isid/ms/2014/1
January 4, 2014
http://www.isid.ac.in/~statmath/eprints

From random matrices to long range dependence

Arijit Chakrabarty, Rajat Subhra Hazra and Deepayan Sarkar

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

FROM RANDOM MATRICES TO LONG RANGE DEPENDENCE

ARIJIT CHAKRABARTY, RAJAT SUBHRA HAZRA, AND DEEPAYAN SARKAR

Abstract. Random matrices whose entries come from a stationary Gaussian process are studied. The limiting behavior of the eigenvalues as the size of the matrix goes to infinity is the main subject of interest in this work. It is shown that the limiting spectral distribution is determined by the absolutely continuous component of the spectral measure of the stationary process, a phenomenon resembling that in the situation where the entries of the matrix are i.i.d. On the other hand, the discrete component contributes to the limiting behavior of the eigenvalues in a completely different way. Therefore, this helps to define a boundary between short and long range dependence of a stationary Gaussian process in the context of random matrices.

1. Introduction

The notion of *long range dependence* is of significant importance in the field of stochastic processes. Consider any stationary stochastic process indexed by $\mathbb{Z}$. If the process is an i.i.d. collection, then it does not have any memory, and hence it is *short range dependent*. For a general stochastic process which is not necessarily i.i.d., whether it is long or short range dependent is determined by how much it resembles an i.i.d. collection. In order to make the idea of resemblance precise, different functionals of the process are studied. If the behavior of a functional of interest is close to that in the i.i.d. setup, then the process is short range dependent, otherwise it is long range dependent. Therefore, the definition of long range dependence varies widely with context, and it is no wonder that there are numerous definitions of this concept in the literature, which are not equivalent. The survey article by Samorodnitsky (2006) describes in detail this notion from various points of view.

The current paper is an attempt to understand long range dependence in yet another context, namely that of random matrices. Let $\{X_{j,k} : j, k \in \mathbb{Z}\}$ be a **real** stationary Gaussian process with zero mean and positive variance. That means,

$$\mathrm{E}(X_{j,k}) = 0\,,$$
$$\mathrm{E}\left(X_{j,k}^2\right) > 0\,,$$

2010 *Mathematics Subject Classification.* Primary 60B20; Secondary 60B10, 46L53.

Key words and phrases. Random matrix, long range dependence, stationary Gaussian process, spectral density.

and

$$\mathrm{E}\left(X_{j,k}X_{j+u,k+v}\right)$$

is independent of j and k for all fixed $u, v \in \mathbb{Z}$. For $N \geq 1$, define a $N \times N$ matrix W_N by

$$W_N(i,j) := X_{i,j} + X_{j,i}\,, \tag{1.1}$$

for all $1 \leq i, j \leq N$. Clearly, W_N is symmetric by construction, and hence its eigenvalues are all real. For any $N \times N$ symmetric matrix A, denote its eigenvalues by $\lambda_1(A) \leq \ldots \leq \lambda_N(A)$, and define its empirical spectral distribution, henceforth abbreviated to ESD, by

$$\mathrm{ESD}(A) := \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j(A)}\,.$$

Section 2 lists the main results of the paper. That section is divided into three subsections. In Subsection 2.1, the results that study the limit of $\mathrm{ESD}(W_N/\sqrt{N})$ as $N \to \infty$ are listed, the main result being Theorem 2.1. In Subsection 2.2, a variant of the ESD called eigen measure is defined. The main result of that subsection, Theorem 2.6, studies the limit of the eigen measure of W_N/N as $N \to \infty$. The above two theorems motivate a natural definition of long range dependence, which is discussed in Subsection 2.3. The proofs of the results mentioned in Section 2 are given in Section 3. Finally, in Section 4, the paper is concluded with a corollary and a few examples.

We end this section by pointing out that Theorem 2.1 is actually an extension of the classical result by Wigner which says that if $X_{i,j}$ are i.i.d. standard normal random variables, then $\mathrm{ESD}(W_N/\sqrt{N})$ converges to the Wigner semicircle law (defined in (2.5)). Relaxation of the independence assumption has previously been investigated by Chatterjee (2006), Götze and Tikhomirov (2005), Hofmann-Credner and Stolz (2008) and Rashidi Far et al. (2008). The articles by Adamczak (2011), Hachem et al. (2005), Naumov (2012), Nguyen and Rourke (2012) and Pfaffel and Schlemm (2012) have studied the sample covariance matrix and non-symmetric matrices after imposing some dependence structures. A work by Anderson and Zeitouni (2008), which is related to the current paper, considered the ESD of Wigner matrices where on and off diagonal elements form a finite-range dependent random field; in particular, the entries are assumed to be independent beyond a finite range, and within the finite range the correlation structure is given by a kernel function. The results of the current paper, however, are more general than those therein.

2. The results

Define

$$R(u,v) := \mathrm{E}\left(X_{0,0}X_{u,v}\right),\ u, v \in \mathbb{Z}\,.$$

The Herglotz representation theorem asserts that there exists a finite measure ν on $(-\pi,\pi]^2$ such that

$$R(u,v)=\int_{(-\pi,\pi]^2} e^{\iota(ux+vy)}\nu(dx,dy) \text{ for all } u,v\in\mathbb{Z}\,, \tag{2.1}$$

where $\iota:=\sqrt{-1}$. Let ν_{ac}, ν_{cs} and ν_d denote the components of ν which are absolutely continuous with respect to the Lebesgue measure, continuous and singular with respect to the Lebesgue measure, that is, supported on a set of measure zero, and discrete, that is, supported on a countable set, respectively. Since ν_{ac} is absolutely continuous with respect to the Lebesgue measure, there exists a function f from $[-\pi,\pi]^2$ to $[0,\infty)$ such that

$$\nu_{ac}(dx,dy)=f(x,y)dxdy\,. \tag{2.2}$$

The one and only **assumption** of this paper is that the continuous and singular component is absent, that is,

$$\nu_{cs}\equiv 0\,.$$

As a consequence, it follows that

$$\nu=\nu_{ac}+\nu_d\,. \tag{2.3}$$

2.1. **The empirical spectral distribution.** Denote

$$\mu_N:=\mathrm{ESD}(W_N/\sqrt{N}),\ N\ge 1\,, \tag{2.4}$$

where W_N is as in (1.1).

The task of this subsection is to list the results that study the limiting spectral distribution (henceforth LSD) of $W_N/\sqrt{N}$, that is, the limit of the random probability measures μ_N as $N\to\infty$. The first result, Theorem 2.1 below, establishes that the limit exists.

Theorem 2.1. *There exists a deterministic probability measure μ_f, determined solely by the spectral density f which is as in (2.2), such that*

$$\mu_N\to\mu_f\,,$$

weakly in probability as $N\to\infty$. By saying that the LSD μ_f is determined by f, the following is meant. If for two stationary processes satisfying assumption (2.3), the absolutely continuous component of the corresponding spectral measures match, then the LSD of the scaled symmetric random matrices formed by them also agree.

The exact description of μ_f is complicated, and will come much later in Remark 3.1. However, a natural question at this stage is "When is μ_f the probability measure degenerate at zero?". The following result answers this question.

Theorem 2.2. *The second moment of the probability measure μ_f is given by*

$$\int_{\mathbb{R}} x^2\mu_f(dx)=2\int_{[-\pi,\pi]^2} f(x,y)dxdy\,.$$

The next result relates some properties of μ_f with those of f. Some new notations will be needed for stating that result, which we now introduce. Fix $m \geq 1$ and $\sigma \in NC_2(2m)$, the set of non-crossing pair partitions of $\{1, \ldots, 2m\}$. Let $(V_1, \ldots, V_{m+1})$ denote the Kreweras complement of σ, which is the maximal partition $\overline{\sigma}$ of $\{\overline{1}, \ldots, \overline{2m}\}$ such that $\sigma \cup \overline{\sigma}$ is a non-crossing partition of $\{1, \overline{1}, \ldots, 2m, \overline{2m}\}$. Although the Kreweras complement is a partition of $\{\overline{1}, \ldots, \overline{2m}\}$, for the ease of notation, $V_1, \ldots, V_{m+1}$ will be thought of as subsets of $\{1, \ldots, 2m\}$, that is, the overline will be suppressed. In order to ensure uniqueness in the notation, we impose the requirement that the blocks $V_1, \ldots, V_{m+1}$ are ordered in the following way. If $1 \leq i < j \leq m+1$, then the **maximal** element of V_i is strictly less than that of V_j. Let $\mathcal{T}_\sigma$ be the unique function from $\{1, \ldots, 2m\}$ to $\{1, \ldots, m+1\}$ satisfying

$$i \in V_{\mathcal{T}_\sigma(i)},\ 1 \leq i \leq 2m\,.$$

For example, if

$$\sigma := \{(1,4), (2,3), (5,6)\}\,,$$

then $\mathcal{T}_\sigma(1) = 2, \mathcal{T}_\sigma(2) = 1, \mathcal{T}_\sigma(3) = 2, \mathcal{T}_\sigma(4) = 4, \mathcal{T}_\sigma(5) = 3, \mathcal{T}_\sigma(6) = 4$. For **any function** f from $[-\pi, \pi]^2$ to $\mathbb{R}$, define the function $L_{\sigma,f}$ from $[-\pi, \pi]^{m+1}$ to $\mathbb{R}$ by

$$L_{\sigma,f}(x) := \prod_{(u,v)\in\sigma} \left[f\left(x_{\mathcal{T}_\sigma(u)}, -x_{\mathcal{T}_\sigma(v)}\right) + f\left(-x_{\mathcal{T}_\sigma(v)}, x_{\mathcal{T}_\sigma(u)}\right) \right]\,,$$

for all $x \in [-\pi, \pi]^{m+1}$.

