A-optimal diallel crosses for test versus control comparisons

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SUMMARY

A-optimality of block designs for control versus test comparisons in diallel crosses is investigated. A sufficient condition for designs to be A-optimal is derived. Type $S_0$ designs are defined and A-optimal type $S_0$ designs are characterized. A lower bound to the A-efficiency of type $S_0$ designs is also given. Using the lower bound to A-efficiency, type $S_0$ designs are shown to yield efficient designs for test versus control comparisons.

Some key words: Type $S$ design; Type $S_0$ design; A-optimality; Inbred lines; A-efficiency.

1. INTRODUCTION

Although designs for varietal trials and factorial experiments have been extensively investigated in literature over the past several decades, it was not until recently that some progress in the design of diallel cross experiments has been made, see e.g. Gupta and Kageyama (1994), Dey and Midha (1996), Mukerjee (1997), Das, Dey and Dean (1998). Designs for control versus test comparisons where the treatments form different levels of a factor have also been extensively investigated in the literature; see Majumdar (1996). The problem of deriving designs appropriate for diallel crosses is quite different from the set-up of designs for varietal trials and factorial experiments. Therefore, here we continue the work of Gupta and Kageyama (1994) for studying optimal block designs for control versus test comparisons among the lines with respect to their general combining ability effects. Recently Choi, Gupta and Kageyama (2002) introduced a class of designs, called type $S$ block designs, for control-test comparisons in a diallel cross experiment. Let $p$ ($>2$), $b$, and $k$ denote the number of test lines, number of blocks and block size respectively. In Section 2 we define a sub-class of type $S$ designs, called type $S_0$ designs, and derive a sufficient condition for designs to be A-optimal. Henceforth, by optimal we mean A-optimal, and by efficiency we mean A-efficiency. We also characterize optimal type $S_0$ designs in Section 2, and show the optimality of some of the designs of Choi, Gupta and Kageyama (2002). A new method of constructing type $S_0$ designs is also provided there. A type $S_0$ design satisfying the derived sufficient condition for optimality does not always exist. Therefore, a lower bound to the efficiency of a type $S_0$ design is defined in Section 3. For type
S₀ designs in a practical range $p \leq 30$, $b \leq 50$, $k \leq p$, 70 designs out of a total of 247 possible types $S₀$ designs are optimal. Of the remaining 177 designs that do not satisfy the sufficient condition for optimality, 175 designs have lower bound to efficiency at least 0.80. Thus, type $S₀$ designs provide highly efficient designs for control-test comparisons. For the sake of brevity, table of efficient type $S₀$ designs is not presented in this paper, and it will be reported elsewhere.

2. Optimal designs

We consider diallel cross experiments involving $p + 1$ inbred lines, giving rise to a total of $n_c = p(p + 1)/2$ distinct crosses. Let a cross between lines $i$ and $j$ be denoted by $(i, j)$, $i < j = 0, 1, \ldots, p$. Suppose line 0 is a control or a standard line and lines $1, \ldots, p$ are test lines. Let $s_{dij}$ denote the total number of times that the $j$th line occurs in the crosses in the design $d$, $j = 0, 1, \ldots, p$. Further let $s_d = (s_{d0}, s_{d1}, \ldots, s_{dp})'$ and let $n$ denote the total number of crosses in the design. Following e.g., Gupta and Kageyama (1994), the model under the block design set-up, for a design $d$ involving $p + 1$ inbred lines and $b$ blocks each containing $k$ crosses, is assumed to be

$$Y_d = µ1_n + Δ_{1d}τ + Δ_{2d}β + ε,$$

where $Y_d$ is the $n \times 1$ vector of responses, $µ$ is the overall mean, $1_t$ is the $t \times 1$ vector of 1’s, $τ = (τ₀, τ₁, \ldots, τₚ)'$ is the vector of $p + 1$ general combining ability effects, $β = (β₁, β₂, \ldots, βₕ)'$ is the vector of $b$ block effects and $Δ_{1d}$ ($Δ_{2d}$) is the corresponding observation versus line (block) design matrix, that is, the $(h, l)$th element of $Δ_{1d}$ ($Δ_{2d}$), is 1 if the $h$th observation pertains to the $l$th line (block), and is zero otherwise, and $ε$ is the $n \times 1$ vector of independent random errors with zero expectation and a constant variance $σ^2$. The coefficient matrix of the reduced normal equations for estimating the vector of general combining ability effects is then given by

