33, 31), (64, 31)\}$ if $k = 12$ and $(n, d) \in \{(3, 31), (34, 31), (35, 43)\}$ if $k = 15$.

For (n, d, k) given by (3), we check that $P(\Delta(n, d, k)) > 2k$. $\square$

Let $n \geq 1, d > 2$ and $k \geq 3$. By Lemma 1, we may restrict to those k for which $2k-1$ is prime. For the exceptions (n, d, k) given in Lemma 5 and $(n, d, k) \in V$ given by Lemma 6, we check that $P(\Delta(n, d, k)) > 2k$. Therefore we assume that (n, d, k) is different from the exceptions in Lemma 5 and $(n, d, k) \notin V$. If $p(d) \leq k$ for $k = 6, 7$ and $p(d) \leq 2k$ for $k \geq 9$, then the assertion follows from Lemma 5 and Corollary 1, respectively. Thus we may suppose that $p(d) > k$ for $k = 6, 7$ and $p(d) > 2k$ for $k \geq 9$. Therefore the assumption of Lemma 8 is satisfied. We shall follow the assumptions stated in this paragraph throughout the paper. We split the proof of Theorem 1 for $k = 3$; $k = 4$; $k = 6, 7, 9, 10$; $k = 12, 15, 16$ and $k \geq 19$ with $2k-1$ prime in sections 3, 4, 5, 6 and 7, respectively.

3. The case $k = 3$

We assume that $P(n(n+d)(n+2d)) \leq 5$ and (n, d) is different from the exceptions given in Theorem 1. Let $5 \nmid \Delta$. Then either

$$n = 1, 1 + d = 2^\alpha, 1 + 2d = 3^\beta \text{ or } n = 2, 2 + d = 3^\beta, 2 + 2d = 2^\alpha.$$

Assume the first possibility. Then $2^{\alpha+1} - 3^\beta = 1$ implying $2^{\alpha+1} = 4, 3^\beta = 3$ by Lemma 2. Thus $d = 1$, a contradiction. Now we turn to the second. We get $3^\beta - 2^{\alpha-1} = 1$. Therefore either $3^\beta = 2, 2^{\alpha-1} = 2$ or $3^\beta = 9, 2^{\alpha-1} = 8$ by Lemma 2. The former is not possible since $d > 1$ and the latter implies that $d = 7$ which is excluded. Hence $5|\Delta$.

Suppose $3 \nmid \Delta$. We observe that $5 \nmid n$ since $\gcd(n+d, n+2d) = 1$. Let $5|n+2d$. Then $n = 1, 1+d = 2^\alpha, 1+2d = 5^\gamma$ implying $2^{\alpha+1} - 5^\gamma = 1$ which is not possible by Lemma 2. Let $5|n + d$. Then $n = 2^\eta, n + d = 5^\gamma, n + 2d = 2^\alpha$ implying $n = 2, 5^\gamma - 2^{\alpha-1} = 1$. Therefore by Lemma 2, we get $n = 2, d = 3$ which is excluded. Hence $3|\Delta$.

Let $15|n+id$ for some $i \in \{0, 1, 2\}$. We observe that $15 \nmid n$ since $\gcd(n+d, n+2d) = 1$. Let $15|n+d$. Then $n = 2, 2+d = 3^\beta 5^\gamma, 2+2d = 2^\alpha$ giving $2^{\alpha-1} - 3^\beta 5^\gamma = -1$ which is not possible by Lemma 4 (i). Let $15|n + 2d$. Then $n = 1, 1 + d = 2^\alpha, 1 + 2d = 3^\beta 5^\gamma$ giving $2^{\alpha+1} - 3^\beta 5^\gamma = 1$. Therefore by Lemma 4 (i), we get $n = 1, d = 7$ which is excluded. Thus $15 \nmid n + id$ for $i = 0, 1, 2$.

