SQUARES IN PRODUCTS IN ARITHMETIC PROGRESSION WITH AT MOST TWO TERMS OMITTED AND COMMON DIFFERENCE A PRIME POWER

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Abstract. It is shown that a product of $k - 2$ terms out of $k \geq 15$ terms in arithmetic progression with common difference a prime power $> 1$ is not a square. In fact it is not of the form $by^2$ where the greatest prime factor of $b$ is less than $k$.

1. Introduction

For an integer $x > 1$, we denote by $P(x)$ and $\omega(x)$ the greatest prime factor of $x$ and the number of distinct prime divisors of $x$, respectively. Further we put $P(1) = 1$ and $\omega(1) = 0$. Let $p_i$ be the $i$-th prime number. Let $k \geq 4$, $t \geq k - 2$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_t$ be integers with $0 \leq \gamma_i < k$ for $1 \leq i \leq t$. Thus $t \in \{k, k - 1, k - 2\}$, $\gamma_t \geq k - 3$ and $\gamma_i = i - 1$ for $1 \leq i \leq t$ if $t = k$. We put $\psi = k - t$. Let $b$ be a positive squarefree integer and we shall always assume, unless otherwise specified, that $P(b) \leq k$. We consider the equation

\begin{equation}
\Delta = \Delta(n, d, k) = (n + \gamma_1d) \cdots (n + \gamma_t d) = by^2
\end{equation}

in positive integers $n, d, k, b, y, t$. We prove

Theorem 1. Let $\psi = 2, k \geq 15$ and $d \nmid n$. Assume that $P(b) < k$ if $k = 17, 19$. Then (1.1) with $\omega(d) = 1$ does not hold.

From Theorem 1, we obtain the following results immediately.

Corollary 1. Let $\psi = 1, k \geq 15$ and $d \nmid n$. Then (1.1) with $\omega(d) = 1$ does not hold.

Corollary 2. Let $\psi = 0, k \geq 15$ and $d \nmid n$. Assume that $P(b) \leq p_{\pi(k)+1}$ if $k = 17, 19$ and $P(b) \leq p_{\pi(k)+2}$ otherwise. Then (1.1) with $\omega(d) = 1$ does not hold.

For the proof of Corollary 1, we may suppose $P(b) = k$ otherwise it follows from (2.1) and Theorem 1. Then we delete the term divisible by $k$ on the left hand side of (1.1) and the assertion follows from Theorem 1. Further Corollary 2 also follow similarly from Theorem 1.

Let $\psi = 0$. If $d = 1$, then (1.1) has been completely solved for $P(b) < k$ by Erdős and Selfridge [ErSe75] and for $P(b) = k$ by Saradha [Sar97]. Let $d > 1$. We observe that (1.1) has infinitely many solutions if $k = 2, 3$ and $b = 1$. Also (1.1) with $k = 4$ and $b = 6$ has infinitely many solutions. It has been conjectured that (1.1) with $\gcd(n, d) = 1$ and $k \geq 5$ does not hold. Let $\omega(d) = 1$. It has been shown in [SaSh03a] for $k > 29$ and [MuSh03] for $4 \leq k \leq 29$ that (1.1) with $\gcd(n, d) = 1$ implies that either $k = 4, (n, d, b, y) = (75, 23, 6, 140)$ or

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k = 5, \( P(b) = k \). In fact we shall derive the preceding result with \( k \geq 10 \) and \( P(b) < k \) from Theorem 1, see Corollary 3.11. We refer to [LaSh06a] for results on (1.1) with \( 1 < \omega(d) \leq 4 \).

Let \( \psi = 1 \). We may assume that \( \gamma_1 = 0 \) and \( \gamma_t = k - 1 \). It has been shown in [SaSh03b] that

\[
\frac{6!}{5} = (12)^2, \quad \frac{10!}{7} = (720)^2
\]

are the only squares that are products of \( k - 1 \) distinct integers out of \( k \) consecutive integers confirming a conjecture of Erdős and Selfridge [ErSe75]. This corresponds to the case \( b = 1 \) and \( d = 1 \) in (1.1). In general, it has been proved in [SaSh03b] that (1.1) with \( d = 1 \) and \( k \geq 4 \) implies that \((b, k, n) = (2, 4, 24)\) under the necessary assumption that the left hand side of (1.1) is divisible by a prime \( > k \). Further it has been shown in [SaSh03a, Theorem 4] and [MuSh04a] that (1.1) with \( d > 1 \), \( \gcd(n, d) = 1 \), \( \omega(d) = 1 \) and \( P(b) < k \) implies that \( k \leq 8 \). Thus we derive the preceding result with \( k \geq 15 \) from Corollary 1. Further the assumption \( P(b) < k \) has been relaxed to \( P(b) \leq k \) and the assumption \( \gcd(n, d) = 1 \) has been replaced by \( d \nmid n \).

Let \( \psi = 2 \). Let \( d = 1 \). Then it has been shown in [MuSh04b, Corollary 3] that a product of \( k - 2 \) distinct terms out of \( k \) consecutive positive integers is a square only if it is given by

\[
\frac{6!}{1.5} = \frac{7!}{5.7} = 12^2, \quad \frac{10!}{1.7} = \frac{11!}{7.11} = 720^2
\]

and

\[
\begin{align*}
\frac{4!}{2.3} &= 2^2, \quad \frac{6!}{4.5} = 6^2, \quad \frac{8!}{2.5.7} = 24^2, \quad \frac{10!}{2.3.4.6.7} = 60^2, \quad \frac{9!}{2.5.7} = 72^2, \\
\frac{10!}{2.3.6.7} &= 120^2, \quad \frac{10!}{7.8} = 180^2, \quad \frac{10!}{7.9} = 240^2, \quad \frac{10!}{47} = 360^2, \\
\frac{21!}{13.17.19} &= 5040^2, \quad \frac{14!}{2.3.4.11.13} = 5040^2, \quad \frac{14!}{2.3.11.13} = 10080^2.
\end{align*}
\]

The above result corresponds to (1.1) with \( b = 1 \). For the general case, we have

**Theorem 2.** Let \( \psi = 2 \), \( d = 1 \) and \( k \geq 6 \). Assume that the left hand side of (1.1) is divisible by a prime \( > k \). Then (1.1) is not valid unless \( k = 6 \) and \( n = 45, 240 \).

We observe that \( n > k^2 \) since the left hand side of (1.1) is divisible by a prime \( > k \). Then the assertion follows immediately from [MuSh04b, Theorem 2]. Therefore we take \( d > 1 \) from now onwards in this paper. For the proof of Theorem 1, we show without loss of generality that \( \gcd(n, d) = 1 \). Let \( \gcd(n, d) > 1 \). Let \( p^\beta = \gcd(n, d) \), \( n' = \frac{n}{p^\beta} \) and \( d' = \frac{d}{p^\beta} \). Then \( d' > 1 \) since \( d \nmid n \). Now, by dividing \((p^\beta)^\epsilon\) on both sides of (1.1), we have

\[
(n' + \gamma_1 d') \cdots (n' + \gamma_t d') = p^\epsilon b'y^2
\]

where \( y' > 0 \) is an integer, \( b' \) squarefree, \( P(b') < k \) when \( k = 17 \) and \( \epsilon \in \{0, 1\} \). Since \( p \nmid d' \) and \( \gcd(n', d') = 1 \), we see that \( p \nmid (n' + \gamma_1 d') \cdots (n' + \gamma_t d') \) giving \( \epsilon = 0 \) and assertion follows.