$$C_d = G_d - \frac{1}{k}N_dN_d',$$  \hspace{1cm} (2.1)

where $N_d = (n_{dij})$, $i = 0, 1, \ldots, p; j = 1, \ldots, b$, is the $(p + 1) \times b$ line versus block incidence matrix, $G_d = (g_{divi})$, $g_{divi} = s_{di}$, and for $i \neq i'$, $g_{divi'}$ is the number of times the cross $(i, i')$ appears in the design.

Let $D(p + 1, b, k)$ denote the set of all connected designs with $p$ test lines, one control line and $bk$ crosses arranged in $b$ blocks each of size $k$. A design $d \in D(p + 1, b, k)$ is said to be optimal for control-test comparisons if it minimizes $\sum_{i=1}^{p} Var(\hat{τ}_d - \hat{τ}_d₀)$, where $\hat{τ}_d - \hat{τ}_d₀$ denotes the best linear unbiased estimator (BLUE) of $τ_i - τ₀$ using $d$. Let $P = (-1_p \ I_p)$ where $I_t$ denotes the identity matrix of order $t$. Then the covariance matrix for the BLUE’s $(\hat{τ}_{d₁} - \hat{τ}_{d₀}, \hat{τ}_{d₂} - \hat{τ}_{d₀}, \ldots, \hat{τ}_{dp} - \hat{τ}_{d₀})$ of the control-test contrast is $σ^2PC_d P'$. If one partitions $C_d$ as

$$C_d = \begin{pmatrix}
c_{d₀₀} & γ'_d \\
γ_d & M_d
\end{pmatrix}$$  \hspace{1cm} (2.2)

then it can be shown (see Gupta, 1989) that $(PC_d P')^{-1} = M_d$, i.e., $M_d$ is the information matrix for the control-test contrasts. For a design $d$ in $D(p + 1, b, k)$, using Kiefer’s (1975)
that are used in the sequel.

A design with the property that the summation being taken over all \((p + 1) \times (p + 1)\) permutation matrices \(\pi\) that correspond to permutations of the \(p\) test treatments only. Partitioning \(\bar{C}_d\) as in (2.2), we see that

\[
\bar{M}_d = (\bar{P} \bar{C}_d^{-1} \bar{P}^{T})^{-1}
\]

is a completely symmetric matrix. In general, there may be no design in \(D(p + 1, b, k)\) for which \(\bar{M}_d\) is the information matrix for the control-test contrasts. If there is such a design, then for this design, call it \(d^*\), \(\bar{M}_{d^*} = \bar{M}_d\) is completely symmetric and \(\gamma_{d^*}\) of (2.2) is a vector with all entries equal. That is, \(d^*\) belongs to a class of designs, called type \(S\) block designs, introduced by Choi, Gupta and Kageyama (2002).

**Definition 2.1.** A design \(d \in D(p + 1, b, k)\) is called a type \(S\) block design if there are positive integers \(g_0, g_1, \lambda_0\) and \(\lambda_1\), such that for \(i \neq i' = 1, \ldots, p\),

\[
g_0 d_0 = g_0, \quad g_{di'} = g_1, \quad \Sigma_{j=1}^{b} n_{d_0j} n_{dij} = \lambda_0, \quad \Sigma_{j=1}^{b} n_{d_0j} n_{dij'} = \lambda_1.
\]