Suppose $2 \nmid \Delta$. Then

$$n = 1, 1 + d = 3^\beta, 1 + 2d = 5^\gamma \text{ or } n = 1, 1 + d = 5^\gamma, 1 + 2d = 3^\beta$$

which imply $5^\gamma - 2 \cdot 3^\beta = -1$ or $3^\beta - 2 \cdot 5^\gamma = -1$, respectively. Therefore $(n, d) = (1, 2)$ or $(1, 4)$ by Lemma 4. This is not possible. Hence $2|\Delta$.

Let $n = 1$. In view of the above conclusions in this section, we have

$$1 + d = 2^\alpha 3^\beta, 1 + 2d = 5^\gamma \text{ or } 1 + d = 2^\alpha 5^\gamma, 1 + 2d = 3^\beta$$

implying $5^\gamma - 2^{\alpha+1} \cdot 3^\beta = -1$ or $3^\beta - 2^{\alpha+1} \cdot 5^\gamma = -1$, respectively, contradicting Lemma 4 since $\alpha \geq 1$. Let $n = 2$. Then $2 + d = 3^\beta, 2 + 2d = 2^\alpha 5^\gamma$ or $2 + d = 5^\gamma, 2 + 2d = 2^\alpha 3^\beta$ implying $3^\beta - 2^{\alpha-1} \cdot 5^\gamma = 1$ or $5^\gamma - 2^{\alpha-1} \cdot 3^\beta = 1$, respectively. By Lemma 4, the first equation gives $d = 79$ and the second one gives $d = 23$ which are excluded. Thus $n > 2$. Now we have

$$\begin{aligned} & n = 2^\alpha, n+d = 3^\beta, n+2d = 2\cdot 5^\gamma \quad \text{or} \quad n = 2^\alpha, n+d = 5^\gamma, n+2d = 2\cdot 3^\beta \\ \text{or} \quad & n = 2\cdot 3^\beta, n+d = 5^\gamma, n+2d = 2^\alpha \quad \text{or} \quad n = 2\cdot 5^\gamma, n+d = 3^\beta, n+2d = 2^\alpha \\ \text{or} \quad & n = 3^\beta, n+d = 2^\alpha, n+2d = 5^\gamma \qquad \text{or} \quad n = 5^\gamma, n+d = 2^\alpha, n+2d = 3^\beta. \end{aligned}$$

By using the identity

$$n + (n+2d) - 2(n+d) = 0, \tag{4}$$

we see that the above relations imply equations of the form given by Lemma 3. Now we use Lemma 3 to find all the pairs (n, d) arising out of the solutions of these equation. Finally we observe that these pairs (n, d) are already excluded.

4. The case $k = 4$

We shall derive Theorem 1 with $k = 4$ from the case $k = 3$ and the following more general result. We put $\Delta_1 = n(n+2d)(n+3d)$ and $\Delta_2 = n(n+d)(n+3d)$. Let

$$S_1 = \{(1, 13), (3, 11), (4, 7), (6, 7), (6, 13), (18, 119), (30, 17)\}$$

and

$$\begin{aligned} S_2 = \{&(1, 3), (1, 5), (1, 8), (1, 53), (3, 2), (3, 5), (3, 17), \\ &(3, 29), (3, 47), (9, 7), (9, 247), (15, 49), (27, 23)\}. \end{aligned}$$

Lemma 9. *We have*

$$P(\Delta_1) \geq 7 \text{ unless } (n, d) \in S_1 \tag{5}$$

and

$$P(\Delta_2) \geq 7 \text{ unless } (n, d) \in S_2. \tag{6}$$

Proof. First we prove (5). Assume that $(n, d) \notin S_1$ and $P(\Delta_1) \leq 5$. Suppose $5 \nmid \Delta_1$. Then either

$$n = 1, 1+2d = 3^\beta, 1+3d = 2^\alpha \text{ or } n = 6, 6+2d = 2^\alpha, 6+3d = 3^\beta.$$

This is not possible by Lemma 2 since $d > 1$. Suppose $3 \nmid \Delta_1$. Then either $n = 1, 1+2d = 5^\gamma, 1+3d = 2^\alpha$ or $n = 2, 2+2d = 2^\alpha, 2+3d = 5^\gamma$. This is again not possible by Lemma 4 (i), (iii). Suppose $2 \nmid \Delta_1$. Then either $n = 1, 1+2d = 3^\beta, 1+3d = 5^\gamma$ or $n = 3, 3+2d = 5^\gamma, 3+3d = 3^\beta$. This is not valid by Lemma 4 (ii), (iii). Hence $2\cdot 3\cdot 5 \mid \Delta_1$.