Theorem 2.3. *1. For $m \geq 2$, the $(2m)$-th moment of μ_f is finite if $\|f\|_m < \infty$. Here $\|f\|_p$ denotes the L^p norm of f for all $p \in [1, \infty]$.*
2. If $\|f\|_\infty < \infty$, then μ_f is compactly supported, and

$$\int_{\mathbb{R}} x^{2m} \mu_f(dx) = (2\pi)^{m-1} \sum_{\sigma \in NC_2(2m)} \int_{[-\pi,\pi]^{m+1}} L_{\sigma,f}(x) dx \text{ for all } m \geq 1\,.$$

Remark 2.1. *It is shown in Example 1 that the converses of the statements above is false.*

The last two results of this subsection gives neat descriptions of μ_f in two special cases. In what follows, $WSL(\gamma)$ for $\gamma > 0$ denotes the Wigner semicircle law with variance γ, that is, it is the law whose density is

$$\frac{1}{2\pi\sqrt{\gamma}} \sqrt{4 - x^2/\gamma}\, \mathbf{1}(|x| \leq 2\sqrt{\gamma})\,. \tag{2.5}$$

Theorem 2.4. *If there exists a function r from $[-\pi, \pi]$ to $[0, \infty)$ such that*

$$\frac{1}{2}\left[f(x,y) + f(y,x)\right] = r(x)r(y) \text{ for almost all } x, y \in [-\pi, \pi]\,,$$

then

$$\mu_f = \eta_r \boxtimes WSL(1)\,,$$

where η_r denotes the law of $2^{3/2}\pi r(U)$, U is an Uniform $(-\pi,\pi)$ random variable, and "$\boxtimes$" denotes the free multiplicative convolution.

Theorem 2.5. *Suppose that there exists finite subsets $A_1, A_2, \ldots$ of $\mathbb{Z}$ such that $A_n \uparrow \mathbb{Z}$. Define*

$$d_{j,k} := \frac{1}{2\sqrt{2}\,\pi}\int_{[-\pi,\pi]^2} e^{-\iota(jx+ky)}\sqrt{f(x,y)+f(y,x)}dxdy,\ j,k\in\mathbb{Z}\,.$$

If it holds that

$$\sum_{k,l\in\mathbb{Z}} d_{k,l}d_{j+k,l}\mathbf{1}(k,l,j+k\in A_n)=0 \text{ for all } j\in\mathbb{Z}\setminus\{0\} \text{ and } n\geq 1\,,$$

then

$$\mu_f = WSL(2\|f\|_1)\,.$$

2.2. **The eigen measure.** Theorem 2.1 shows that the discrete component of the spectral measure does not have a bearing on the limiting behavior of the ESD. Therefore, it is imperative to come up with a variant of the ESD that would capture the role of this component. That end is achieved in this subsection. The first task is to define the proper variant, which we now proceed towards.

It should be remembered that a symmetric matrix always means a $N\times N$ symmetric matrix for some finite N. A symmetric matrix A is to be thought of as a Hermitian operator $\overline{A}$ of finite rank acting on the first N coordinates of l^2, where

$$l^p := \left\{(a_n : n\in\mathbb{N})\subset\mathbb{R} : \sum_n |a_n|^p < \infty\right\},\ p\in[1,\infty)\,.$$

If $\lambda_1 \leq \ldots \leq \lambda_N$ are the eigenvalues of A counted with multiplicity, then the spectrum of $\overline{A}$ is $\{0,\lambda_1,\ldots,\lambda_N\}$, where 0 has infinite multiplicity. Motivated by this, we define the **eigen measure** of A, denoted by $\mathrm{EM}(A)$, by

$$\mathrm{EM}(A) := \infty\delta_0 + \sum_{j=1}^N \delta_{\lambda_j}\,.$$

The measure $\mathrm{EM}(A)$ is to be viewed as an element of the set $\mathcal{P}$ of point measures ξ of the form

$$\xi := \infty\delta_0 + \sum_{j=1}^\infty \delta_{\theta_j}\,,$$

where $(\theta_j : j\geq 1)$ is some sequence of real numbers. It is not hard to see why $\mathrm{EM}(A)$ is an element of $\mathcal{P}$ for a symmetric matrix A because θ_j can be taken to be zero after a stage. For $p\in[1,\infty)$, define a subfamily $\mathcal{C}_p$ of $\mathcal{P}$ by

$$\mathcal{C}_p := \left\{\mu\in\mathcal{P} : \int_{\mathbb{R}} |x|^p\mu(dx) < \infty\right\}\,.$$

Once again, it is easy to see that for any symmetric matrix A,

$$\mathrm{EM}(A) \in \mathcal{C}_p \text{ for all } p \geq 1\,.$$

Fix $p \geq 1$ and $\xi \in \mathcal{C}_p$. Clearly, there exist unique real numbers

$$\alpha_1(\xi) \geq \alpha_2(\xi) \geq \ldots \geq 0\,,$$

and

$$\alpha_{-1}(\xi) \leq \alpha_{-2}(\xi) \leq \ldots \leq 0\,,$$

such that

$$\xi = \infty\delta_0 + \sum_{j\neq 0} \delta_{\alpha_j(\xi)}\,,$$

where $\sum_{j\neq 0}$ means the sum over all non-zero integers. Define

$$d_p(\xi_1, \xi_2) := \left[\sum_{j\neq 0} |\alpha_j(\xi_1) - \alpha_j(\xi_2)|^p\right]^{1/p}, \; \xi_1, \xi_2 \in \mathcal{C}_p\,.$$

Given the natural bijection between $\mathcal{C}_p$ and l^p, it is immediate that $(\mathcal{C}_p, d_p)$ is a complete metric space. It is also worth noting that

$$\left|\left[\int_{\mathbb{R}} |x|^p \xi_1(dx)\right]^{1/p} - \left[\int_{\mathbb{R}} |x|^p \xi_2(dx)\right]^{1/p}\right| \leq d_p(\xi_1, \xi_2),\; \xi_1, \xi_2 \in \mathcal{C}_p\,. \tag{2.6}$$

The main result of this subsection is the following.

Theorem 2.6. *Under the assumption (2.3), there exists a random point measure ξ which is almost surely in $\mathcal{C}_2$ such that*

$$d_4\left(\mathrm{EM}(W_N/N), \xi\right) \xrightarrow{P} 0\,, \tag{2.7}$$

as $N \to \infty$, where W_N is as defined in (1.1). Furthermore, the distribution of ξ is determined by ν_d.

Remark 2.2. *It is trivial to see that $\mathcal{C}_2 \subset \mathcal{C}_4$, and hence one can talk about the d_4 distance between two point measures in $\mathcal{C}_2$.*

Remark 2.3. *There is a notion of convergence different from that in (2.7), namely "vague convergence". Suppose that $(\xi_n : 1 \leq n \leq \infty)$ are measures on $\mathbb{R}$ such that*

$$\xi_n(\mathbb{R} \setminus (-\varepsilon, \varepsilon)) < \infty \text{ for all } \varepsilon > 0, 1 \leq n \leq \infty\,.$$

Then ξ_n converges vaguely to ξ_∞ if for all $x < 0 < y$ with $\xi_\infty(\{x, y\}) = 0$, it holds that

$$\lim_{n\to\infty} \xi_n(\mathbb{R} \setminus (x, y)) = \xi_\infty(\mathbb{R} \setminus (x, y))\,.$$

The vague convergence defined above is same as the vague convergence on $[-\infty, \infty] \setminus \{0\}$ discussed on page 171 in Resnick (2007), for example. It can be proved without much difficulty that if $(\xi_n : 1 \leq n \leq \infty) \subset \mathcal{C}_p$ for some p such that

$$\lim_{n\to\infty} d_p(\xi_n, \xi_\infty) = 0\,, \tag{2.8}$$

then ξ_n converges to ξ_∞ vaguely. The converse is, however, not true, that is, (2.8) *is strictly stronger than vague convergence.*

If Theorem 2.6 is seen as an analogue of Theorem 2.1, then the next natural question should be the analogue of that answered in Theorem 2.2, namely whether ξ restricted to $\mathbb{R}\backslash\{0\}$ is non-null and necessarily random. Both these questions are answered in the affirmative in the case when $\nu_d((-\pi,\pi]^2) > 0$ by the following result.

Theorem 2.7. *If $\nu_d((-\pi,\pi]^2) > 0$, then the random variable*

$$\int_{\mathbb{R}} x^2 \xi(dx)$$

is positive almost surely, and non-degenerate.