2. Notations and Preliminaries

We assume (1.1) with \( \gcd(n, d) = 1 \) in this section. Then we have

\[
n + \gamma_i d = a_n x_n^2 \quad \text{for } 1 \leq i \leq t
\]
with \(a_{\gamma_i}\) squarefree such that \(P(a_{\gamma_i}) \leq \max(k-1, P(b))\). Thus (1.1) with \(b\) as the squarefree part of \(a_{\gamma_1} \cdots a_{\gamma_t}\) is determined by the \(t\)-tuple \((a_{\gamma_1}, \ldots, a_{\gamma_t})\). Also

\[
(2.2) \quad n + \gamma_id = A_{\gamma_i}X_{\gamma_i}^2 \quad \text{for } 1 \leq i \leq t
\]

with \(P(A_{\gamma_i}) \leq k\) and \(\gcd(X_{\gamma_i}, \prod_{p \leq k} p) = 1\). Further we write

\[
b_i = a_{\gamma_i}, \quad B_i = A_{\gamma_i}, \quad y_i = x_{\gamma_i}, \quad Y_i = X_{\gamma_i}.
\]

Since \(\gcd(n, d) = 1\), we see from (2.1) and (2.2) that

\[
(2.3) \quad (b_1, d) = (B_1, d) = (y_1, d) = (Y_1, d) = 1 \quad \text{for } 1 \leq i \leq t.
\]

Let

\[
R = \{b_i : 1 \leq i \leq t\}.
\]

For \(b_{i_0} \in R\), let \(\nu(b_{i_0}) = |\{j : 1 \leq j \leq t, b_j = b_{i_0}\}|.\) Let

\[
T = \{1 \leq i \leq t : Y_i = 1\}, \quad T_1 = \{1 \leq i \leq t : Y_i > 1\}, \quad S_1 = \{B_i : i \in T_1\}.
\]

Note that \(Y_i > k\) for \(i \in T_1\). For \(i_0 \in T_1\), we denote by \(\nu(B_{i_0}) = |\{j \in T_1 : B_j = B_{i_0}\}|.\)

Let

\[
(2.4) \quad \delta = \min(3, \ord_2(d)), \quad \delta' = \min(1, \ord_2(d)),
\]

\[
(2.5) \quad \eta = \begin{cases} 1 & \text{if } \ord_2(d) \leq 1, \\ 2 & \text{if } \ord_2(d) \geq 2, \end{cases}
\]

\[
(2.6) \quad \rho = \begin{cases} 3 & \text{if } 3 \mid d, \\ 1 & \text{if } 3 \nmid d. \end{cases}
\]

and

\[
(2.7) \quad \theta = \begin{cases} 1 & \text{if } d = 2, 4 \\ 0 & \text{otherwise.} \end{cases}
\]

Let \(d = p^\alpha\). Then we say \((d_1, d_2)\) is a partition of \(d\) if \(d = d_1d_2\) and \(\gcd(d_1, d_2) = \eta\) and we take \((1, 2)\) as the partition of \(d = 2\). Further (2.2) is the only partition if \(d = 4\). For \(d \neq 2, 4\), we see that \((\eta, \frac{d}{\eta})\) and \((\frac{d}{\eta}, \eta)\) are the only distinct partitions of \(d\). Let \(b_i = b_j, i > j\).

Then from (2.1) and (2.3), we have

\[
(2.8) \quad \frac{(\gamma_i - \gamma_j)}{b_i} = \frac{y_i^2 - y_j^2}{d} = \frac{(y_i - y_j)(y_i + y_j)}{d}
\]

such that \(\gcd(d, y_i - y_j, y_i + y_j) = 2^{\delta'}\). Thus a pair \((i, j)\) with \(i > j\) and \(b_i = b_j\) corresponds to a partition \((d_1, d_2)\) of \(d\) such that \(d_1|(y_i - y_j)\) and \(d_2|(y_i + y_j)\) and this partition is unique. Similarly, we have unique partition of \(d\) corresponding to every pair \((i, j)\) with \(i > j, i, j \in T_1\) and \(B_i = B_j\).
Let \( q \) be a prime \( \leq k \) and coprime to \( d \). Then the number of \( i \)'s for which \( b_i \) are divisible by \( q \) is at most \( \sigma_q = \left\lceil \frac{k}{q} \right\rceil \). Let \( \sigma_q' = |\{b_i : q|b_i\}| \). Then \( \sigma_q' \leq \sigma_q \). Let \( r \geq 3 \) be any positive integer. Define \( F(k, r) \) and \( F'(k, r) \) as

\[
F(k, r) = |\{\gamma_i : P(b_i) > p_r\}| \quad \text{and} \quad F'(k, r) = \sum_{i=r+1}^{\pi(k)} \sigma_{p_i}.
\]

Then \( |\{b_i : P(b_i) > p_r\}| \leq F(k, r) \leq F'(k, r) - \sum_{p|d,p>p_r} \sigma_p \). Let

\[
B_r = \{b_i : P(b_i) \leq p_r\}, \quad I_r = \{\gamma_i : b_i \in B_r\} \quad \text{and} \quad \xi_r = |I_r|.
\]

We have
\[
(2.9) \quad \xi_r \geq t - F(k, r) \geq t - F'(k, r) + \sum_{p|d,p>p_r} \sigma_p
\]
and
\[
(2.10) \quad t - |R| \geq t - |\{b_i : P(b_i) > p_r\}| - |\{b_i : P(b_i) \leq p_r\}|
\]
\[
(2.11) \quad \geq t - F(k, r) - |\{b_i : P(b_i) \leq p_r\}|
\]
\[
(2.12) \quad \geq t - F'(k, r) + \sum_{p|d,p>p_r} \sigma_p - |\{b_i : P(b_i) \leq p_r\}|
\]
\[
(2.13) \quad \geq t - F'(k, r) + \sum_{p|d,p>p_r} \sigma_p - 2^r.
\]

We write \( S := S(r) \) for the set of positive squarefree integers composed of primes \( \leq p_r \). Let \( p = 2^\delta \) if \( d \) is even and \( p = P(d) \) if \( d \) is odd. Let \( p = 2^\delta \). Then \( b_i \equiv n(\mod 2^\delta) \). Considering modulo \( 2^\delta \) for elements of \( S(r) \), we see by induction on \( r \) that

\[
(2.14) \quad |\{b_i : P(b_i) \leq p_r\}| \leq 2^{r-\delta} =: g_{2^\delta}.
\]

Let \( p = P(d) \). Then all \( b_i \)'s are either quadratic residues \( \mod p \) or non-quadratic residues \( \mod p \). We consider two sets

\[
S_1(p, r) = \{s \in S : \left( \frac{s}{p} \right) = 1\},
\]
\[
S_2(p, r) = \{s \in S : \left( \frac{s}{p} \right) = -1\}
\]
and define

\[
(2.15) \quad g_p(r) = \max(|S_1(p, r)|, |S_2(p, r)|).
\]

Then

\[
(2.16) \quad |\{b_i : P(b_i) \leq p_r\}| \leq g_p.
\]

In view of (2.14) and (2.17), the inequality (2.12) is improved as

\[
(2.17) \quad t - |R| \geq k - \psi - F'(k, r) + \sum_{p|d,p>p_r} \sigma_p - g_p.
\]

(2.18)
Let $r = 3, 4, 2 < p \leq 220$. Then we calculate

\[
g_p(r) = \begin{cases} 
2^{r-2} & \text{if } p \leq p_r \\
2^{r-1} & \text{if } p > p_r 
\end{cases}
\]

except when $r = 3, p \in \{71, 191\}$ where $g_p = 2^r$. We close this section with the following Lemmas which are independent of (1.1). The first Lemma is an estimate on $\pi(x)$ due to Dusart [Dus99].