Let $n = 1$. Then either $1+2d = 3^\beta 5^\gamma, 1+3d = 2^\alpha$ or $1+2d = 3^\beta, 1+3d = 2^\alpha 5^\gamma$. The first possibility is excluded by Lemma 4 (i) and second possibility implies $d = 13$ by Lemma 4 (ii). Let $n = 2$. Then $2+2d = 2^\alpha 3^\beta, 2+3d = 5^\gamma$ which is not possible by Lemma 4 (iii). Let $n = 3$. Then $3+2d = 5^\gamma, 3+3d = 2^\alpha 3^\beta$ implying $d = 11$ by Lemma 4 (iii). Let $n = 6$. Then either $6+2d = 2^\alpha 5^\gamma, 6+3d = 3^\beta$ or $6+2d = 2^\alpha, 6+3d = 3^\beta 5^\gamma$. The first possibility implies $d = 7$ by Lemma 4 (ii) and second implies $d = 13$ by Lemma 4 (i).

Let $n = 4, 5$ or $n > 6$. We observe that $n = 2^{\delta_1}5^{\gamma}$ with $\delta_1 \geq 1$ or $3^{\delta_2}5^{\gamma}$ with $\delta_2 \geq 1$ are not possible since otherwise $P(n+3d) > 5$ or $P(n+2d) > 5$, respectively. Let $n = 2^{\delta_1}3^{\delta_2}$ or $n = 2^{\delta_1}3^{\delta_2}5^{\gamma}$ with $\delta_1 \geq 1, \delta_2 \geq 1$. Then

$$\delta_1 = 1, n = 2 \cdot 3^{\beta}, n+2d = 2^{\alpha}, n+3d = 3 \cdot 5^{\gamma}$$
$$\text{or } \delta_2 = 1, n = 3 \cdot 2^{\alpha}, n+2d = 2 \cdot 5^{\gamma}, n+3d = 3^{\beta}.$$

if $n = 2^{\delta_1}3^{\delta_2}$ and

$$\delta_1 = 1, \delta_2 = 1, n = 6 \cdot 5^{\gamma}, n+2d = 2^{\alpha}, n+3d = 3^{\beta}$$

if $n = 2^{\delta_1}3^{\delta_2}5^{\gamma}$. Further

$$n+2d = 2 \cdot 3^{\beta}, n+3d = 5^{\gamma} \text{ if } n = 2^{\alpha}$$
$$n+2d = 5^{\gamma}, n+3d = 3 \cdot 2^{\alpha} \text{ if } n = 3^{\beta}$$
$$n+2d = 3^{\beta}, n+3d = 2^{\alpha} \text{ if } n = 5^{\gamma}.$$

This exhaust all the possibilities. For each of the above relations, we use the identity

$$n + 2(n+3d) - 3(n+2d) = 0 \tag{7}$$

to obtain an equation of the form given by Lemma 3. Finally we apply Lemma 3 as in the preceding section to conclude that $(n, d) \in S_1$, a contradiction.

The proof of (6) is similar to that of (5). Here we use the identity $2n + (n+3d) - 3(n+d) = 0$ in place of (7). $\square$

Now we turn to the proof of Theorem 1 for $k = 4$. We assume $P(\Delta) \leq 7$. In view of the case $k = 3$, we may assume that $7|n+d$ or $7|n+2d$. Thus $P(\Delta_1) \leq 5$ if $7|n+d$ and $P(\Delta_2) \leq 5$ if $7|n+2d$. Now we conclude from Lemma 9 that $(n, d) \in S_1$ if $7|n+d$ and $(n, d) \in S_2$ if $7|n+2d$. Finally we check that $P(\Delta) \geq 11$ for $(n, d) \in S_1 \cup S_2$ unless $(n, d) \in \{(1, 3), (1, 13), (3, 11)\}$.