2.3. **Long range dependence.** In this subsection, we make the connection between the random matrix models and the long range dependence mentioned in Section 1. Recalling the fact that for a family of i.i.d. Gaussian random variables, the spectral measure is absolutely continuous, Theorem 2.1 can be interpreted as a result about the "short range dependent" component of the process $\{X_{j,k} : j,k \in \mathbb{Z}\}$. Indeed, the LSD μ_f is completely determined by the absolutely continuous component of the spectral measure.

On the other hand, Theorem 2.6 establishes the connection between the discrete component of the spectral measure and the limiting eigen measure ξ. In the presence of atoms in the spectral measure, a stationary Gaussian process is considered to have a long memory for several reasons. For example, in that case, the process is non-ergodic; see Cornfeld et al. (1982). A trivial example of such a process is the following. Let G be a $N(0,1)$ random variable, and set $X_{j,k} := G$ for all j,k.

It is also worth noting that in addition to the transition from ESD to EM, the scaling also changes from $\sqrt{N}$ to N when passing from the former result to the latter. Therefore, it is clear that the absolutely continuous and discrete components of the process contribute only towards the LSD and the limiting eigen measure of W_N respectively, albeit with different scalings. The above observation suggests naturally the following definition of short and long range dependence.

Definition 1. *A mean zero stationary Gaussian process with positive variance indexed by $\mathbb{Z}^2$ is short range dependent if the corresponding spectral measure is absolutely continuous, and the same is long range dependent if the spectral measure is discrete, that is, supported on a countable set.*

The above definitions, of course, are not exhaustive in that there may be processes whose range of dependence is neither short nor long. That can be hoped to be resolved partially if the role of the component ν_{cs} is understood. This we leave aside for future research.

We conclude this discussion by pointing out that there are other contexts in which long and short range dependence is defined based on absolute continuity of the spectral measure. For example, Section 5 of Samorodnitsky

(2006) approaches long range dependence for a stationary second order process indexed by $\mathbb{Z}$ from the point of view of the growth rate of the variance of its partial sums. In particular, the definition given in (5.14) on page 194 therein is close to the definition given above, though not exactly the same.

3. Proofs

3.1. Proofs of Theorems 2.1-2.5. We now proceed towards the proof of Theorem 2.1. The proof is by the classical method of moments. However, as illustrated later by Example 3, the moments of the LSD need not be finite. Hence, some work is needed to get around that.

Define a map T from $(-\pi, \pi]$ to itself by

$$T(x) = -x\mathbf{1}(x < \pi) + \pi\mathbf{1}(x = \pi),\ -\pi < x \le \pi\,.$$

Since the integral on the right hand side of (2.1) is real for all u and v, it follows that ν is invariant under the transformation $(x, y) \mapsto (T(x), T(y))$, and in particular

$$\nu(\{(x, y)\}) = \nu(\{(T(x), T(y))\}) \text{ for all } x, y\,.$$

Since the measure ν_d is concentrated on a countable set, and

$$\nu_d(\{(x, y)\}) = \nu(\{(x, y)\}) \text{ for all } x, y\,,$$

it follows that ν_d is also invariant under the map T. By (2.3), it follows that ν_{ac} is also invariant under that map, that is,

$$f(x, y) = f(-x, -y) \text{ for almost all } (x, y) \in [-\pi, \pi]^2\,. \tag{3.1}$$

Therefore, for $k, l \in \mathbb{Z}$, $c_{k,l}$ defined by

$$c_{k,l} := (2\pi)^{-1} \int_{[-\pi,\pi]^2} e^{-\iota(kx+ly)} \sqrt{f(x, y)}\, dxdy\,, \tag{3.2}$$

is a real number. By Parseval's identity, it follows that

$$\sum_{k,l\in\mathbb{Z}} c_{k,l}^2 < \infty\,.$$

Let $(U_{i,j} : i, j \in \mathbb{Z})$ be i.i.d. $N(0, 1)$ random variables. Define

$$Y_{i,j} := \sum_{k,l\in\mathbb{Z}} c_{k,l} U_{i-k,j-l},\ i, j \in \mathbb{Z}\,. \tag{3.3}$$

An important result, on which the current paper is built, is the following fact which is well known in the literature of stationary processes.

Fact 3.1. *The process $(Y_{i,j} : i, j \in \mathbb{Z})$ defined in (3.3) is a stationary Gaussian process with*

$$\mathrm{E}\left(Y_{i,j} Y_{i+u,j+v}\right) = \int_{[-\pi,\pi]^2} e^{\iota(ux+vy)} f(x, y) dx\, dy,\ \textit{for all } u, v \in \mathbb{Z}\,.$$

Since ν_d is T invariant, it follows that

$$\nu_d = \sum_{j\geq 1} \frac{a_j}{2}\left(\delta_{(x_j,y_j)} + \delta_{(T(x_j),T(y_j))}\right),$$

for some at most countable set $\{(x_1,y_1),(x_2,y_2),\ldots\} \subset (-\pi,\pi]^2$ and non-negative numbers $a_1,a_2,\ldots$ such that $\sum_j a_j < \infty$. Since some of the a_j's can be zero, we can and do assume without loss of generality that the above set is countably infinite. Let $(V_{i,j} : i = 1,2,\, j \geq 1)$ be a family of i.i.d. $N(0,1)$ random variables which is independent of the family $(U_{i,j} : i,j \in \mathbb{Z})$. Define

$$Z_{i,j} := \sum_{k=1}^{\infty} \sqrt{a_k}\left[V_{1,k}\cos(ix_k + jy_k) + V_{2,k}\sin(ix_k+jy_k)\right],\, i,j \in \mathbb{Z}\,. \tag{3.4}$$

It can be verified by calculating the covariances that

$$(X_{i,j} : i,j \in \mathbb{Z}) \stackrel{d}{=} (Y_{i,j} + Z_{i,j} : i,j \in \mathbb{Z})\,.$$

Therefore, without loss of generality, we assume that

$$X_{i,j} = Y_{i,j} + Z_{i,j},\, i,j \in \mathbb{Z}\,. \tag{3.5}$$

Fix $n \geq 1$, and define

$$Y_{i,j,n} := \sum_{k,l=-n}^{n} c_{k,l} U_{i-k,j-l},\, i,j \in \mathbb{Z}\,, \tag{3.6}$$

and similarly,

$$Z_{i,j,n} := \sum_{k=1}^{n} \sqrt{a_k}\left[V_{1,k}\cos(ix_k+jy_k) + V_{2,k}\sin(ix_k+jy_k)\right],\, i,j,n \geq 1\,.$$

Set

$$\begin{aligned} \hat{f}_n(u,v) &:= \mathrm{E}\left[Y_{i,j,n}Y_{i+u,j+v,n}\right] \qquad (3.7)\\ &= \sum_{k,l\in\mathbb{Z}} c_{k,l}c_{k+u,l+v}\mathbf{1}(|k| \vee |l| \vee |k+u| \vee |l+v| \leq n)\,, \end{aligned}$$

for all $u,v \in \mathbb{Z}$. For $N,n \geq 1$, define the following $N \times N$ symmetric matrices:

$$\begin{aligned}
W_{N,n}(i,j) &:= Y_{i,j,n} + Y_{j,i,n}\,, && (3.8)\\
W_{N,\infty}(i,j) &:= Y_{i,j} + Y_{j,i}\,, && (3.9)\\
\overline{W}_{N,n}(i,j) &:= Y_{i,j,n} + Y_{j,i,n} + Z_{i,j,n} + Z_{j,i,n}\,, && (3.10)\\
\widetilde{W}_{N}(i,j) &:= Z_{i,j} + Z_{j,i}\,, && (3.11)\\
\widetilde{W}_{N,n}(i,j) &:= Z_{i,j,n} + Z_{j,i,n}\,, && (3.12)
\end{aligned}$$

for all $1 \leq i,j \leq N$.

Fix $m \geq 1$, and $\sigma \in NC_2(2m)$. Let $(V_1, \ldots, V_{m+1})$ denote the Kreweras complement of σ. For $1 \leq i \leq m+1$, denote

$$V_i := \{v_1^i, \ldots, v_{l_i}^i\} \,. \tag{3.13}$$

Define

$$S(\sigma) := \left\{ (k_1, \ldots, k_{2m}) \in \mathbb{Z}^{2m} : \sum_{j=1}^{l_s} k_{v_j^s} = 0,\ s = 1, \ldots, m+1 \right\}, \tag{3.14}$$

and

$$\beta_{n,2m} := \sum_{\sigma \in NC_2(2m)} \sum_{k \in S(\sigma)} \prod_{(u,v) \in \sigma} \left[\hat{f}_n(k_u, -k_v) + \hat{f}_n(k_v, -k_u) \right],\ m, n \geq 1 \,. \tag{3.15}$$

Notice that even though the set $S(\sigma)$ has infinite cardinality, only finitely many summands on the right hand side above are non-zero, because $\hat{f}_n(u, v)$ is 0 if $|u| \vee |v| > 2n$.

Our first step towards proving Theorem 2.1 is the following proposition.