**Lemma 2.1.** We have

\[
\pi(x) \leq \frac{x}{\log x} \left(1 + \frac{1.2762}{\log x}\right) \text{ for } x > 1.
\]

The following lemma is contained in [LaSh04, Theorem 1].

**Lemma 2.2.** Let $k \geq 9$, $\gcd(n, d) = 1$ if $d = 2$ and $(n, d, k) \notin V$ where $V$ is given by

\[
\begin{cases} 
n = 1, d = 3, k = 9, 10, 11, 12, 19, 22, 24, 31; \\
n = 2, d = 3, k = 12; n = 4, d = 3, k = 9, 10; \\
n = 2, d = 5, k = 9, 10; \\
n = 1, d = 7, k = 10.
\end{cases}
\]

Then

\[
W(n(n + d) \cdots (n + (k - 1)d)) := |\{i : 0 \leq i < k, P(n + id) > k\}| \geq \pi(2k) - \pi_d(k).
\]

Let $d = 2$ and $n \leq k$. Then

\[
W(n(n + d) \cdots (n + (k - 1)d)) \geq \pi(2k) - \pi_d(k) - 1.
\]

The following lemma is contained in [Lai06, Lemma 8].

**Lemma 2.3.** Let $s_i$ denote the $i$-th squarefree positive integer. Then

\[
\prod_{i=1}^{l} s_i \geq (1.6)^l l! \text{ for } l \geq 286.
\]

3. **Lemmas for the equation (1.1)**

All the lemmas in this section are under the assumption that (1.1) with $\omega(d) = 1$ is valid and we shall suppose it without reference.

**Lemma 3.1.** Let $\psi$ be fixed. Suppose that (1.1) with $P(b) \leq k$ has no solution at $k = k_1$ with $k_1$ prime. Then (1.1) with $P(b) \leq k$ and $k_1 \leq k < k_2$ has no solution where $k_1, k_2$ are consecutive primes.

**Proof.** Let $k_1, k_2$ be consecutive primes such that $k_1 \leq k < k_2$. Suppose $(n, d, b, y)$ is a solution of

\[
(n + \gamma_1 d) \cdots (n + \gamma_l d) = by^2
\]

with $P(b) \leq k$. Then $P(b) \leq k_1$. We observe that $\gamma_{k_1 - \psi} < k_1$ and by (2.1),

\[
(n + \gamma_1 d) \cdots (n + \gamma_{k_1 - \psi} d) = b'y^2
\]

holds for some $b'$ with $P(b') \leq k_1$ giving a solution of (1.1) at $k = k_1$. This is a contradiction. \qed
In view of Lemma 3.1, there is no loss of generality in assuming that \( k \) is prime whenever \( k \geq 23 \) in the proof of Theorem 1. Therefore we suppose from now onward without reference that \( k \) is prime if \( k \geq 23 \). The following Lemma gives a lower bound for \( |T_1| \), see [LaSh06a, Lemma 4.1].

**Lemma 3.2.** Let \( k \geq 4 \). Then
\[
|T_1| > t - \frac{(k - 1) \log (k - 1) - \sum_{p|d, p < k} \max \left( 0, \frac{(k-1-p)\log p}{p-1} - \log (k-2) \right)}{\log (n + (k-1)d)} - \pi_d(k) - 1.
\]

We apply Lemmas 2.2 and 3.2 to derive the following result.

**Corollary 3.3.** Let \( k \geq 9 \). Then we have
\[
|T_1| > 0.1754k \quad \text{for} \quad k \geq 81.
\]
and
\[
n + \gamma d > \eta^2 k^2.
\]

**Proof.** We observe that \( \pi(2k) - \pi(k) > 2 \) since \( k \geq 9 \). Therefore \( P(\Delta) > k \) by Lemma 2.2. Now we see from (1.1) that
\[
n + \gamma d > k^2.
\]
By (3.1), \( t \geq k - 2, \pi_d(k) \leq \pi(k) \) and Lemma 2.1, we get
\[
|T_1| > k - 3 - \frac{(k - 1) \log k}{2 \log k} - \frac{k}{\log k} \left( 1 + \frac{1.2762}{\log k} \right).
\]
Since the right hand side of the above inequality exceeds 0.1754k for \( k \geq 81 \), the assertion (3.2) follows.

Now we turn to the proof of (3.3). By (3.4), it suffices to consider \( d = 2^\alpha \) with \( \alpha > 1 \). From Lemma 2.2 and (1.1), we have \( n + (k-1)d > p_{\pi(2k)}^2 \). Now we see from (3.1) that
\[
|T_1| + \pi_d(k) - \pi(2k) > k - 3 - \frac{(k - 1) \log (k-1) - (k-3) \log 2 + \log (k-2)}{2 \log p_{\pi(2k)}^2} - \pi(2k)
\]
and
\[
|T_1| + \pi_d(k) - \pi(2k) > k - 3 - \frac{(k - 1) \log k - (k-3) \log 2 + \log k}{2 \log k} - \frac{2k}{\log 2k} \left( 1 + \frac{1.2762}{\log 2k} \right)
\]
by Lemma 2.1. When \( k \geq 60 \), we observe that the right hand side of the preceding inequality is positive. Therefore \( |T_1| + \pi_d(k) > \pi(2k) \) implying \( n + \gamma d > 4k^2 \) for \( k \geq 60 \). Thus we may assume \( k < 60 \). Now we check that the right hand side of (3.5) is positive for \( k \geq 33 \). Therefore we may suppose that \( k < 33 \) and \( n + (k-3)d \leq n + \gamma d \leq 4k^2 \). Hence \( d = 2^\alpha < \frac{4k^2}{k-3} \).

For \( n, d, k \) satisfying \( k < 33, d < \frac{4k^2}{k-3}, n + (k-3)d \leq 4k^2 \) and \( n + (k-1)d \geq p_{\pi(2k)}^2 \), we check that there are at least three \( i \) with \( 0 \leq i < k \) such that \( n + id \) is divisible by a prime \( > k \) to the first power. This is not possible. \( \square \)

The next Lemma follows from (3.3) and [LaSh06a, Corollaries 3.5, 3.7].
Lemma 3.4. For any pair \((i, j)\) with \(b_i = b_j\), the partition \((\eta^{-1}, \eta)\) of \(d\) is not possible. Further \(\nu(b_{i0}) \leq 2^{1-\theta}\) and \(\nu(B_{i0}) \leq 2^{1-\theta}\).

The following Lemma follows from (3.3), Lemma 3.4 and [LaSh06a, Corollary 3.9].