5. The cases $k = 6, 7, 9, 10$

We assume $P(\Delta) \leq 2k$. Further by Lemma 8, we may assume that

$$n + d > \begin{cases} 20 \cdot 3^5 & \text{if } k = 6, 7, \\ 40 \cdot 3^6 & \text{if } k = 9, 10. \end{cases} \tag{8}$$

There are at most $1 + [\frac{k-1}{p}]$ terms in Δ divisible by a prime p. After removing all the terms in Δ divisible by $p \geq 7$, we are left with at least 4 terms divisible by $2, 3$ and 5 only. After deleting the terms in which $2, 3, 5$ appear to maximal power, we are left with a term $n + i_0 d$ with $0 \leq i_0 < k$ such that $P(n + i_0 d) \leq 5$ and $n + i_0 d$ is at most $4 \cdot 3 \cdot 5$ if $k = 6, 7$; $8 \cdot 3 \cdot 5$ if $k = 9$ and $8 \cdot 9 \cdot 5$ if $k = 10$. If $i_0 > 0$, we get $n + d \leq 360$ contradicting (8). Thus we may suppose that $i_0 = 0$ and the terms in which $2, 3, 5$ appear to maximal power are different. Let $n + i_2 d$ and $n + i_3 d$ be the terms in which 2 and 3 appear to maximal power, respectively. Since 5 can divide at most 2 terms, we see that 5 can divide at most one of $n+i_2d$ and $n+i_3d$. Also $5\nmid n$ if $5|(n+i_2d)(n+i_3d)$. We write

$$n+i_2d=2^{\alpha_2}3^{\beta_2}5^{\gamma_2}, n+i_3d=2^{\alpha_3}3^{\beta_3}5^{\gamma_3} \tag{9}$$

with $(\gamma_2,\gamma_3)\in\{(0,0),(1,0),(0,1)\}$. We observe that α_3 is at most 2 and 3 if $k=6,7$ and $k=9,10$, respectively, and β_2 is at most 1 and 2 if $k=6,7,9$ and $k=10$, respectively. If $k=6,7$, then $\alpha_2\geq 7$ otherwise $n+d\leq n+i_2d\leq 2^6\cdot 3\cdot 5$ contradicting (8). Similarly we derive $\beta_3\geq 6$ if $k=6,7$ and $\alpha_2\geq 8, \beta_3\geq 7$ if $k=9,10$. From $i_3(n+i_2d)-i_2(n+i_3d)=(i_3-i_2)n$, we get

$$i_32^{\alpha_2}3^{\beta_2}5^{\gamma_2}-i_22^{\alpha_3}3^{\beta_3}5^{\gamma_3}=(i_3-i_2)n \tag{10}$$

Let

$$\alpha=\text{ord}_2\left(\frac{i_32^{\alpha_2}}{i_22^{\alpha_3}}\right),\ \beta=\text{ord}_3\left(\frac{i_23^{\beta_3}}{i_33^{\beta_2}}\right). \tag{11}$$

We show that $\alpha\geq\alpha_2-\delta$ where $\delta=2$ if $k=6,7$ and $\delta=3$ if $k=9,10$. It suffices to prove $\text{ord}_2(\frac{i_3}{i_22^{\alpha_3}})\geq-\delta$. If $\text{ord}_2(i_3)\geq\text{ord}_2(i_2)$, then it is clear. Thus we may assume that $\text{ord}_2(i_3)<\text{ord}_2(i_2)$. From (9), we get $(i_2-i_3)d=2^{\alpha_3}(2^{\alpha_2-\alpha_3}O_2-O_3)$ with O_2,O_3 odd. Therefore $\alpha_3=\text{ord}_2(i_2-i_3)$ since $\alpha_2>\alpha_3$. Thus $\text{ord}_2(i_3)=\alpha_3$. Since $i_2<k$, we get the desired inequality $\text{ord}_2(\frac{i_3}{i_22^{\alpha_3}})\geq-\delta$. Hence $\alpha\geq\alpha_2-\delta\geq 5$. Similarly we derive $\beta\geq 5$.