Proposition 3.1. *For fixed $n \geq 1$, there exists a compactly supported symmetric probability measure $\mu_{f,n}$ whose $2m$-th moment is $\beta_{n,2m}$ for all $m \geq 1$. Furthermore,*

$$\mathrm{ESD}(W_{N,n}/\sqrt{N}) \to \mu_{f,n} \,,$$

weakly in probability, as $N \to \infty$.

Proof. The proof is by the method of moments. As is now standard in the literature, for executing the proof, it is sufficient to show that

$$\lim_{N \to \infty} N^{-(m+1)} \mathrm{E} \left[\mathrm{Tr} \left(W_{N,n}^{2m} \right) \right] = \beta_{n,2m} \text{ for all } m \geq 1 \,, \tag{3.16}$$

$$\lim_{N \to \infty} N^{-2(m+1)} \mathrm{Var} \left[\mathrm{Tr} \left(W_{N,n}^{2m} \right) \right] = 0 \text{ for all } m \geq 1 \,, \tag{3.17}$$

and

$$\limsup_{m \to \infty} \beta_{2m}^{1/2m} < \infty \,. \tag{3.18}$$

It is worth mentioning that the odd moments of the ESD can be safely ignored, because that they go to zero, is now routine.

We start with showing (3.16). To that end, fix $m \geq 1$, and for $i := (i_1, \ldots, i_{2m}) \in \mathbb{Z}^{2m}$, define

$$E_i := \mathrm{E} \left[\prod_{j=1}^{2m} \left(Y_{i_{j-1}, i_j, n} + Y_{i_j, i_{j-1}, n} \right) \right],$$

with the convention that $i_0 := i_{2m}$ for all $i \in \mathbb{Z}^{2m}$, a convention that will be followed throughout this proof. Recall that

$$\mathrm{E} \left[\mathrm{Tr} \left(W_{N,n}^{2m} \right) \right] = \sum_{i \in \{1, \ldots, N\}^{2m}} E_i \,.$$

Fix $\sigma \in NC_2(2m)$ and denote its Kreweras complement by $K(\sigma)$. For a tuple $i \in \mathbb{Z}^{2m}$, call i to be σ-Catalan if

$$|i_{u-1} - i_v| \vee |i_u - i_{v-1}| \le 2n \text{ for all } (u,v) \in \sigma\,,$$

and

$$|i_u - i_v| > 4n \text{ whenever } u,v \text{ are in distinct blocks of } K(\sigma)\,.$$

For $N \ge 1$, denote

$$Cat(\sigma, N) := \{i \in \{1,\dots,N\}^{2m} : i \text{ is } \sigma\text{-Catalan}\}\,.$$

In view of standard combinatorial arguments, it suffices to show that

$$\lim_{N\to\infty} N^{-(m+1)} \sum_{i\in Cat(\sigma,N)} E_i \tag{3.19}$$

$$= \sum_{k\in S(\sigma)} \prod_{(u,v)\in\sigma} \left[\hat f_n(k_u, -k_v) + \hat f_n(k_v, -k_u)\right]\,,$$

for all fixed σ in $NC_2(2m)$.

To that end, fix a $\sigma \in NC_2(2m)$. Let $V_1,\dots,V_{m+1}$ denote the blocks of $K(\sigma)$. Write

$$V_u = \{v_1^u, \dots, v_{l_u}^u\},\ u = 1,\dots,m+1\,,$$

where

$$v_1^u \le \dots \le v_{l_u}^u\,.$$

Then it can be shown that for all $i \in Cat(\sigma, N)$, there exist unique tuples $k(i) \in S(\sigma) \cap \{-2n,\dots,2n\}^{2m}$ and $j(i) \in \mathbb{Z}^{2m}$ such that

$$(j(i)_{u-1}, j(i)_u) = (j(i)_v, j(i)_{v-1}) \text{ for all } (u,v) \in \sigma\,, \tag{3.20}$$

and

$$i_{v_x^u} = j(i)_{v_x^u} + \sum_{w=1}^{x} k(i)_{v_w^u},\ x = 1,\dots,l_u,\ u = 1,\dots,m+1\,. \tag{3.21}$$

As a consequence, it follows that for all $(u,v) \in \sigma$,

$$i_u - i_{v-1} = k(i)_u\,, \tag{3.22}$$

$$\text{and } i_v - i_{u-1} = k(i)_v\,. \tag{3.23}$$

Notice that for fixed $(u,v) \in \sigma$ with $1 \le u < v \le 2m$,

$$|i_{v-1} - i_v| > 4n\,,$$

because $v-1$ and v cannot belong to the same block of $K(\sigma)$. Therefore,

$$|i_u - i_v| \ge |i_v - i_{v-1}| - |i_{v-1} - i_u| > 2n\,,$$

and hence

$$\mathrm{E}\left[Y_{i_{u-1},i_u,n} Y_{i_{v-1},i_v,n}\right] = \mathrm{E}\left[Y_{i_u,i_{u-1},n} Y_{i_v,i_{v-1},n}\right] = 0\,.$$

Furthermore, if $(u,v) \notin \sigma$, then by a similar reasoning, it can be shown that

$$\mathrm{E}\left[\left(Y_{i_{u-1},i_u,n} + Y_{i_u,i_{u-1},n}\right)\left(Y_{i_{v-1},i_v,n} + Y_{i_v,i_{v-1},n}\right)\right] = 0\,.$$

As a consequence of the above two identities, it follows that

$$\begin{aligned} E_i &= \prod_{(u,v)\in\sigma} \mathrm{E}\left[\left(Y_{i_{u-1},i_u,n} + Y_{i_u,i_{u-1},n}\right)\left(Y_{i_{v-1},i_v,n} + Y_{i_v,i_{v-1},n}\right)\right] \\ &= \prod_{(u,v)\in\sigma} \left[\hat{f}_n(k(i)_v, -k(i)_u) + \hat{f}_n(k(i)_u, -k(i)_v)\right], \end{aligned}$$

the last equality following from (3.22) and (3.23). It is, once again, easy to check that for fixed $k \in S(\sigma)$,

$$\lim_{N\to\infty} N^{-(m+1)} \#\{i \in Cat(\sigma, N) : k(i) = k\} = 1\,,$$

and hence (3.19) follows, which establishes (3.16). Proof of (3.17) follows by a similar combinatorial analysis which is analogous to the proof by method of moments for the classical Wigner matrix. Hence we omit that.

The proof will be complete if (3.18) can be shown. To that end observe that

$$\beta_{n,2m} \le \left(32n^2 \max_{|u|\vee|v|\le 2n} |\hat{f}_n(u,v)|\right)^m \#NC_2(2m)\,.$$

It can be shown by Stirling's approximation that

$$\#NC_2(2m) = O\left(4^m\right),$$

and hence (3.18) follows. This completes the proof. □

Recall the $N \times N$ random matrix $\overline{W}_{N,n}$ from (3.10). The second step in the proof of Theorem 2.1 is the following lemma.

Lemma 3.1. *For fixed $n \ge 1$, as $N \to \infty$,*

$$\mathrm{ESD}(\overline{W}_{N,n}/\sqrt{N}) \to \mu_{f,n}\,,$$

weakly in probability, where $\mu_{f,n}$ is as in the statement of Proposition 3.1.

For the proof of the above result, we shall use the following fact which follows from Theorem A.43 on page 503 in Bai and Silverstein (2010).

Fact 3.2. *If L denotes the Lévy distance between two probability measures, then for any two $N \times N$ real symmetric matrices A and B,*

$$L\left(\mathrm{ESD}(A), \mathrm{ESD}(B)\right) \le \frac{1}{N}\,\mathrm{Rank}(A - B)\,.$$

Proof of Lemma 3.1. All that needs to be shown is that

$$L\left(\mathrm{ESD}(\overline{W}_{N,n}/\sqrt{N}), \mu_{f,n}\right) \xrightarrow{P} 0\,,$$

as $N \to \infty$. In view of Proposition 3.1, it suffices to show that

$$L\left(\mathrm{ESD}(\overline{W}_{N,n}/\sqrt{N}), \mathrm{ESD}(W_{N,n}/\sqrt{N})\right) \xrightarrow{P} 0\,,$$

as $N \to \infty$. To that end, notice that by Fact 3.2,

$$L\left(\mathrm{ESD}(\overline{W}_{N,n}/\sqrt{N}), \mathrm{ESD}(W_{N,n}/\sqrt{N})\right) \leq \frac{1}{N}\,\mathrm{Rank}\left(\overline{W}_{N,n} - W_{N,n}\right).$$

It is easy to see that the rank of the $N \times N$ matrix whose (i,j)-th entry is $Z_{i,j,n}$ is at most $4n$. Therefore,

$$\mathrm{Rank}\left(\overline{W}_{N,n} - W_{N,n}\right) \leq 8n\,.$$

This completes the proof. □

For the final step in the proof of Theorem 2.1, we shall use the following fact which is also well known.