Lemma 3.5. Let \(z_0 \in \{2, 3, 5\}\). Assume that either \(d\) is odd or \(8|d\) and \(z_0 = 5\) if \(8|d\). Further let \(d = \theta_1(k-1)^2, n = \theta_2(k-1)^3\) with \(\theta_1 > 0\) and \(\theta_2 > 0\). Suppose that \(t - |R| \geq z_0\). Then we have the partition \((\eta, \eta^{-1})\) of \(d\) such that

\[
d\eta^{-1} < \frac{4(k-1)}{q_2}
\]

and

\[
\frac{1}{2} \left\{ \frac{1}{q_1q_2} - \frac{1}{\sqrt{(q_1q_2)^2 + \frac{\theta_1}{q_1q_2}}} \right\} \leq \frac{1}{2} \theta_2 < \frac{1}{2} \left\{ \frac{1}{q_1q_2} - \frac{1}{\sqrt{(q_1q_2)^2 + \frac{\theta_1}{q_1q_2}}} \right\}
\]

hold with \(q_1 \geq Q_1, q_2 \geq Q_2\) where \((Q_1, Q_2)\) is given by \((1, 1), (2, 2), (4, 4)\) according as \(z_0 = 2, 3, 5\), respectively when \(d\) is odd and \((Q_1, Q_2) = (2, 8)\) when \(z_0 = 5, 8|d\).

Lemma 3.6. Let \(z_1 > 1\) be a real number, \(h_0 > i_0 \geq 0\) to be integers such that \(\prod_{b_i \in R} b_i \geq z_1^{\varepsilon_{i_0}} |R|^{-i_0}\) for \(|R| \geq h_0\). Suppose that \(t - |R| < g\) and let \(g_1 = k - t + g - 1 + i_0\). For \(k \geq h_0 + g_1\) and for any real number \(m > 1\), we have

\[
g_1 > \frac{k \log \left( \frac{z_1^{h_0}}{2.71838} \prod_{p \leq m} p^{\frac{|R|}{p+d}} \right) + (k + \frac{1}{2}) \log(1 - \frac{q_1}{k})}{\log(k - g_1) - 1 + \log z_1} - (1.5|\text{im}| - .5 \ell - 1) \log k + \log \left( \frac{n_1^{-1} n_2 \prod_{p \leq m} p^{5+\frac{2}{p+d}}}{\log(k - g_1) - 1 + \log z_1} \right)
\]

where

\[
\ell = \left| \{p \leq m : p|d\} \right|, \quad n_0 = \prod_{p|d, p \leq m} p^{\frac{1}{p+d}}, \quad n_1 = \prod_{p|d, p \leq m} p^{\frac{2}{p+d}} \quad \text{and} \quad n_2 = \begin{cases} 2^{\frac{1}{2}} & \text{if } 2 \nmid d \\ 1 & \text{otherwise.} \end{cases}
\]

For a proof, see [LaSh06a, Lemma 5.3]. The assumption \(\omega(d) = 1\) is not necessary for Lemmas 3.1, 3.2, 3.6 and Corollary 3.3.

Lemma 3.7. We have

\[
t - |R| \geq \begin{cases} 5 \text{ for } k \geq 81 \\ 5 - \psi \text{ for } k \geq 55 \\ 4 - \psi \text{ for } k \geq 28, k \neq 31 \\ 3 - \psi \text{ for } k = 31. \end{cases}
\]

Proof. Suppose \(t - |R| < 5\) and \(k \geq 292\). Then \(|R| \geq 286\) since \(t \geq k - 2\) and \(\prod_{b_i \in R} b_i \geq (1.6)^{|R|} |R|!\) by (2.23). We observe that (3.8) hold for \(k \geq 292\) with \(i_0 = 0, h_0 = 286, z_1 = 1.6, g_1 = 6, m = 17, \ell = 0, n_0 = 1, n_1 = 1\) and \(n_2 = 2^{\frac{1}{2}}\). We check that the right hand side of (3.8) is an increasing function of \(k\) and it exceeds \(g_1\) at \(k = 292\) which is a contradiction.
Comparing the upper and lower bounds of (2.21), we have

\[ t - |R| \geq k - \psi - F'(k, r) - 2^r \geq 7 - \psi, \quad 5 - \psi, \quad 4 - \psi \] for \( k \geq 81, 55, 28 \), respectively except at \( k = 29, 31, 43, 47 \) where \( t - |R| \geq k - \psi - F(k, r) - 2^r \geq k - \psi - F'(k, r) - 2^r = 3 - \psi \). We may suppose that \( k = 29, 43, 47 \), \( t - |R| = 3 - \psi \) and \( F(k, r) = F'(k, r) \). Further we may assume that for each prime \( 7 \leq p \leq k \), there are exactly \( \sigma_p \) number of \( i \)'s for which \( p|b_i \) and for any \( i, \) \( pq \mid b_i \) whenever \( 7 \leq q \leq k, q \neq p \). Now we get a contradiction by considering the \( i \)'s for which \( b_i \)'s are divisible by primes \( 7, 13; 7, 41; 23, 11 \) when \( k = 29, 43, 47 \), respectively. For instance let \( k = 29 \). Then \( 7|b_i \) for \( i \in \{ 0, 7, 14, 21, 28 \} \). Then \( 13|b_i \) for \( i \in \{ h + 13j : 0 \leq j \leq 2 \} \) with \( h = 0, 1, 2 \). This is not possible. \( \square \)

**Lemma 3.8.** Let \( 9 \leq k \leq 23 \) and \( d \) odd. Suppose that \( t - |R| \geq 3 \) for \( k = 23 \) and \( t - |R| \geq 2 \) for \( k < 23 \). Then (1.1) does not hold.

**Proof.** Suppose (1.1) holds. Let \( Q = 2 \) if \( k = 23 \) and \( Q = 1 \) if \( k < 23 \). We now apply Lemma 3.5 with \( z_0 = 3 \) for \( k = 23 \) and \( z_0 = 2 \) for \( k < 23 \) to get \( d < \frac{4}{Q} (k - 1), \) \( \theta_1 < \frac{4}{Q(k - 1)} \) and

\[ \theta_1 + \theta_2 < \frac{1}{2} \left( \frac{1}{Q^2} + \frac{4}{Q(k - 1)} + \sqrt{\frac{1}{Q^4} + \frac{4}{Q^3(k - 1)}} \right) \leq \Omega(k - 1). \]

Further from (2.21), we have \( n + (k - 1)d \geq n + \gamma_id \geq p_i^{2(k-2)} \). Therefore \( p_i^x = d < \frac{4}{Q} (k - 1) \) and \( p_i^{2(k-2)} \leq n + (k - 1)d < (k - 1)^3 \Omega(k - 1) \). For these possibilities of \( n, d, k \), we check that there are at least three \( i \) with \( 0 \leq i < k \) such that \( n + id \) is divisible by a prime \( > k \) to an odd power. This contradicts (1.1). \( \square \)

**Lemma 3.9.** Equation (1.1) with \( k \geq 9 \) implies that \( t - |R| \leq 1 \).

**Proof.** Assume that \( k \geq 9 \) and \( t - |R| \geq 2 \). Let \( d = 2, 4 \). Then \( |R| \leq t - 2 \) contradicting \( |R| = t \) by Lemma 3.4. Thus \( d \neq 2, 4 \). By Lemma 3.4, we have \( \nu(b_{0a}) \leq 2 \) and \( \nu(B_{0a}) \leq 2 \).