We denote a type \(S\) block design with parameters \(p, b, k, g_0, g_1, \lambda_0\) and \(\lambda_1\) by \(S(p, b, k, g_0, g_1, \lambda_0, \lambda_1)\). For an \(S(p, b, k, g_0, g_1, \lambda_0, \lambda_1)\) design \(d\) it holds that \(s_{di} = pg_0, s_{d1} = p, s_{d2} = \ldots = s_{dp} = g_0 + (p - 1)g_1, bk = (s_{d0} + ps_{d1})/2, \)

\[
\text{Var}(\hat{\tau}_{di} - \hat{\tau}_{d0}) = \frac{k(a_1 - (p - 2)b_1)\sigma^2}{(a_1 + b_1)(a_1 - (p - 1)b_1)}, \quad i = 1, \ldots, p,
\]

\[
\text{Cov}(\hat{\tau}_{di} - \hat{\tau}_{d0}, \hat{\tau}_{di'} - \hat{\tau}_{d0}) = \frac{kb_1\sigma^2}{(a_1 + b_1)(a_1 - (p - 1)b_1)}, \quad i \neq i' = 1, \ldots, p,
\]

where \(a_1 = \lambda_0 - kg_0 + (p - 1)b_1\) and \(b_1 = \lambda_1 - kg_1\).

**Definition 2.2.** A type \(S_0\) block design denoted by \(S_0(p, b, k, g_0, g_1, \lambda_0, \lambda_1)\) is a type \(S\) block design with the property that \(|n_{d_0j} - n_{d_0j'}| \leq 1, |n_{dij} - n_{dij'}| \leq 1\) for \(i, i' = 1, \ldots, p, j, j' = 1, \ldots, b\).

Using \([z]\) to denote the largest integer not exceeding \(z\), we now introduce some notations that are used in the sequel.

\[
a(s) = (2bk - s)(2x + 1) - pbx(x + 1), \quad x = \left\lfloor \frac{2bk - s}{pb} \right\rfloor,
\]

\[
h(s) = s(2y + 1) - by(y + 1), \quad y = \left\lfloor \frac{s}{b} \right\rfloor,
\]

\[
g(s; p, b, k) = \frac{p}{sh(s)/k} + \frac{(p - 1)^2}{2bk - s - a(s)/k - (s - h(s))/k}/p.
\]

For an \(S_0(p, b, k, g_0, g_1, \lambda_0, \lambda_1)\) design it can be shown that \(n_{d_0j} = \left\lfloor \frac{pg_0}{b} \right\rfloor\) or \(\left\lfloor \frac{pg_0}{b} \right\rfloor + 1, 2ks_0 = h(s_0) + p\lambda_0, 2ks_1 = h(s_1) + (p - 1)\lambda_1 + \lambda_0,\) and \(n_{dij} = \left\lfloor \frac{2bk - pg_0}{pb} \right\rfloor\) or \(\left\lfloor \frac{2bk - pg_0}{pb} \right\rfloor + 1, i = 1, \ldots, p; j = 1, \ldots, b.\)

We require the following lemmas for characterizing optimal type \(S_0\) designs.
Lemma 2.1. If \( d \in \mathcal{D}(p+1,b,k) \) then \( \bar{M}_d \) has eigenvalues \( \mu_{d1}, \mu_{d2} = \cdots = \mu_{dp} \) with
\[
\mu_{d1} = \frac{ks_d0 - \sum_{j=1}^b n_{d0j}^2}{pk}, \quad \mu_{d2} = \frac{2bk - s_d0 - \frac{1}{k} \sum_{j=1}^b n_{dij}^2}{p-1} - \mu_{d1}.
\]

Proof. From (2.1) and (2.2), the entries of \( \bar{M}_d \) are
\[
m_{div'} = \left\{
\begin{array}{ll}
s_{di'j} - \frac{1}{k} \sum_{j=1}^b n_{dij}^2 & (i = i') \\
g_{div'} - \frac{1}{k} \sum_{j=1}^b n_{dij}^2 & (i \neq i')
\end{array}
\right.
\]
and the sum of the entries in the \( i \)th row (or \( i' \)th column) is
\[
\sum_{i'=1}^p m_{div'} = -g_{dii} + \frac{1}{k} \sum_{j=1}^b n_{dij} n_{div'j}.
\]
The lemma then follows by noting that the \( \bar{M}_d = \xi_1 I_p + \xi_2 p I_p' \) with \( \xi_2 = \frac{1}{p(p-1)} \sum_{1 \leq i \neq i' \leq p} m_{div'} \) and \( \xi_1 = \frac{1}{p} \sum_{i=1}^p m_{di} - m_{d2} \).