Fact 3.3. *Let (Σ, d) be a complete metric space, and let $(\Omega, \mathcal{A}, P)$ be a probability space. Suppose that $(X_{mn} : (m,n) \in \{1,2,\ldots,\infty\}^2 \setminus \{\infty,\infty\})$ is a family of random elements in Σ, that is, measurable maps from Ω to Σ, the latter being equipped with the Borel σ-field induced by d. Assume that*

(1) *for all fixed $1 \leq m < \infty$,*
$$d(X_{mn}, X_{m\infty}) \xrightarrow{P} 0\,,$$
as $n \to \infty$,

(2) *and, for all $\varepsilon > 0$,*
$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left[d(X_{mn}, X_{\infty n}) > \varepsilon\right] = 0\,.$$

Then, there exists a random element $X_{\infty\infty}$ of Σ such that

$$d(X_{m\infty}, X_{\infty\infty}) \xrightarrow{P} 0\,, \tag{3.24}$$

as $m \to \infty$, and

$$d(X_{\infty n}, X_{\infty\infty}) \xrightarrow{P} 0\,,$$

as $n \to \infty$. Furthermore, if $X_{m\infty}$ is deterministic for all m, then so is $X_{\infty\infty}$, and then (3.24) simplifies to

$$\lim_{m\to\infty} d(X_{m\infty}, X_{\infty\infty}) = 0\,. \tag{3.25}$$

Proof of Theorem 2.1. The space of probability measures on $\mathbb{R}$ is a complete metric space when equipped with the Lévy distance $L(\cdot,\cdot)$. In view of Lemma 3.1 and Fact 3.3, all that needs to be shown to complete the proof is that

$$\lim_{n\to\infty} \limsup_{N\to\infty} P\left(L\left(\mathrm{ESD}(W_N/\sqrt{N}), \mathrm{ESD}(\overline{W}_{N,n}/\sqrt{N})\right) > \varepsilon\right) = 0\,, \tag{3.26}$$

for all $\varepsilon > 0$. To that end, fix $\varepsilon > 0$ and observe that

$$\begin{aligned} & P\left(L\left(\mathrm{ESD}(W_N/\sqrt{N}), \mathrm{ESD}(\overline{W}_{N,n}/\sqrt{N})\right) > \varepsilon\right) \\ \leq\ & \varepsilon^{-3}\mathrm{E}\left[L^3\left(\mathrm{ESD}(W_N/\sqrt{N}), \mathrm{ESD}(\overline{W}_{N,n}/\sqrt{N})\right)\right] \\ \leq\ & \varepsilon^{-3}N^{-2}\mathrm{E}\,\mathrm{Tr}\left[(W_N - \overline{W}_{N,n})^2\right] , \end{aligned}$$

the inequality in the last line following from the Hoffman-Wielandt inequality; see Corollary A.41 on page 502 in Bai and Silverstein (2010). Clearly, by (3.5), it follows that

$$\begin{aligned} & \mathrm{E}\,\mathrm{Tr}\left[(W_N - \overline{W}_{N,n})^2\right] && (3.27) \\ \leq\ & 4\sum_{i,j=1}^{N}\left[\mathrm{E}\left[(Y_{i,j} - Y_{i,j,n})^2\right] + \mathrm{E}\left[(Z_{i,j} - Z_{i,j,n})^2\right]\right] \\ =\ & 4N^2\left[\sum_{k=n+1}^{\infty} a_k + \sum_{i,j\in\mathbb{Z}:|i|\vee|j|>n} c_{i,j}^2\right] . && (3.28) \end{aligned}$$

This establishes (3.26). Fact 3.3 ensures the existence of a deterministic probability measure μ_f such that

$$L\left(\mathrm{ESD}(W_N/\sqrt{N}), \mu_f\right) \xrightarrow{P} 0\,,$$

as $N \to \infty$.

Furthermore, assertion (3.25) ensures that

$$\mu_{f,n} \xrightarrow{w} \mu_f \text{ as } n \to \infty\,. \tag{3.29}$$

From the definition, it is easy to see that $\mu_{f,n}$ is determined by f for every $n \geq 1$, and hence so is μ_f. This completes the proof of Theorem 2.1. □

Remark 3.1. *Since $\mu_{f,n}$ are compactly supported for each n, its characteristic function is*

$$\int_{\mathbb{R}} e^{itx}\mu_{f,n}(dx) = 1 + \sum_{m=1}^{\infty}(-1)^m\frac{\beta_{n,2m}}{(2m)!}t^{2m},\ t \in \mathbb{R}\,.$$

Thus, the characteristic function of μ_f is

$$\int_{\mathbb{R}} e^{itx}\mu_f(dx) = 1 + \lim_{n\to\infty}\sum_{m=1}^{\infty}(-1)^m\frac{\beta_{n,2m}}{(2m)!}t^{2m},\ t \in \mathbb{R}\,.$$

It is worth noting that exchanging the sum and limit above does not make sense because $\lim_{n\to\infty}\beta_{n,2m}$ may or may not be finite. Example 3 is one where the limit is infinite for all $m \geq 2$.

Proof of Theorem 2.2. Denote the probability space on which we were working so far by $(\Omega, \mathcal{A}, P)$. In particular, the random matrices $W_{N,n}$ are defined on this probability space. Consider the interval $(0, 1)$ equipped with the standard Borel σ-field $\mathcal{B}((0, 1))$ and the Lebesgue measure *Leb* which when restricted to $(0,1)$ becomes a probability measure. Define a master probability space

$$(\Omega \times (0,1), \mathcal{A} \times \mathcal{B}((0,1)), \mathbb{P} := P \times Leb) .$$

Denote the expectation with respect to $\mathbb{P}$ by $\mathbb{E}$. By Proposition 3.1 and the Cantor diagonalization principle, one can choose positive integers $N_1 < N_2 < N_3 < \ldots$ such that for all fixed $n \ge 1$,

$$\mathrm{ESD}(W_{N_k,n}/\sqrt{N_k}) \to \mu_{f,n} \text{ as } k \to \infty ,$$

weakly **almost surely**, that is,

$$\lim_{k\to\infty} L\left(\mathrm{ESD}(W_{N_k,n}/\sqrt{N_k}), \mu_{f,n}\right) = 0 \text{ almost surely,} \tag{3.30}$$

for all fixed $n \ge 1$, where L is the Lévy distance. For $1 \le k, n < \infty$, we define a random variables $\chi_{k,n}$ on $\Omega \times (0,1)$ by

$$\chi_{k,n}(\omega, x) := N_k^{-1/2} \lambda_{\lceil N_k x \rceil} \left(W_{N_k,n}(\omega)\right), \, \omega \in \Omega, x \in (0,1) .$$

Furthermore, for all k, define

$$\chi_{k,\infty}(\omega, x) := N_k^{-1/2} \lambda_{\lceil N_k x \rceil} \left(W_{N_k,\infty}(\omega)\right), \, \omega \in \Omega, x \in (0,1) ,$$

where $W_{N,\infty}$ is as in (3.9). Finally, for all $n \ge 1$, define

$$\chi_{\infty,n}(\omega, x) := F_n^{\leftarrow}(x), \, \omega \in \Omega, x \in (0,1) ,$$

where $F_n(\cdot)$ is the c.d.f. corresponding to $\mu_{f,n}$, and for any c.d.f. $F(\cdot)$, $F^{\leftarrow}(\cdot)$ is defined by

$$F^{\leftarrow}(y) := \inf \{x \in \mathbb{R} : F(x) \ge y\}, \, 0 < y < 1 .$$

Our first goal is to show that for all fixed $1 \le n < \infty$,

$$\chi_{k,n} \to \chi_{\infty,n} \ \mathbb{P}\text{-almost surely, as } k \to \infty . \tag{3.31}$$

To that end, define the set

$$A := \left\{ \omega \in \Omega : \lim_{k\to\infty} L\left(\mathrm{ESD}(W_{N_k,n}(\omega)/\sqrt{N_k}), \mu_{f,n}\right) = 0 \text{ for all } n \ge 1 \right\} .$$

By (3.30), it follows that $P(A) = 1$. Therefore, for establishing (3.31), it suffices to show that for all $\omega \in A$,

$$\chi_{k,n}(\omega, x) \to \chi_{\infty,n}(\omega, x) \text{ as } k \to \infty \text{ for almost all } x \in (0,1) . \tag{3.32}$$

To that end, fix $\omega \in A$. If $F_{k,n}$ denotes the c.d.f. of $\mathrm{ESD}(W_{N_k,n}(\omega)/\sqrt{N_k})$, then it is easy to see that

$$\chi_{k,n}(\omega, x) = F_{k,n}^{\leftarrow}(x) .$$

By the choice of the set A, it follows that for fixed $1 \le n < \infty$,

$$\lim_{k\to\infty} F_{k,n}(x) = F_n(x)$$

for all x which is a continuity point of F_n. Therefore, by standard analytic arguments (see for example the proof of Theorem 25.6, page 333 in Billingsley (1995)), (3.32) follows, which in turn establishes (3.31).