Let \( k \geq 81 \). Then \( t - |R| \geq 5 \) by Lemma 3.7. Now we derive from Lemma 3.5 with \( z_0 = 5 \) that \( d < k - 1 \) giving \( \theta_1 < \frac{1}{k - 1} \) and hence

\[ n + (k - 1)d = (\theta_1 + \theta_2)(k - 1)^3 < \frac{(k - 1)^3}{2} \left\{ \frac{1}{16} + \frac{1}{k - 1} + \sqrt{\frac{1}{(16)^2} + \frac{1}{16(k - 1)}} \right\}. \]

On the other hand, we get from (3.2) and \( \nu(B_{0a}) \leq 2 \) that \( n + (k - 1)d \geq \frac{0.1754k}{2} k^2 \geq 0.1754\frac{k^3}{2} \). Comparing the upper and lower bounds of \( n + (k - 1)d \), we obtain

\[ 0.1754 < \left\{ \frac{1}{16} + \frac{1}{k - 1} + \sqrt{\frac{1}{(16)^2} + \frac{1}{16(k - 1)}} \right\} \leq 0.144 \]

since \( k \geq 81 \). This is a contradiction.

Thus \( k < 81 \). Let \( d \) be even. Then \( 8|d \) and we see from \( \nu(a_i) \leq 2 \) and (2.14) that \( \xi_r \leq 2g_{2^s} \leq 2^{r-2} \). Let \( r = 3 \). From (2.9), we get \( k - 2 - F'(k, r) \leq \xi_r \leq 2^{r-2} \). We find \( k - 2 - F'(k, r) \leq 2^{r-2} \) by computation. This is a contradiction.

Thus \( d \) is odd. Since \( \psi \leq 2 \), we get from Lemmas 3.7 and 3.5 with \( z_0 = 3, 2 \) that \( d < 2(k - 1) \) if \( k \geq 55 \) and \( d < 4(k - 1) \) if \( k < 55 \). Since \( g_{2^s}(r) \leq 2^{r-1} \) for \( r = 4, p < 220 \) by (2.19), we get from (2.18) with \( r = 4 \) that \( t - |R| \geq k - 2 - F'(k, r) - 2^{r-1} \) which is \( \geq 5 \) for \( k \geq 29 \) and \( \geq 3 \) for \( k = 23 \).
Let $k \geq 29$. Then we get from Lemma 3.5 with $z_0 = 5$ that $d < k - 1$. By taking $r = 3$ for $k < 53$ and $r = 4$ for $53 \leq k < 81$, we derive from (2.17), (2.19), $\nu(a_i) \leq 2$ and (2.9) that $k - 2 - F'(k, r) \leq \xi_r \leq 2g_p \leq 2^r$. On the other hand, we check by computation that $k - 2 - F'(k, r) > 2^r$. This is a contradiction.

Thus $k \leq 23$. Then $t - |R| \geq 3$ for $k = 23$ and $t - |R| \geq 2$ for $k < 23$. By Lemma 3.8, this is not possible.

**Corollary 3.10.** Let $k \geq 9$. Equation (1.1) implies that either $k \leq 23$ or $k = 31$. Also $P(d) > k$.

**Proof.** By Lemmas 3.7 and 3.9, we see that either $k \leq 23$ or $k = 31$. Suppose that $P(d) \leq k$. Since $g_{P(d)}(r) \leq 2^{r-1}$ for $r = 3$ by (2.19), we get from (2.18) with $r = 3$ that $t - |R| \geq k - 2 - F'(k, r) - 2^{r-1} \geq 2$ except at $k = 9$ where $t - |R| = 1$. This contradicts Lemma 3.9 for $k > 9$. Let $k = 9$. By taking $r = 4$, we get from $g_{P(d)}(r) \leq 2^{r-2}$ by (2.19) and (2.18) that $t - |R| \geq k - 2 - F'(k, 4) - 2^{r-2} \geq 2$. This contradicts Lemma 3.9.

As a consequence, we derive the following Corollary which is [SaSh03a, Theorem 1 (ii)].

**Corollary 3.11.** Let $\psi = 0$. Equation (1.1) with $P(b) < k$ implies that $k \leq 9$.

**Proof.** Let $k \geq 10$. By Corollary 3.10, we see that either $k \leq 23$ or $k = 31$. Let $k = 10$. Then we get from (2.13) with $r = 2$ that $t - |R| \geq k - F'(k, r) - 2^r = 2$ contradicting Lemma 3.9. Thus (1.1) does not hold at $k = 10$. By induction, we may assume $k \in \{12, 14, 18, 20\}$ and further there is at most one $i$ for which $p|a_i$ with $p = k - 1$. We take $r = 2$ for $k = 12, 14$ and $r = 3$ for $k = 18, 20$. Now we get from $|\{b_i : P(b_i) > p_r\}| \leq F'(k, r) - 1$ and (2.10) that $t - |R| \geq k - F'(k, r) + 1 - 2^r \geq 2$. This contradicts Lemma 3.9.

4. **Proof of Theorem 1**

Suppose that the assumptions of Theorem 1 are satisfied and assume (1.1) with $\omega(d) = 1$. By Corollary 3.10, we have $P(d) > k$ and further we restrict to $k \leq 23$ and $k = 31$. Also $t - |R| \leq 1$ by Lemma 3.9. Further it suffices to prove the assertion for $k \in \{15, 18, 20, 23, 31\}$ since the cases $k = 16, 17; k = 19$ and $k = 21, 22$ follows from those of $k = 15, 18$ and 20, respectively.

We shall arrive at a contradiction by showing $t - |R| \geq 2$. For a prime $p \leq k$, we observe that $p \nmid d$ and let $i_p$ be such that $0 \leq i_p < p$ and $p|n + i_pd$. For any subset $\mathcal{I} \subseteq [0, k) \cap \mathbb{Z}$ and primes $p_1$ and $p_2$, we define

$$\mathcal{I}_1 = \{i \in \mathcal{I} : \left(\frac{i - i_{p_1}}{p_1}\right) = \left(\frac{i - i_{p_2}}{p_2}\right)\} \text{ and } \mathcal{I}_2 = \{i \in \mathcal{I} : \left(\frac{i - i_{p_1}}{p_1}\right) \neq \left(\frac{i - i_{p_2}}{p_2}\right)\}. $$

Then from $\left(\frac{a_i}{p}\right) = \left(\frac{i - i_p}{p}\right)$, we see that either

(4.1) \hspace{1cm} \left(\frac{a_i}{p_1}\right) \neq \left(\frac{a_i}{p_2}\right) \hspace{0.5cm} \text{for all } i \in \mathcal{I}_1 \text{ and } \left(\frac{a_i}{p_1}\right) = \left(\frac{a_i}{p_2}\right) \hspace{0.5cm} \text{for all } i \in \mathcal{I}_2$

or

(4.2) \hspace{1cm} \left(\frac{a_i}{p_1}\right) \neq \left(\frac{a_i}{p_2}\right) \hspace{0.5cm} \text{for all } i \in \mathcal{I}_2 \text{ and } \left(\frac{a_i}{p_1}\right) = \left(\frac{a_i}{p_2}\right) \hspace{0.5cm} \text{for all } i \in \mathcal{I}_1$. 