Lemma 2.2 [Cheng, 1978]. For given positive integers \( v \) and \( t \), the minimum of \( \sum_{i=1}^v n_i^2 \) subject to \( \sum_{i=1}^v n_i = t \), where \( n_i \)'s are non-negative integers, is obtained when \( t - v[t/v] \) of the \( n_i \)'s are equal to \( [t/v] + 1 \) and \( v - t + v[t/v] \) are equal to \( [t/v] \). The corresponding minimum of \( \sum_{i=1}^v n_i^2 \) is \( t(2[t/v] + 1) - v[t/v]([t/v] + 1) \).

Lemma 2.3. Let \( d \in \mathcal{D}(p+1,b,k) \) and \( n_{d01}, \ldots, n_{d0b} \) be fixed quantities. Then
\[
tr(M_{d1}^{-1}) \geq \mu_{d1}^{-1} + (p-1)^2 \{2bk - s_d0 - a(s_d0)/k - \mu_{d1}\}^{-1} (= \theta_d, \text{ say}). \tag{2.4}
\]

Proof. Using Lemma 2.2 we have \( \sum_{i=1}^p \sum_{j=1}^b n_{dij}^2 \geq a(s_d0) \). Thus from (2.3) we get
\[
tr(M_{d1}^{-1}) \geq \mu_{d1}^{-1} + (p-1)\mu_{d2}^{-1} \geq \mu_{d1}^{-1} + (p-1)^2 \{2bk - s_d0 - a(s_d0)/k - \mu_{d1}\}^{-1}.
\]

Lemma 2.4. Suppose \( d \in \mathcal{D}(p+1,b,k) \) satisfies \( \sum_{i=1}^p \sum_{j=1}^b n_{dij}^2 = a(s_d0) \) and has \( s_d0 > b[k^2] \). Then there exists a design \( d^* \) satisfying (i) \( \sum_{i=1}^p \sum_{j=1}^b n_{d^*ij}^2 = a(s_{d^*0}) \) and (ii) \( s_{d^*0} \leq b[k^2] \) such that \( \theta_d \geq \theta_{d^*} \) unless (i) \( p = 5, k = 3 \), (ii) \( p = 4, k \) odd and (iii) \( p = 3 \).

Proof. We replace \( d \) by a \( d^* \) which is such that
\[
n_{d^*0j} = n_{d0j} \quad \text{if} \quad n_{d0j} \leq \left\lfloor \frac{k}{2} \right\rfloor \quad \text{and} \quad n_{d^*0j} = k - n_{d0j} \quad \text{if} \quad n_{d0j} > \left\lfloor \frac{k}{2} \right\rfloor.
\]
Clearly, \( s_{d^*0} < s_{d0} \), and \( s_{d^*0} \) satisfies \( s_{d^*0} \leq b[k^2] \). Also, \( \mu_{d^*1} = \mu_{d1} \). The result then follows by noting that the function \( \psi(s_{d0}) = 2bk - s_{d0} - a(s_{d0})/k \) decreases as \( s_{d0} \) increases except when (i) \( p = 5, k = 3 \), (ii) \( p = 4, k \) odd and (iii) \( p = 3 \).

Lemma 2.5. Let \( d \in \mathcal{D}(p+1,b,k) \). Then
\[
\theta_d \geq pk\{(ks_0 - h(s_{d0}))^{-1} + (p-1)^2(pk(2bk - s_{d0}) - pa(s_{d0}) - ks_{d0} + h(s_{d0}))^{-1}\} \tag{2.5}
\]
where \( \theta_d \) is the same as in (2.4).