The next task is to show that for fixed $1 \le n < \infty$, the family

$$\{\chi_{k,n}^2 : 1 \le k < \infty\} \text{ is uniformly integrable.}$$

To that end it suffices to show that

$$\sup_{1\le k<\infty} \mathbb{E}\left(\chi_{k,n}^4\right) < \infty\,.$$

Fix n and notice that

$$\begin{aligned}\mathbb{E}\left(\chi_{k,n}^4\right) &= N_k^{-3}\mathrm{E}\,\mathrm{Tr}\left(W_{N_k,n}^4\right)\\ &\to \beta_{n,4} \text{ as } k\to\infty\,,\end{aligned}$$

the last step following by (3.16). This establishes the uniform integrability, which along with (3.31), proves that

$$\lim_{k\to\infty} \mathbb{E}\left[\left(\chi_{k,n}-\chi_{\infty,n}\right)^2\right] = 0 \text{ for all } 1\le n<\infty\,. \tag{3.33}$$

Our final claim is that

$$\lim_{n\to\infty}\limsup_{k\to\infty} \mathbb{E}\left[\left(\chi_{k,n}-\chi_{k,\infty}\right)^2\right] = 0\,. \tag{3.34}$$

To that end, notice that

$$\begin{aligned}\mathbb{E}\left[\left(\chi_{k,n}-\chi_{k,\infty}\right)^2\right] &= N_k^{-2}\mathrm{E}\sum_{j=1}^{N_k}\left[\lambda_j(W_{N_k,n})-\lambda_j(W_{N_k,\infty})\right]^2\\ &\le N_k^{-2}\mathrm{E}\,\mathrm{Tr}\left[\left(W_{N_k,n}-W_{N_k,\infty}\right)^2\right]\\ &\le C\sum_{m,l\in\mathbb{Z}:|m|\vee|l|>n} c_{m,l}^2\,,\end{aligned}$$

for some finite constant C. The inequality in the second line is the Hoffman-Wielandt inequality; see Lemma 2.1.19 on page 21 in Anderson et al. (2010). This completes the proof of (3.34).

Fact 3.3 along with (3.33) and (3.34) shows that there exists $\chi_{\infty,\infty} \in L^2(\Omega\times(0,1))$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\chi_{\infty,n}-\chi_{\infty,\infty}\right)^2\right] = 0\,. \tag{3.35}$$

It is easy to see that for all $n<\infty$, $\chi_{\infty,n}$ has law $\mu_{f,n}$. Therefore, by (3.29) and (3.35), it follows that law of $\chi_{\infty,\infty}$ is μ_f. Equation (3.35) furthermore ensures that

$$\begin{aligned}\int_{\mathbb{R}} x^2\mu_f(dx) &= \lim_{n\to\infty}\int_{\mathbb{R}} x^2\mu_{f,n}(dx)\\ &= \lim_{n\to\infty} 2\mathrm{E}(Y_{0,0,n}^2)\\ &= 2\mathrm{E}(Y_{0,0}^2)\\ &= 2\int_{[-\pi,\pi]^2} f(x,y)dxdy\,,\end{aligned}$$

where $Y_{0,0}$ and $Y_{0,0,n}$ are as in (3.3) and (3.6) respectively. This completes the proof. □

We now proceed towards the proof of Theorem 2.3. For that, we shall need the following two facts, the first of which is a simple consequence of the Holder's inequality.

Fact 3.4. *Suppose for some integer $k \geq 1$ and a measure space (Σ, Ξ, m), the functions $\{f_{in} : 1 \leq i \leq k, 1 \leq n \leq \infty\}$ are in $L^k(\Sigma)$. Furthermore, assume that for all fixed $1 \leq i \leq k$,*

$$f_{in} \to f_{i\infty}, \text{ as } n \to \infty \text{ in } L^k\,.$$

Then, $f_{1n} \dots f_{kn} \in L^1(\Sigma)$ for all $1 \leq n \leq \infty$, and

$$f_{1n} \dots f_{kn} \to f_{1\infty} \dots f_{k\infty}, \text{ as } n \to \infty \text{ in } L^1\,.$$

The second fact is a restatement of Theorem 3.4.4, page 146 in Krantz (1999).

Fact 3.5. *Assume that for some $p \in (1, \infty)$, $h \in L^p\left([-\pi, \pi]^2, \mathbb{C}\right)$, that is, it is a function from $[-\pi, \pi]^2$ to $\mathbb{C}$ with finite L^p norm. Define*

$$\hat{h}_{jk} := \frac{1}{2\pi} \int_{[-\pi,\pi]^2} e^{-\iota(jx+ky)} h(x, y) dx dy, \; j, k \in \mathbb{Z}\,.$$

Then,

$$\frac{1}{2\pi} \sum_{j,k=-n}^{n} \hat{h}_{j,k} e^{\iota(jx+ky)} \to h(x, y) \text{ in the } L^p \text{ norm, as } n \to \infty\,.$$

The first step towards proving Theorem 2.3 is the following lemma.

Lemma 3.2. *If f is a non-negative trigonometric polynomial defined on $[-\pi, \pi]^2$, that is,*

$$f(x, y) := \sum_{j,k=-n}^{n} a_{jk} e^{\iota(jx+ky)} \geq 0\,,$$

for some finite $n \geq 1$, and real numbers $(a_{jk} : 1 \leq j, k \leq n)$, then for all fixed $m \geq 1$,

$$\begin{aligned} &\int_{\mathbb{R}} x^{2m} \mu_f(dx) \\ &= \sum_{\sigma \in NC_2(2m)} \sum_{k \in S(\sigma)} \prod_{(u,v) \in \sigma} \int_{[-\pi,\pi]^2} e^{\iota(k_u x + k_v y)} [f(x, -y) + \\ &\qquad f(-y, x)] dx dy \qquad (3.36) \\ &= (2\pi)^{m-1} \sum_{\sigma \in NC_2(2m)} \int_{[-\pi,\pi]^{m+1}} L_{\sigma,f}(x) dx\,, \qquad (3.37) \end{aligned}$$

where $S(\sigma)$ is as in (3.14).

Proof. Since f is a trigonometric polynomial, it is integrable, and hence there exists a stationary Gaussian process $(G_{i,j} : i, j \in \mathbb{Z})$ with mean zero, and

$$R_G(k,l) := \mathrm{E}(G_{0,0}G_{k,l}) = \int_{[-\pi,\pi]^2} e^{\iota(kx+ly)} f(x,y)dxdy,\ k,l \in \mathbb{Z}\,.$$

The hypothesis ensures that $R_G(k,l) = 0$ if $|k| \vee |l| > n$. Hence, exactly same arguments as those in the proof of Proposition 3.1 will show that for fixed $m \geq 1$,

$$\begin{aligned}\int_{\mathbb{R}} x^{2m}\mu_f(dx) &= \sum_{\sigma\in NC_2(2m)} \sum_{k\in S(\sigma)} \prod_{(u,v)\in\sigma} [R_G(k_u,-k_v) + R_G(k_v,-k_u)] \\ &= \sum_{\sigma\in NC_2(2m)} \sum_{k\in S(\sigma)} \prod_{(u,v)\in\sigma} \overline{R}_G(k_u,k_v)\,,\end{aligned}$$

where

$$\overline{R}_G(u,v) := R_G(u,-v) + R_G(v,-u)\,.$$

Defining

$$g(x,y) := f(x,-y) + f(-y,x)\,,$$

it is easy to see from (3.1) that

$$\int_{[-\pi,\pi]^2} e^{\iota(ux+vy)} g(x,y)dxdy = \overline{R}_G(u,v),\ u,v \in \mathbb{Z}\,, \tag{3.38}$$

which shows (3.36).

Therefore, to complete the proof, it suffices to show that

$$\sum_{k\in S(\sigma)} \prod_{(u,v)\in\sigma} \overline{R}_G(u,v) = (2\pi)^{m-1} \int_{[-\pi,\pi]^{m+1}} L_{\sigma,f}(x)dx\,, \tag{3.39}$$

for all $\sigma \in NC_2(2m)$. To that end, fix σ, and notice that (3.38) implies that

$$g(x,y) = (2\pi)^{-2} \sum_{k,l=-n}^{n} \overline{R}_G(k,l)e^{-\iota(kx+ly)}\,,$$

for almost all x, y. Observe that for all $k \in \{-n,\ldots,n\}^{2m}$ and $x \in [-\pi,\pi]^{m+1}$,

$$\sum_{(u,v)\in\sigma} \left[k_u x_{\mathcal{T}_\sigma(u)} + k_v x_{\mathcal{T}_\sigma(v)}\right] = \sum_{l=1}^{m+1} x_l \sum_{j\in V_l} k_j\,,$$

and hence

$$\begin{aligned}
&\int_{[-\pi,\pi]^{m+1}} L_{\sigma,f}(x)dx \\
= &\int_{[-\pi,\pi]^{m+1}} \left[\prod_{(u,v)\in\sigma} g\left(x_{\mathcal{T}_\sigma(u)}, x_{\mathcal{T}_\sigma(v)}\right)\right] dx \\
= &(2\pi)^{-2m} \int_{[-\pi,\pi]^{m+1}} \Bigg[\sum_{k\in\{-n,\ldots,n\}^{2m}} \exp\left(\sum_{l=1}^{m+1} x_l \sum_{j\in V_l} k_j\right) \\
&\prod_{(u,v)\in\sigma} \overline{R}_G(u,v)\Bigg] dx \\
= &(2\pi)^{1-m} \sum_{k\in\{-n,\ldots,n\}^{2m}\cap S(\sigma)} \prod_{(u,v)\in\sigma} \overline{R}_G(u,v) \\
= &(2\pi)^{1-m} \sum_{k\in S(\sigma)} \prod_{(u,v)\in\sigma} \overline{R}_G(u,v)\,.
\end{aligned}$$

This shows (3.39) which in turn establishes (3.37) and hence completes the proof. □

The following lemma will also be needed for the proof of Theorem 2.3.