We define \((M, B) = (I_1, I_2)\) in the case (4.1) and \((M, B) = (I_2, I_1)\) in the case (4.2). We call \((I_1, I_2, M, B) = (I_1^k, I_2^k, M^k, B^k)\) when \(I = \{0, k\} \cap \mathbb{Z}\). Then for any \(I \subseteq \{0, k\} \cap \mathbb{Z}\), we have
\[
I_1 \subseteq I_1^k, I_2 \subseteq I_2^k, M \subseteq M^k, B \subseteq B^k
\]
and
\[
|M| \geq |M^k| - (k - |I|), \quad |B| \geq |B^k| - (k - |I|).
\]

By taking \(m = n + \gamma t d\) and \(\gamma' = \gamma t - \gamma t - i + 1\), we re-write (1.1) as
\[
(m - \gamma' d) \cdots (m - \gamma' d) = by^2.
\]
The equation (4.4) is called the mirror image of (1.1). The corresponding \(t\)-tuple \((a_{\gamma_1}, a_{\gamma_2}, \ldots, a_{\gamma_t})\) is called the mirror image of \((a_{\gamma_1}, \ldots, a_{\gamma_t})\).

4.1. The case \(k = 15\). Then \(\sigma' = 3\) implies that \(7|a_j\) for \(j = 0, 1, 2\) and \(\sigma' = 2\) if \(7 \nmid a_0 a_2 a_{14}\). Similarly \(\sigma' = 2\) implies \(13|a_0, 13|a_{13}\) or \(13|a_{1}, 13|a_{14}\) and \(\sigma' = 1\) otherwise. Thus \(|\{a_i : 7|a_i\text{ or } 13|a_i\}| \leq 4\). It suffices to have
\[
|\{a_i : p|a_i\text{ for } 5 \leq p \leq 13\}| \leq 7
\]
since then \(t - |R| \geq k - 2 - |\{a_i : p|a_i\text{ for } 5 \leq p \leq 13\}| - 4 \geq 2\) by (2.10) with \(r = 2\), a contradiction.

Let \(p_1 = 11, p_2 = 13\) and \(I = \{\gamma_1, \gamma_2, \ldots, \gamma_t\}\). We observe that \(P(a_i) \leq 7\) for \(i \in M \cup B\).
Since \(\frac{3}{11} \neq \frac{3}{13}\) but \(\frac{3}{11} = \frac{3}{13}\) for a prime \(q < k\) other than \(5, 11, 13\), we observe that \(5|a_i\) whenever \(i \in M\). Since \(\sigma_5 \leq 3\) and \(|I| = k - 2\), we obtain from (4.3) that \(|M^k| \leq 5\) and \(5|a_i\) for at least \(|M^k| - 2\) \(a_i\)’s with \(i \in M^k\). Further \(5 \nmid a_i\) for \(i \in B\).

By taking the mirror image (4.4) of (1.1), we may suppose that \(0 \leq i_{13} \leq 7\). For each possibility \(0 \leq i_{11} < 11\) and \(0 \leq i_{13} \leq 7\), we compute \(|I_1^k|, |I_2^k|\) and restrict to those pairs \((i_{11}, i_{13})\) with \(\min(|I_1^k|, |I_2^k|) \leq 5\). We see from \(\max(|I_1^k|, |I_2^k|) \geq 6\) that \(M^k\) is exactly one of \(I_1^k\) or \(I_2^k\) with minimum cardinality and hence \(B^k\) is the other. Now we restrict to those pairs \((i_{11}, i_{13})\) for which there are at most two elements \(i \in M^k\) such that \(5 \nmid a_i\). There are 31 such pairs. By counting the multiples of 11 and 13 and also the maximum multiples of 5 in \(M^k\) and the maximum number of multiples of 7 in \(B^k\), we again restrict to those pairs \((i_{11}, i_{13})\) which do not satisfy (4.5). With this procedure, all pairs \((i_{11}, i_{13})\) are excluded other than
\[
(0, 6), (1, 3), (2, 4), (3, 5), (4, 6), (5, 3).
\]
We first explain the procedure by showing how \((i_{11}, i_{13}) = (0, 0)\) is excluded. Now \(M^k = \{5, 10\}\) and \(B^k = \{2, 4, 7, 8, 9, 10, 12, 13, 14\}\). Then there are 3 multiples of 11 and 13, at most 2 multiples of 5 in \(M^k\) and at most 2 multiples of 7 in \(B^k\) implying (4.5). Thus \((i_{11}, i_{13}) = (0, 0)\) is excluded.

Let \((i_{11}, i_{13}) = (5, 3)\). Then \(M^k = \{1, 6, 11\}\) and \(B^k = \{2, 4, 7, 8, 9, 10, 12, 13, 14\}\) giving \(i_5 = 1\) and \(5|a_1 a_6 a_{11}\). We may assume that \(7|a_i\) for \(i \in \{0, 7, 14\}\) otherwise (4.5) holds. By taking \(p_1 = 5, p_2 = 11\) and \(I = B^k\), we get \(I_1 = \{4, 10, 13\}\) and \(I_2 = \{2, 4, 7, 8, 9, 12, 14\}\). Since \(\frac{3}{5} = \frac{3}{11}\), \(\frac{2}{5} = \frac{2}{11}\) and \(\frac{3}{5} \neq \frac{3}{11}\), we observe that \(3|a_i\) for \(i \in I_1 \cap B\) and \(3 \nmid a_i\) for \(i \in I_2 \cap B\). Thus \(a_i \in \{3, 6\}\) for \(i \in I_1 \cap B\) and \(a_i \in \{1, 2, 7, 14\}\) for \(i \in I_2 \cap B\). Now from \(\frac{3}{5} = \frac{3}{11}\), \(\frac{2}{5} = \frac{2}{11}\) and \(\frac{3}{5} = \frac{3}{11}\), we see that at least one of 4, 10, 13 is not in \(B\) implying \(i \notin B\) for at most one \(i \in I_2\). Therefore there are distinct pairs \((i_1, i_2)\) and \((j_1, j_2)\) with
$i_1, i_2, j_1, j_2 \in I_3 \cap B$ such that $a_{i_1} = a_{i_2}, i_1 > i_2$ and $a_{j_1} = a_{j_2}, j_1 > j_2$ giving $t - |R| \geq 2$. This is a contradiction. Similarly, all other pairs $(i_{11}, i_{13})$ in (4.6) are excluded.

4.2. The case $k = 18$. We may assume that $\sigma'_{17} = 1$ and $17 \mid a_0 a_1 a_2 a_{15} a_{16} a_{17}$ otherwise the assertion follows the case $k = 15$. If $\{|a_i: P(a_i) = 5\}| = 4$, we see from $\{|a_i: P(a_i) = 5\} \subseteq \{5, 10, 15, 30\}$ that $a_5 a_{15} + 5 a_{16} + 10 a_{17}$ is a square, contradicting Eulers’ result for $k = 4$. Thus we have $\{|a_i: P(a_i) = 5\}| \leq 3$. Further for each prime $7 \leq p \leq 13$, we may also assume that $\sigma'_p = \sigma_p$ and for any $i, p q \nmid a_i$ whenever $7 \leq q \leq 17, q \neq p$ otherwise $t - |R| \geq k - 2 - \sum_{7 \leq p \leq 17} \sigma'_p - 3 - 4 \geq 2$ by (2.10) with $r = 2$.