Proof. From Lemma 2.1 and equation (2.4) we have
\[
\theta_d = pk\{(ks_0 - \Sigma_{j=1}^b n_{d0j}^2)^{-1} + (p-1)^2(pk(2bk - s_{d0}) - pa(s_{d0}) - ks_{d0} + \Sigma_{j=1}^b n_{d0j}^2)^{-1}\}
= pk\{(ks_0 - q)^{-1} + (p-1)^2(w + q)^{-1}\},
\]
where \( q = \sum_{i=1}^{b} n_{i0} \) and \( w = pk(2bk - s_{d0}) - pa(s_{d0}) - ks_{d0}. \) For fixed \( s_{d0}, \frac{s_{d0}^2}{b} \leq q \leq ks_{d0}. \) We shall prove that
\[
\frac{\delta \theta_d}{\delta q} \geq 0 \text{ for all } q \in \left[ \frac{s_{d0}^2}{b}, ks_{d0} \right].
\] (2.6)

Inequality (2.5) will then follow from (2.6) since a sharp lower bound for \( q \) is \( h(s_{d0}) \). To prove (2.6), it is enough to show that \((w + q)^2 \geq (p - 1)^2(ks_{d0} - q)^2\). Equivalently, since both \( ks_{d0} - q \) and \( w + q \) are non-negative, we need only show that \( w + q \geq (p - 1)(ks_{d0} - q), \) i.e., \( q \geq 2ks_{d0} + a(s_{d0}) - 2bk^2 \). Thus, since \( q \geq \frac{s_{d0}^2}{b} \), it is sufficient to show that \( \frac{s_{d0}^2}{b} \geq 2ks_{d0} + a(s_{d0}) - 2bk^2 \).

Now we consider separately the two cases (i) \( 2k \leq p \) and (ii) \( 2k > p \).

**Case (i):** \( 2k \leq p \). Here \( a(s_{d0}) = 2bk - s_{d0} \) since \( x = \frac{2bk - s_{d0}}{pb} = 0. \) Therefore, we need to show \( \frac{s_{d0}^2}{b} \geq 2ks_{d0} + 2bk - s_{d0} - 2bk^2 \), or \( s_{d0} \{s_{d0} - b(2k - 1)\} + 2b^2k(k - 1) \geq 0. \) This inequality holds because \( s_{d0} \leq bk/2 \), a consequence of Lemma 2.4.

**Case (ii):** \( 2k > p \). Here \( a(s_{d0}) \leq \frac{2bk^2}{pb} + \frac{pb}{4} - 2bk^2 \), or \( 4(p - 1)s_{d0}^2 - 8bk(p - 2)s_{d0} + 8b^2k^2(p - 2) - p^2b^2 \geq 0. \) This inequality holds whenever \( s_{d0} \leq bk/2 \). It is easy to see that the inequality also holds for the particular cases (i) \( p = 5, k = 3 \), (ii) \( p = 4, k \) odd and (iii) \( p = 3 \). Using Lemma 2.4, the inequality holds for the other cases as well.

Finally, the following theorem can be established using Lemmas 2.1 - 2.5.

**Theorem 2.1.** Suppose \( s_0 \) is an integer defined by
\[
g(s_0; p, b, k) = \min_{1 \leq s \leq c} g(s; p, b, k),
\] (2.7)

where \( c = bk \) if (i) \( p = 5, k = 3 \), (ii) \( p = 4, k \) odd or (iii) \( p = 3, \) else \( c = b\lfloor \frac{k}{2} \rfloor \). Then a type \( S_0 \)-block design \( S_0(p, b, k, g_0, g_1, \lambda_0, \lambda_1) \) with \( g_0 = \frac{s_0}{p}, \) \( g_1 = \frac{s_1 - s_0}{p - 1}, \) \( \lambda_0 = \frac{2ks_0 - h(s_0)}{p} \) and \( s_1 = \frac{2bk - s_0}{p} \) is optimal in \( D(p + 1, b, k). \)

The integer \( s \) which minimizes \( g(s; p, b, k) \) can easily be found using a computer.