We obtain from (10) the equation

$$i2^\alpha-j3^\beta=t \tag{12}$$

with

$$\alpha\geq 5,\ \beta\geq 5, \tag{13}$$

$i,j\in\{1,5,7,25,35\}, t\in\{\pm1,\pm5,\pm7,\pm25,\pm35\}$ and $\gcd(i,j)=\gcd(i,t)=\gcd(j,t)=1$. From Lemmas 2, 3 and 4, we see that equations of the form

$$\begin{aligned}
&2^\alpha-3^\beta=\pm1, && 2^\alpha-3^\beta=\pm5,\pm25,\\
&2^\alpha-5\cdot3^\beta=\pm1, && 5\cdot2^\alpha-3^\beta=\pm1,\\
&2^\alpha-25\cdot3^\beta=\pm1, && 25\cdot2^\alpha-3^\beta=\pm1
\end{aligned}$$

are not possible by (13). Let the equations given by (12) be different from the above. Each of the equation gives rise to a Thue equality

$$X^3+AY^3=B \tag{14}$$

with integers $X,Y,A>0,B>0$ given by

	Equation	A	B	X	Y
(i)	$2^\alpha - 3^\beta = \pm 7$	$2^{a'}3^{b'}$	$7 \cdot 2^{a'}$	$\pm 2^{\frac{\alpha+a'}{3}}$	$\pm 3^{\frac{\beta-b'}{3}}$
(ii)	$7 \cdot 2^\alpha - 3^\beta = \pm 1, \pm 5, \pm 25$	$7 \cdot 2^{a'}3^{b'}$	$3^{b'}, 5 \cdot 3^{b'}, 25 \cdot 3^{b'}$	$\pm 3^{\frac{\beta+b'}{3}}$	$\pm 2^{\frac{\alpha-a'}{3}}$
(iii)	$2^\alpha - 7 \cdot 3^\beta = \pm 1, \pm 5, \pm 25$	$7 \cdot 2^{a'}3^{b'}$	$2^{a'}, 5 \cdot 2^{a'}, 25 \cdot 2^{a'}$	$\pm 2^{\frac{\alpha+a'}{3}}$	$\pm 3^{\frac{\beta-b'}{3}}$
(iv)	$25 \cdot 2^\alpha - 3^\beta = \pm 7$	$5 \cdot 2^{a'}3^{b'}$	$35 \cdot 2^{a'}$	$\pm 5 \cdot 2^{\frac{\alpha+a'}{3}}$	$\pm 3^{\frac{\beta-b'}{3}}$
(v)	$2^\alpha - 25 \cdot 3^\beta = \pm 7$	$5 \cdot 2^{a'}3^{b'}$	$35 \cdot 3^{b'}$	$\pm 5 \cdot 3^{\frac{\beta+b'}{3}}$	$\pm 2^{\frac{\alpha-a'}{3}}$
(vi)	$5 \cdot 2^\alpha - 7 \cdot 3^\beta = \pm 1$	$25 \cdot 7 \cdot 2^{a'}3^{b'}$	$25 \cdot 2^{a'}$	$\pm 5 \cdot 2^{\frac{\alpha+a'}{3}}$	$\pm 3^{\frac{\beta-b'}{3}}$
(vii)	$7 \cdot 2^\alpha - 5 \cdot 3^\beta = \pm 1$	$25 \cdot 7 \cdot 2^{a'}3^{b'}$	$25 \cdot 3^{b'}$	$\pm 5 \cdot 3^{\frac{\beta+b'}{3}}$	$\pm 2^{\frac{\alpha-a'}{3}}$
$(viii)$	$2^\alpha - 5 \cdot 3^\beta = \pm 7$	$5 \cdot 2^{a'}3^{b'}$	$7 \cdot 2^{a'}$	$\pm 2^{\frac{\alpha+a'}{3}}$	$\pm 3^{\frac{\beta-b'}{3}}$
(ix)	$5 \cdot 2^\alpha - 3^\beta = \pm 7$	$5 \cdot 2^{a'}3^{b'}$	$7 \cdot 3^{b'}$	$\pm 3^{\frac{\beta+b'}{3}}$	$\pm 2^{\frac{\alpha-a'}{3}}$
(x)	$35 \cdot 2^\alpha - 3^\beta = \pm 1$	$35 \cdot 2^{a'}3^{b'}$	$3^{b'}$	$\pm 3^{\frac{\beta+b'}{3}}$	$\pm 2^{\frac{\alpha-a'}{3}}$
(xi)	$2^\alpha - 35 \cdot 3^\beta = \pm 1$	$35 \cdot 2^{a'}3^{b'}$	$2^{a'}$	$\pm 2^{\frac{\alpha+a'}{3}}$	$\pm 3^{\frac{\beta-b'}{3}}$
(xii)	$2^\alpha - 3^\beta = \pm 35$	$2^{a'}3^{b'}$	$35 \cdot 2^{a'}$	$\pm 2^{\frac{\alpha+a'}{3}}$	$\pm 3^{\frac{\beta-b'}{3}}$
$(xiii)$	$7 \cdot 2^\alpha - 25 \cdot 3^\beta = \pm 1$	$5 \cdot 7 \cdot 2^{a'}3^{b'}$	$5 \cdot 3^{b'}$	$\pm 5 \cdot 3^{\frac{\beta+b'}{3}}$	$\pm 2^{\frac{\alpha-a'}{3}}$
(xiv)	$25 \cdot 2^\alpha - 7 \cdot 3^\beta = \pm 1$	$5 \cdot 7 \cdot 2^{a'}3^{b'}$	$5 \cdot 2^{a'}$	$\pm 5 \cdot 2^{\frac{\alpha+a'}{3}}$	$\pm 3^{\frac{\beta-b'}{3}}$