Let $p_1 = 11, p_2 = 13$ and $I = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$. Since $(\frac{1}{11}) \neq (\frac{5}{11})$ and $(\frac{11}{11}) \neq (\frac{17}{11})$ but $(\frac{11}{17}) = (\frac{17}{17})$ for $q < k, q \neq 17, 11, 13$, we observe that for $i \in M$, exactly one of $5 a_i$ or $17 a_i$ holds. Thus $5 \cdot 17 \mid a_i$ whenever $i \in M$. For $i \in B$, either $5 \mid a_i, 17 \mid a_i$ or $5 | a_5, 17 | a_5$. Thus for $i \in B$, we have $P(a_i) \leq 7$ except possibly for one $i$ for which $5 \cdot 17 | a_5$. Since $\sigma_5 \leq 4$ and $\sigma_{17} \leq 1$, we obtain $|M^k| \leq 7$ and $5 a_i$ for at least $|M^k| - 3$’s with $i \in M^k$. Hence $|M^k| = 7$ implies that either

$|a + 5 j : 0 \leq j \leq 3| \leq I^k_1$ or $|b + 5 j : 0 \leq j \leq 3| \leq I^k_2$ for some $a, b \in \{0, 1, 2\}$.

Since $\sigma'_{i_1} = 2$ and $\sigma'_{i_3} = 2$, we may suppose that $0 \leq i_{11} \leq 6$ and $0 \leq i_{13} \leq 4$. Further $i_{11} \neq i_{13}$ and $i_{11} + 11 \neq i_{13} + 13$. We observe that either $\min(|I^k_1|, |I^k_2|) \leq 6$ or $|I^k_1| = |I^k_2| = 7$. For pairs $(i_{11}, i_{13})$ with $|I^k_1| = |I^k_2| = 7$, we check that (4.7) is not valid. Thus we restrict to those pairs satisfying $\min(|I^k_1|, |I^k_2|) \leq 6$. There are 16 such pairs. Further we see from $\max(|I^k_1|, |I^k_2|) \geq 8$ that $M^k$ is exactly one of $I^k_1$ or $I^k_2$ with minimum cardinality and hence $B^k$ is the other one. Now we restrict to those pairs $(i_{11}, i_{13})$ for which $5 | a_i$ for at least 3 elements $i \in M^k$ otherwise $t - |R| \geq k - 2 - \sum_{7 \leq p \leq 17} \sigma'_p - 3 - 4 \geq 2$ by (2.10) with $r = 2$. We find that $(i_{11}, i_{13}) \in \{(1, 3), (2, 4), (4, 0), (5, 1)\}$. For these pairs $(i_{11}, i_{13})$, we check that there are at most 4 multiples of 5 and 17 with $i \in M^k \cup B^k$. Thus if $\{|i : i \in B, 7 | a_i\} \leq 2$, then $t - |R| \geq 2$ by (2.10) with $r = 2$. Therefore we may assume that $\{|i : i \in B, 7 | a_i\} = 3$ and hence $\{|i : i \in B^k, 7 | a_i\} = 3$. We now restrict to those pairs $(i_{11}, i_{13})$ for which $\{|i : i \in B^k, 7 | a_i\} = 3$. They are given by $(i_{11}, i_{13}) \in \{(2, 4), (4, 0)\}$.

Let $(i_{11}, i_{13}) = (2, 4)$. Then by taking $p_1 = 11$ and $p_2 = 13$ as above, we have $M^k = \{1, 6, 8, 11\} \cup B^k = \{0, 3, 5, 7, 9, 10, 12, 14, 15, 16\}$ giving $i_5 = 1$ and $5 | a_5 a_6 a_{11}$. We may assume that $17 | a_8$ since $17 \nmid a_{16}$. Hence $P(a_i) \leq 7$ for $i \in B$. Consequently $P(a_i) \leq 7$ for exactly 8 elements $i \in B^k$ and other 2 elements are not in $B$. Further $7 | a_i$ for $i \in \{0, 7, 14\}$ and $0, 7, 14 \in B$. Now we take $p_1 = 5, p_2 = 11$ and $I = B^k$ to get $I_1 = \{0, 5, 7, 9\}$ and $I_2 = \{3, 10, 12, 14, 15\}$. Since $(\frac{5}{11}) = (\frac{5}{17}), (\frac{7}{11}) = (\frac{7}{17})$ and $(\frac{5}{17}) = (\frac{7}{17})$, we observe that either $3 | a_i$ for $i \in I_1 \cap B$ or $3 | a_i$ for $i \in I_2 \cap B$. The former possibility is excluded since $0, 7 \in I_1 \cap B$ and the latter is not possible since $14 \in I_2 \cap B$. The other case $(i_{11}, i_{13}) = (4, 0)$ is excluded similarly.

4.3. The case $k = 20$. We may assume that $\sigma'_{i_9} = 1$ and $19 \nmid a_0 a_{19}$ otherwise the assertion follows from the case $k = 18$. Also we have $\{|a_i: P(a_i) = 5\}| \leq 3$ by Eulers’ result for $k = 4$. Further for each prime $7 \leq p \leq 17$, we may also assume that $\sigma'_p = \sigma_p$ and for any $i, p q \nmid a_i$ whenever $7 \leq p < q \leq 19$ otherwise $t - |R| \geq k - 2 - \sum_{7 \leq p \leq 17} \sigma'_p - 3 - 4 \geq 2$ by (2.10) with $r = 2$. 

Let $p_1 = 11$, $p_2 = 13$ and $\mathcal{I} = \{\gamma_1, \gamma_2, \cdots, \gamma_t\}$. Then as in the case $k = 18$, we observe that for $i \in \mathcal{M}$, exactly one of $5a_i$ or $17a_i$ holds but $5 \cdot 17 \nmid a_i$. For $i \in \mathcal{B}$, either $5 \nmid a_{i}$, $17 \nmid a_{i}$ or $5|a_{i}$, $17|a_{i}$. Since $\sigma_5 \leq 4$ and $\sigma_{17} \leq 2$, we obtain $|\mathcal{M}^k| \leq 8$ and $5|a_{i}$ for at least $|\mathcal{M}^k| - 4$ $i$'s with $i \in \mathcal{M}^k$. Hence $|\mathcal{M}^k| = 8$ implies that either

$$\{a + 5j : 0 \leq j \leq 3\} \subseteq \mathcal{I}_1^k$$

or $\{b + 5j : 0 \leq j \leq 3\} \subseteq \mathcal{I}_2^k$

for some $a, b \in \{0, 1, 2, 3, 4\}$.