Lemma 3.3. *Suppose that for all $1 \le n \le \infty$, g_n is a non-negative, integrable and even function on $[-\pi,\pi]^2$ such that as $n \to \infty$,*

$$g_n \to g_\infty \text{ in } L^1\,.$$

Then,

$$\mu_{g_n} \xrightarrow{w} \mu_{g_\infty} \text{ as } n \to \infty\,.$$

Proof. The hypothesis can be restated as

$$\sqrt{g_n} \to \sqrt{g_\infty} \text{ in } L^2\,. \tag{3.40}$$

Let $(G_{i,j} : i,j \in \mathbb{Z})$ be a family of i.i.d. $N(0,1)$ random variables. Define

$$d_{k,l,n} := (2\pi)^{-1} \int_{[-\pi,\pi]^2} e^{-\iota(kx+ly)} \sqrt{g_n(x,y)}dxdy,\ k,l \in \mathbb{Z}, 1 \le n \le \infty\,,$$

and

$$H_{i,j,n} := \sum_{k,l\in\mathbb{Z}} d_{k,l,n} G_{i-k,j-l},\ i,j \in \mathbb{Z}, 1 \le n \le \infty\,.$$

By Fact 3.1, it follows that for all $1 \le n \le \infty$, the family $(H_{i,j,n} : i,j \in \mathbb{Z})$ is a stationary Gaussian process whose spectral density is g_n. For every $1 \le n \le \infty$ and $1 \le N < \infty$, define a $N \times N$ matrix $A_{N,n}$ by

$$A_{N,n}(i,j) := \left(H_{i,j,n} + H_{j,i,n}\right)/\sqrt{N},\ 1 \le i,j \le N\,.$$

By Theorem 2.1, it follows that for all $1 \le n \le \infty$,

$$L\left(\mathrm{ESD}(A_{N,n}), \mu_{g_n}\right) \xrightarrow{P} 0 \text{ as } N \to \infty\,. \tag{3.41}$$

Notice that for fixed $1 \le N, n < \infty$, by arguments similar to those leading to (3.28) from (3.27),

$$\begin{aligned}\mathrm{E}\,\mathrm{Tr}\left[(A_{N,n} - A_{N,\infty})^2/N\right] &\le 4\sum_{k,l\in\mathbb{Z}} (d_{k,l,n} - d_{k,l,\infty})^2 \\ &= 4\int_{[-\pi,\pi]^2} \left(\sqrt{g_n(x,y)} - \sqrt{g_\infty(x,y)}\right)^2 dxdy\,,\end{aligned}$$

the last equality following from Parseval. Therefore, by (3.40), it holds that for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} P\left[L\left(\mathrm{ESD}(A_{N,n}), \mathrm{ESD}(A_{N,\infty})\right) > \varepsilon\right] = 0\,.$$

The above, along with (3.41) and Fact 3.3 proves the claim of the lemma. ☐

Proof of Theorem 2.3. For the first part, fix $m \ge 2$, and assume that $\|f\|_m < \infty$. Let c_{kl} be as in (3.2), and define for $n \ge 1$,

$$f_n(x,y) := \left[\frac{1}{2\pi}\sum_{k,l=-n}^{n} c_{kl} e^{\iota(kx+ly)}\right]^2,\ -\pi \le x, y \le \pi\,. \tag{3.42}$$

By Fact 3.5, it follows that

$$f_n \to f \text{ in } L^m \text{ norm, as } n \to \infty\,, \tag{3.43}$$

which, with an appeal to Fact 3.4, implies that

$$\lim_{n\to\infty} \int_{[-\pi,\pi]^{m+1}} L_{\sigma,f_n}(x)dx = \int_{[-\pi,\pi]^{m+1}} L_{\sigma,f}(x)dx,\ \sigma \in NC_2(2m)\,. \tag{3.44}$$

Equation (3.43) along with Lemma 3.3 and the observation that f_n is a non-negative even function implies that

$$\mu_{f_n} \xrightarrow{w} \mu_f \text{ as } n \to \infty\,. \tag{3.45}$$

Therefore, by Fatou's lemma, it follows that

$$\begin{aligned}\int_{\mathbb{R}} x^{2m}\mu_f(dx) &\le \liminf_{n\to\infty} \int_{\mathbb{R}} x^{2m}\mu_{f_n}(dx) \\ &= \liminf_{n\to\infty} (2\pi)^{m-1} \sum_{\sigma\in NC_2(2m)} \int_{[-\pi,\pi]^{m+1}} L_{\sigma,f_n}(x)dx \\ &= (2\pi)^{m-1} \sum_{\sigma\in NC_2(2m)} \int_{[-\pi,\pi]^{m+1}} L_{\sigma,f}(x)dx < \infty\,,\end{aligned}$$

the equality in the last two lines following from Lemma 3.2 and (3.44) respectively. This completes the proof of the first part.

For the second part, assume that $\|f\|_\infty < \infty$. By the arguments above, it follows that

$$\lim_{n\to\infty} \int_{\mathbb{R}} x^{2m} \mu_{f_n}(dx) =$$

$$(2\pi)^{m-1} \sum_{\sigma\in NC_2(2m)} \int_{[-\pi,\pi]^{m+1}} L_{\sigma,f}(x)dx \text{ for all } m \geq 1\,,$$

and

$$\begin{aligned}
&\limsup_{m\to\infty} \left[(2\pi)^{m-1} \sum_{\sigma\in NC_2(2m)} \int_{[-\pi,\pi]^{m+1}} L_{\sigma,f}(x)dx \right]^{1/2m} \\
\leq\ & \limsup_{m\to\infty} \left[(2\pi)^{2m} (2\|f\|_\infty)^{m+1} \# NC_2(2m) \right]^{1/2m} \\
=\ & 4\sqrt{2}\,\pi \sqrt{\|f\|_\infty}\,.
\end{aligned}$$

This shows that there exists a compactly supported even probability measure μ^* such that

$$\int_{\mathbb{R}} x^{2m} \mu^*(dx) = \sum_{\sigma\in NC_2(2m)} \int_{[-\pi,\pi]^{m+1}} L_{\sigma,f}(x)dx \text{ for all } m \geq 1\,,$$

and

$$\mu_{f_n} \xrightarrow{w} \mu^* \text{ as } n \to \infty\,.$$

This, along with (3.45) completes the proof of the second part. □

Next, we proceed towards the proof of Theorem 2.4. The following lemma, which is the first step towards that, proves the result for a special case.

Lemma 3.4. *Suppose that $(G_{i,j} : i,j \in \mathbb{Z})$ is a stationary Gaussian process whose covariance kernel $R_G(\cdot,\cdot)$ defined by*

$$R_G(u,v) := \mathrm{E}\left[G_{0,0}G_{u,v}\right],\ u,v\in\mathbb{R}\,,$$

satisfies

$$R_G(u,v) = \int_{[-\pi,\pi]^2} e^{\iota(ux+vy)} f_G(x) f_G(y)\, dxdy,\ u,v\in\mathbb{R}\,,$$

for some non-negative $f_G(\cdot)$ defined on $[-\pi,\pi]$, and there exists n such that

$$R_G(u,v) = 0 \text{ if } |u| \vee |v| > n\,. \tag{3.46}$$

Then ESD *of the $N\times N$ matrix whose (i,j)-th entry is $G_{i,j}/\sqrt{N}$ converges weakly in probability to*

$$\eta_G \boxtimes WSL(1)\,,$$

where η_G is the law of $f_G(U)\pi 2\sqrt{2}$ and U is an Uniform $(-\pi,\pi)$ random variable.

Proof. By Theorem 2.1, it follows that the limit exists, say μ_G, and furthermore by the hypothesis (3.46), and claim (3.36) of Lemma 3.2, it holds that

$$\int_{\mathbb{R}} x^{2m}\mu_G(dx) = \tag{3.47}$$

$$\sum_{\sigma\in NC_2(2m)}\sum_{k\in S(\sigma)}\prod_{(u,v)\in\sigma}\left[R_G(k_u,-k_v)+R_G(k_v,-k_u)\right],\ m\geq 1\,.$$

Our first claim is that

$$R_G(u,-v)+R_G(v,-u)=r_G(u)r_G(v),\ u,v\in\mathbb{Z}\,, \tag{3.48}$$

where

$$r_G(u):=\sqrt{2}\int_{-\pi}^{\pi}e^{\iota ux}f_G(x)dx\,. \tag{3.49}$$

To that end, notice that by (3.1), it follows that

$$f_G(-x)f_G(-y)=f_G(x)f_G(y)\text{ for almost all }x,y\in[-\pi,\pi]\,.$$

Integrating out y, it follows that $f_G(\cdot)$ is an even function. Therefore,

$$\begin{aligned} R_G(u,-v)+R_G(v,-u) &= \int_{[-\pi,\pi]^2}2\cos(ux-vy)f_G(x)f_G(y)dxdy \\ &= r_G(u)r_G(v)\,, \end{aligned}$$

where the fact that $f_G(\cdot)$ is even has been used for the last equality. This establishes (3.48).