where $0 \le a', b' < 3$ are such that X, Y are integers. Further

$$\max\{\mathrm{ord}_2(X), \mathrm{ord}_3(X)\} \ge 2, \ \max\{\mathrm{ord}_2(Y), \mathrm{ord}_3(Y)\} \ge 1 \tag{15}$$

by (13). Using Magma, we compute all the solutions in integers X, Y of the above Thue equations. We find that all the solutions of Thue equations other than (ii) and $(viii)$ do not satisfy (15). Further we check that the solutions of (ii) and $(viii)$ satisfy (15) but they do not satisfy (13).

6. The cases $k = 12, 15, 16$

We assume $P(\Delta) \le 2k$. Let $k = 12, 15$. Then $P((n+d)\cdots(n+(k-1)d)) \le 2k$. After deleting the terms from $\{n+d, \cdots, n+(k-1)d\}$ divisible by primes p with $7 \le p \le 2k$, we get at least 4 terms $n + id$ composed of $2, 3$ and 5 only. This is also the case when $k = 16$ since 7 and 13 together divide at most 4 terms. Therefore there exists an i with $1 \le i \le k-1$ such that $n + id$ divides $8 \cdot 9 \cdot 5$. Thus $n + d \le 360$. Now the assertion follows from Lemma 8.

7. The case $k \ge 19$ with $2k - 1$ prime

It suffices to prove $W(\Delta) \ge \pi(2k) - \pi(k) + 1$ since $\pi(k) = \pi_d(k)$ by our assumption. We may suppose that $W(\Delta) = \pi(2k) - \pi(k)$ by Lemma 6.