Since $\sigma'_{11} = 2$ and $\sigma'_{13} = 2$, we may suppose that $0 \leq i_{11} \leq 8$ and $0 \leq i_{13} \leq 6$. Further $i_{11} \neq i_{13}$ and $i_{11} + 11 \neq i_{13} + 13$. We observe that either $\min(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \leq 7$ or $|\mathcal{I}_1^k| = |\mathcal{I}_2^k| = 8$. For pairs $(i_{11}, i_{13})$ with $|\mathcal{I}_1^k| = |\mathcal{I}_2^k| = 8$, we check that (4.8) is not valid. Thus we restrict to those pairs satisfying $\min(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \leq 7$. There are 40 such pairs. Further we see from $\max(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \geq 8$ that $\mathcal{M}^k$ is the one of $\mathcal{I}_1^k$ or $\mathcal{I}_2^k$ with minimum cardinality and hence $\mathcal{B}^k$ is the other. Now we restrict to those pairs $(i_{11}, i_{13})$ for which $5|a_{i}$ for at least 3 elements $i \in \mathcal{M}^k$ otherwise $t - |R| \geq k - 2 - 1 - \sum_{7 \leq p \leq 17} \sigma'_p - 2 - 4 \geq 2$ by (2.10) with $r = 2$. We are left with 22 such pairs. Further by (4.3) and $|\mathcal{I}| = k - 2$, we restrict to those pairs $(i_{11}, i_{13})$ for which there are at least $|\mathcal{M}^k| - 2$ elements $i \in \mathcal{M}^k$ such that $5|a_{i}$ or $17|a_{i}$. There are 12 such pairs $(i_{11}, i_{13})$ and for these pairs, we check that there are at most 4 multiples of $a_{i}$ of 5 and 17 with $i \in \mathcal{M}^k \cup \mathcal{B}^k$. This implies $t - |R| \geq k - 2 - 1 - \sum_{11 \leq p \leq 13} \sigma'_p - 4 \geq 2$ by (2.10) with $r = 2$. For instance, let $(i_{11}, i_{13}) = (3, 5)$. Then $\mathcal{M}^k = \{2, 7, 9, 12\}$ and $\mathcal{B}^k = \{0, 1, 4, 6, 8, 10, 11, 13, 15, 16, 17, 19\}$. Since $5|a_{i}$ for at least three elements $i \in \mathcal{M}^k$, we get $5|a_i$ for $i \in \{2, 7, 12\}$ giving $i_5 = 2$. Further $17|a_9$ or $5 \cdot 17|a_{17}$ giving 4 multiples of $a_{i}$ of 5 and 17 with $i \in \mathcal{M}^k \cup \mathcal{B}^k$. Thus $t - |R| \geq 2$ as above.

4.4. **The case** $k = 23$. We may assume that $\sigma'_{23} = 1$ and $23 \nmid a_{i}$ for $0 \leq i \leq 2$ and $20 \leq i < 23$ otherwise the assertion follows from the case $k = 20$. We have $\sigma'_{13} = 3$ if $11|a_{11j}$ with $j = 0, 1, 2$ and $\sigma'_{11} = 2$ if $11 \nmid a_{01}a_{11}a_{22}$. Also $\sigma'_{4} = 4$ implies that $7|a_{7j}$ or $7|a_{1+7j}$ with $0 \leq j \leq 3$ and $\sigma'_{7} \leq 3$ otherwise. Thus $|\{a_{i} : 7|a_{i} \text{ or } 11|a_{i}\}| \leq 6$. Further by Euler's result for $k = 4$, we obtain $|\{a_{i} : P(a_{i}) = 5\}| \leq 4$. If

$$|\{a_{i} : p|a_{i}, 5 \leq p \leq 23\}| \leq 4 + \sum_{7 \leq p \leq 23} \sigma_p - 1 - 2 = 15,$$

then we get from (2.10) with $r = 2$ that $t - |R| \geq k - 2 - 15 - 4 = 2$, a contradiction. Therefore we have

$$4 + \sum_{7 \leq p \leq 23} \sigma_p - 2 \leq |\{a_{i} : p|a_{i}, 5 \leq p \leq 23\}| \leq 4 + \sum_{7 \leq p \leq 19} \sigma_p - 1.$$
is a contradiction. For example, let \((i_{11}, i_{13}) = (0, 2)\). Then \(M^k = \{4, 6, 9, 18, 19, 20\}\) and \(B^k = \{1, 3, 5, 7, 8, 10, 12, 13, 14, 16, 17, 21\}\) giving \(5|a_i\) for \(i \in \{4, 9, 19\}\), \(i_5 = 4\). Further \(17|a_i\) for exactly one \(i \in \{6, 18, 20\}\) and other two \(i\)'s in \(\{6, 18, 20\}\) deleted. Thus \(5 \cdot 17 \nmid a_{i4}\) so that (4.9) is not valid. For another example, let \((i_{11}, i_{13}) = (4, 0)\). Then \(M^k = \{6, 9, 11, 16, 21\}\) and \(B^k = \{1, 2, 3, 5, 7, 8, 10, 12, 14, 17, 18, 19, 20, 22\}\) giving \(5|a_i\) for \(i \in \{6, 11, 16, 21\}\), \(i_5 = 1\). Further we have either \(17|a_9\), \(\gcd(5 \cdot 17, a_1) = 1\) or \(9 \notin M, 5 \cdot 17|a_1\). Now \(7|a_i\) for at most 3 elements \(i \in B^k\) so that (4.9) is not satisfied. This is a contradiction.

4.5. The case \(k = 31\). From \(t - |R| \geq k - 2 - \sum_{7 \leq p \leq 31} \sigma'_p - 8 \geq k - 2 - \sum_{7 \leq p \leq 31} \sigma_p - 8 = 1\) by (2.10) and (2.13) with \(r = 3\), we may assume for each prime \(7 \leq p \leq 31\) that \(\sigma'_p = \sigma_p\) and for any \(i, pq \nmid a_i\) whenever \(7 \leq p < q \leq 31\). Let \(I = \{\gamma_1, \gamma_2, \ldots, \gamma_t\}\). By taking the mirror image (4.4) of (1.1) and \(\sigma_{19} = \sigma_{29} = 2\), we may assume that \(i_{29} = 0\) and \(1 \leq i_{19} \leq 11, i_{19} \neq 10\). For \(p \leq 31\) with \(p \neq 19, 29\), since \(\left(\frac{p}{19}\right) \neq \left(\frac{p}{29}\right)\) if and only if \(p = 11, 13, 17\), we observe that for \(i \in M\), either \(11|a_i\) or \(13|a_i\) or \(17|a_i\). Since \(\sigma_{11} + \sigma_{13} + \sigma_{17} \leq 8\), we obtain \(|M^k| \leq 10\) and \(p|a_i\) for at least \(|M^k| - 2\) elements \(i \in M^k\) and \(p \in \{11, 13, 17\}\). Now for each of the pair \((i_{19}, i_{29})\) given by \(i_{29} = 0, 1 \leq i_{19} \leq 11, i_{19} \neq 10\), we compute \(|I'_1|, |I'_2|\). Since \(\max(|I'_1|, |I'_2|) \geq 14\), we restrict to those pairs \((i_{19}, i_{29})\) with \(\min(|I'_1|, |I'_2|) \leq 10\). Then we are left with the only pair \((i_{19}, i_{29}) = (1, 0)\). Further noticing that \(M^k\) is exactly one of \(I'_1\) or \(I'_2\) with minimum cardinality, we get \(M^k = \{3, 5, 6, 7, 11, 14, 15, 19, 24, 25\}\) and \(B^k = \{2, 4, 8, 9, 10, 12, 13, 16, 17, 18, 21, 22, 23, 26, 27, 28, 30\}\). We find that there are at most 7 elements \(i \in M^k\) for which either \(11|a_i\) or \(13|a_i\) or \(17|a_i\). This is not possible.

References


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