**Example 2.1.** For \( p = 5, b = 10, k = 2 \), the \( g(s; 5, 10, 2) \) is minimized for \( s = 10 \). Thus the following \( S_0(5, 10, 2, 2, 1, 6, 3) \) design \( d^* \) with \( s_{d^*} = s_0 = 10 \) is optimal over \( D(6, 10, 2) : \)
\[
\{(3,5) (0,1)\}, \{(1,4) (0,2)\}, \{(2,5) (0,3)\}, \{(1,3) (0,4)\}, \{(2,4) (0,5)\},
\{(4,5) (0,1)\}, \{(1,5) (0,2)\}, \{(1,2) (0,3)\}, \{(2,3) (0,4)\}, \{(3,4) (0,5)\}.
\]

Theorem 2.1 is useful in checking the optimality of type \( S_0 \) designs. Choi, Gupta and Kageyama (2002) gave some series of type \( S \) designs. Designs of their Series 1, 3 and 4 are type \( S_0 \) designs as they also satisfy the requirements of Definition 2.2. Among designs for \( p \leq 30 \), Series 1 designs for \( p = 3, 5, 7, 9 \) are optimal. Clearly, not all type \( S_0 \) designs are optimal. In the next section we give a lower bound \( e_{sd} \) to efficiency of a type \( S_0 \) design \( d \), and show that type \( S_0 \) designs are highly efficient for control-test comparisons.
We now present a new method of constructing type $S_0$ designs. Following Gupta and Kageyama (1994), let $d_n$ be a universally optimal block design for diallel crosses obtained using a nested balanced incomplete block design with parameters $v = p, b_n, r_n, k_n < p, \lambda_n$, the nest having block size two. Let $B_i$ denote the $i$th block of the nested balanced incomplete block design, and let $\bar{B}_i$ denote the corresponding complementary block such that the contents of $B_i$ and $\bar{B}_i$ taken together form one replication of the lines $i = 1, \ldots, p$. Let $i_1, i_2, \ldots, i_{p-k_n}$ denote the contents of $B_i$. Then, appending to the $i$th block of $d_n$ the crosses $(0, i_1), (0, i_2), \ldots, (0, i_{p-k_n})$ yields a type $S_0$ design of the following theorem.

**Theorem 2.2.** The existence of a nested balanced incomplete block design with parameters $v = p, b_n, r_n, k_n, \lambda_n$, the nest having block size two, implies the existence of an $S_0(p, b = b_n, k = p - k_n/2, g_0 = b_n(1 - k_n/p), g_1 = \frac{b_n k_n}{p(p - 1)}, \lambda_0 = b_n(p - k_n), \lambda_1 = b_n)$.

Several series of universally optimal block designs $d_n$ are available in literature, see e.g. Das, Dey and Dean (1998). Using each of these series, a corresponding series of type $S_0$ designs can be derived using Theorem 2.2. For instance, Das, Dey and Dean (1998) gave Family 1 designs with parameters $p = 4t + 1, b_n = t(4t + 1), k_n = 4, \lambda_n = 3$, where $t$ is a positive integer, and $p$ is a prime or prime power. Using this family of designs, Theorem 2.2 yields a series of type $S_0$ designs with parameters $p = 4t + 1, b = t(4t + 1), k = 4t - 1, g_0 = t(4t - 3), g_1 = 1$, where $t \geq 1$, and $p$ is a prime or prime power.