We observe that $d > 2k$ since $p(d) > 2k$. We follow the proof of Lemma 6. Taking $R = \pi(2k) - \pi(k)$, we apply the fundamental inequality of Sylvester and Erdős [6, Lemma 1, (14)] to conclude that

$$d^{k-\pi(2k)-1} \leq (k-2)\cdots(k-\pi(2k)) \tag{16}$$

and hence

$$2k < d < (k-2)^{\frac{\pi(2k)-1}{k-\pi(2k)-1}}. \tag{17}$$

Using Lemma 7, we see that

$$k - 2\pi(2k) \geq \frac{k}{\log 2k}\left(\log 2k - 4(1 + \frac{1.2762}{\log 2k})\right) \geq 0$$

for $k \geq 76$. With exact values of π function, we see that $k \geq 2\pi(2k)$ for $60 \leq k < 76$. This implies $\pi(2k) - 1 \leq k - \pi(2k) - 1$ for $k \geq 60$. Therefore for $k \geq 60$, we see that (17) does not hold. Thus $k < 60$. From (16), we see that $d \leq 2k$ for $k \geq 30, k \neq 31$. Thus it remains to consider $k = 19, 21, 22, 24, 27, 31$. We see that $d \leq 71$ if $k = 27, 31$; $d \leq 83$ if $k = 19, 21$ and $d \leq 113$ if $k = 22, 24$.

The next argument is analogous to [6, (41), (42)] where $k - \pi(2k) + 1$ has been replaced by $k - \pi(2k)$. Let n_e, d_e, n_o and d_o be positive integers with n_e even and n_o odd. For (n, d, k) with n even, $n \geq n_e, d \leq d_e$, we have

(18)
$$d^{k-\pi(2k)-1}\prod_{i=1}^{A_e-1}\left(\frac{n_e}{2d_e}+i\right)\prod_{j=1}^{k-\pi(2k)-A_e}\left(\frac{n_e}{d_e}+2j-1\right) \leq \min\left(1, \frac{k-1}{n_e}2^{-\theta+1}\right)(k-2)!$$
$$\times\, 2^{\mathrm{ord}_2([\frac{k-2}{2}]!)-\mathrm{ord}_2((k-2)!)}$$

where A_e =min$(k - \pi(2k), \lceil\frac{2}{3}(k - \pi(2k)) + \frac{n_e}{6d_e} - \frac{1}{3}\rceil)$, $\theta = 1$ if k is odd, 0 otherwise. For (n, d, k) with n odd, $n \geq n_o, d \leq d_o$, we have

(19)
$$d^{k-\pi(2k)-1}\prod_{i=1}^{A_o}\left(\frac{n_o}{2d_o}+i-\frac{1}{2}\right)\prod_{j=1}^{k-\pi(2k)-A_o-1}\left(\frac{n_o}{d_o}+2j\right) \leq \min\left(1, \frac{k-1}{n_o}\right)(k-2)!$$
$$\times\, 2^{\mathrm{ord}_2([\frac{k-2}{2}]!)-\mathrm{ord}_2((k-2)!)}$$

where A_o =min$(k - \pi(2k), \lceil\frac{2}{3}(k - \pi(2k)) + \frac{n_o}{6d_o} - \frac{5}{6}\rceil)$. Here we have used $k - \pi(2k) \leq [\frac{k-2}{2}]$ for the expressions given by A_e and A_o. We take $n_e = 2, n_o = 1, d_e = d_o = 83$ if $k = 19, 21, 27, 31$ and $n_e = 2, n_o = 1, d_e = d_o = 113$ if $k = 22, 24$. We get a contradiction for $k = 27, 31$ since $d > 2k$. Thus we may assume that $k \in \{19, 21, 22, 24\}$. We obtain $d \leq D_e$ if n is even where $D_e = 47, 47, 67$ and 61 according as $k = 19, 21, 22$ and 24, respectively. If n is odd, then $d \leq D_o$ where $D_o = 53, 47, 71$ and 67 according as $k = 19, 21, 22$ and 24, respectively. By taking $n_e = 4k, d_e = D_e$ and $n_o = 4k + 1, d_o = D_o$, we derive from (18) and (19) that $d < 2k$. This is a contradiction. Thus $n < 4k$. For these values of n, d and k, we check that $P(\Delta(n, d, k)) > 2k$ is valid. This completes the proof.

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