Morgan, Preece and Rees (2001) tabulated all possible nested balanced incomplete block designs for $p \leq 16$, and $r_n \leq 30$. Using designs in their table with nested design having block size 2, Theorem 2.2 yields a total of 24 type $S_0$ designs. Type $S_0$ designs obtained from designs in their table at serial numbers 6, 13, c, 40, 50, c and 62 are optimal as they satisfy the condition of Theorem 2.1. Of the remaining 19 designs, 7 designs have $e_{Ad} \geq 0.95$, 6 designs have $0.80 \leq e_{Ad} < 0.95$, 5 designs have $0.70 \leq e_{Ad} < 0.80$, and 1 design has $e_{Ad} = 0.66$. The $e_{Ad}$ is the lower bound to efficiency as defined in the next section.

### 3. Efficiency

The efficiency of a design $d \in D(p + 1, b, k)$ for control-test comparisons compared to an optimal design $d_A \in D(p + 1, b, k)$ is defined as

$$E_{Ad} = \frac{\sum_{i=1}^{p} Var(\tilde{\tau}_{d_{Ai}} - \hat{\tau}_{d_{A0}})}{\sum_{i=1}^{p} Var(\tilde{\tau}_{d_{i}} - \hat{\tau}_{d_{0}})}.$$

Clearly, $\sum_{i=1}^{p} Var(\tilde{\tau}_{d_{Ai}} - \hat{\tau}_{d_{A0}}) \geq g(s_0, p, b, k)$, where $s_0$ is as in Theorem 2.1. Therefore, based on Theorem 2.1, a lower bound to the efficiency of a type $S_0$ design $d$ with parameters $p, b, k, g_0, g_1, \lambda_0, \lambda_1$ is given by

$$e_{Ad} = g(s_0; p, b, k)/B_{0d},$$

where $B_{0d} = g(s_{d_0}; p, b, k)$. Since the efficiency of a design $d$ is greater than or equal to $e_{Ad}$, high values of $e_{Ad}$ indicate that the design $d$ is highly efficient, and hence approximately optimal for control-test comparisons. The design $d$ is optimal if $e_{Ad} = 1.0$. 

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Example 3.1. The following type $S_0$ design $d$ for $p = 8$, $b = 10$, $k = 6$ with $e_{Ad} = 0.983$ is approximately optimal:

$\{(1,2) \ (3,5) \ (4,7) \ (0,8) \ (0,1)\}$, $\{(2,3) \ (4,6) \ (5,8) \ (0,7) \ (0,1) \ (0,2)\}$,
$\{(3,4) \ (5,7) \ (1,6) \ (0,8) \ (0,2) \ (0,3)\}$, $\{(4,5) \ (6,8) \ (2,7) \ (0,1) \ (0,3) \ (0,4)\}$,
$\{(5,6) \ (1,7) \ (3,8) \ (0,2) \ (0,4) \ (0,5)\}$, $\{(6,7) \ (2,8) \ (1,4) \ (0,3) \ (0,5) \ (0,6)\}$,
$\{(7,8) \ (1,3) \ (2,5) \ (0,4) \ (0,6) \ (0,7)\}$, $\{(1,8) \ (2,4) \ (3,6) \ (0,5) \ (0,7) \ (0,8)\}$,
$\{(1,5) \ (2,6) \ (0,3) \ (0,4) \ (0,7) \ (0,8)\}$, $\{(3,7) \ (4,8) \ (0,1) \ (0,2) \ (0,5) \ (0,6)\}$.

The lower bound $e_{Ad}$ was computed using (3.1) for all type $S_0$ designs in the practical range $p \leq 30$, $b \leq 50$, $k \leq p$, $g_0 \leq 10$, $1 < g_0/g_1 \leq 5$. Of the 247 possible type $S_0$ designs in this range, 70 designs are optimal, i.e. have $e_{Ad} = 1.0$. Of the remaining 177 designs that do not satisfy condition (2.7) for optimality, 96 designs have $e_{Ad} \geq 0.95$, 52 designs have $0.90 \leq e_{Ad} < 0.95$, 27 designs have $0.81 \leq e_{Ad} < 0.90$, and 2 designs have $e_{Ad} = 0.77$, and 0.70 respectively. For the sake of brevity, these highly efficient type $S_0$ designs are not tabulated here, and they will be reported elsewhere.

References