Our next claim is that

$$f_G(x)=\frac{1}{\pi 2\sqrt{2}}\sum_{k=-n}^{n}r_G(k)e^{-\iota kx},\text{ for almost all }x\in[-\pi,\pi]\,. \tag{3.50}$$

The above follows from (3.49) using Fourier inversion and the fact that for $|u|>n$, $r_G(u)=0$ which is a consequence of (3.46).

By (3.47) and (3.48), it follows that for fixed $m\geq 1$,

$$\begin{aligned}\int x^{2m}\mu_G(dx) &= \sum_{\sigma\in NC_2(2m)}\sum_{k\in S(\sigma)}\prod_{(u,v)\in\sigma}r_G(k_u)r_G(k_v) \\ &= \sum_{\sigma\in NC_2(2m)}\sum_{k\in S(\sigma)}\prod_{j=1}^{2m}r_G(k_j)\,.\end{aligned}$$

Fix $\sigma\in NC_2(2m)$, and let for $i=1,\ldots,m+1$, l_i be the size of V_i which are the blocks of the Kreweras complement of σ, as in (3.13). Then, it is easy to see that

$$\sum_{k\in S(\sigma)}\prod_{j=1}^{2m} r_G(k_j) = \prod_{i=1}^{m+1}\sum_{k\in\mathbb{Z}^{l_i}:k_1+\ldots+k_{l_i}=0}\prod_{j=1}^{l_i} r_G(k_j)$$
$$= \prod_{i=1}^{m+1}(2\pi)^{-1}\int_{-\pi}^{\pi}\left[f_G(x)\pi 2\sqrt{2}\right]^{l_i}dx$$
$$= \prod_{i=1}^{m+1}\int_{\mathbb{R}} x^{l_i}\eta_G(dx)\,,$$

the second last equality being a consequence of (3.50). Therefore, it follows that

$$\int_{\mathbb{R}} x^{2m}\mu_G(dx) = \sum_{\sigma\in NC_2(2m)}\prod_{i=1}^{m+1}\int_{\mathbb{R}} x^{l_i}\eta_G(dx)$$
$$= \int_{\mathbb{R}} x^{2m}\eta_G\boxtimes WSL(1)\,,$$

the last equality following from Theorem 14.4 in Nica and Speicher (2006). This shows that $\mu_G=\eta_G\boxtimes WSL(1)$, and thus completes the proof. □

The next step towards the proof of Theorem 2.4 is the following.

Lemma 3.5. *If*

$$g(x,y):=\frac{1}{2}\left[f(x,y)+f(y,x)\right],\ -\pi\le x,y\le\pi\,,$$

then

$$\mu_f=\mu_g\,.$$

Proof. For $n\ge1$, let f_n be as in (3.42), and define and

$$g_n(x,y):=\frac{1}{2}\left[f_n(x,y)+f_n(y,x)\right],\ -\pi\le x,y\le\pi\,.$$

Noticing that for all $\sigma\in\bigcup_{m=1}^{\infty}NC_2(2m)$,

$$L_{\sigma,f_n}=L_{\sigma,g_n}\text{ a.e.}\,,$$

it follows by Lemma 3.2 that

$$\mu_{f_n}=\mu_{g_n}\text{ for all } n\ge1\,.$$

Using (3.43) with $m=1$, which is valid because $\|f\|_1<\infty$, it follows that

$$f_n\to f\text{ in } L^1\text{ as } n\to\infty\,,$$

from which it follows that

$$g_n\to g\text{ in } L^1\,.$$

Lemma 3.3 completes the proof. □

Proof of Theorem 2.4. Define

$$g(x,y) := \frac{1}{2}\left[f(x,y)+f(y,x)\right], \, -\pi \le x, y \le \pi \, .$$

In view of Lemma 3.5, it suffices to show that

$$\mu_g = \eta_r \boxtimes WSL(1) \, . \tag{3.51}$$

To that end, define

$$d_k := (2\pi)^{-1/2}\int_{-\pi}^{\pi} e^{-\iota kx}\sqrt{r(x)}dx, \; k \in \mathbb{Z} \, .$$

Then, it is easy to see that

$$(2\pi)^{-1}\int_{[-\pi,\pi]^2} e^{-\iota(kx+ly)}\sqrt{g(x,y)}dxdy = d_k d_l, \; k,l \in \mathbb{Z} \, . \tag{3.52}$$

Define

$$g_n(x,y) := \left[(2\pi)^{-1}\sum_{k,l=-n}^{n} d_k d_l e^{\iota(kx+ly)}\right]^2, \, -\pi \le x, y \le \pi \, .$$

Clearly,

$$g_n(x,y) = r_n(x) r_n(y) \text{ for all } -\pi \le x, y \le \pi \, ,$$

where

$$r_n(x) := \left[(2\pi)^{-1/2}\sum_{k=-n}^{n} d_k e^{\iota kx}\right]^2 \, .$$

Arguments similar to those in the proof of Lemma 3.4 show that $g_n(\cdot,\cdot)$ and $r_n(\cdot)$ take values in the non-negative half line. By the same lemma, it follows that

$$\mu_{g_n} = \eta_{r_n} \boxtimes WSL(1), \, n \ge 1 \, ,$$

where η_{r_n} is the law of $2^{3/2}\pi r_n(U)$, U being an Uniform $(-\pi,\pi)$ random variable. By the Fourier inversion theorem, it follows that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} |r_n(x) - r(x)| \, dx = 0 \, ,$$

and hence by Corollary 6.7 of Bercovici and Voiculescu (1993) and Lemma 8 of Arizmendi and Pérez-Abreu (2009), it follows that

$$\eta_{r_n} \boxtimes WSL(1) \xrightarrow{w} \eta_r \boxtimes WSL(1) \text{ as } n \to \infty \, .$$

Again, Fourier inversion and (3.52) tells us that

$$\lim_{n\to\infty}\int_{[-\pi,\pi]^2} |g_n(x,y) - g(x,y)| \, dxdy = 0 \, .$$

An appeal to Lemma 3.3 establishes (3.51), and thus completes the proof. □

Finally, we prove Theorem 2.5.

Proof of Theorem 2.5. Define

$$g(x,y) := \frac{1}{2}\left[f(x,y)+f(y,x)\right],\ -\pi \le x,y \le \pi\,.$$

By Lemma 3.5, it suffices to show that

$$\mu_g = WSL(2\|f\|_1)\,. \tag{3.53}$$

To that end, set

$$h_n(x,y) := (2\pi)^{-1}\sum_{j,k\in\mathbb{Z}} d_{j,k}\mathbf{1}(j,k\in A_n)e^{\iota(jx+ky)},\ -\pi \le x,y \le \pi\,.$$

Since $d_{j,k}$ are the Fourier coefficients of $\sqrt{g}$, they are real numbers, and furthermore by the Parseval's identity, it follows that

$$\sum_{j,k\in\mathbb{Z}} d_{j,k}^2 < \infty\,,$$

and hence in view of the assumption that $A_n \uparrow \mathbb{Z}$, it follows that as $n\to\infty$,

$$h_n \to \sqrt{g} \text{ in } L^2\,.$$

Define

$$g_n(\cdot,\cdot) := |h_n(\cdot,\cdot)|^2,\ n\ge 1\,,$$

where the modulus is necessary because $h_n(\cdot)$ is $\mathbb{C}$-valued. Therefore,

$$g_n \to g \text{ in } L^1\,. \tag{3.54}$$

Fix $n \ge 1$. Since $d_{j,k}$ is real, it is easy to see that

$$\begin{aligned} & g_n(x,y) \\ = \ & (2\pi)^{-2}\sum_{i,j,k,l\in\mathbb{Z}} d_{i,j}d_{k,l}\mathbf{1}(i,j,k,l\in A_n)e^{\iota((i-k)x+(j-l)y)} \\ = \ & (2\pi)^{-2}\sum_{u,v\in\mathbb{Z}} e^{-\iota(ux+vy)}\sum_{i,j\in\mathbb{Z}} d_{i,j}d_{i+u,j+v}\mathbf{1}(i,j,i+u,j+v\in A_n)\,. \end{aligned}$$

Since A_n is a finite set, g_n is a trigonometric polynomial. By (3.36) of Lemma 3.2 and the observation that $g_n(x,y)=g_n(y,x)$, it follows that for all $m\ge 1$,

$$\int_{\mathbb{R}} x^{2m}\mu_{g_n}(dx) = \tag{3.55}$$

$$\sum_{\sigma\in NC_2(2m)}\sum_{k\in S(\sigma)}\prod_{(u,v)\in\sigma} 2\int_{[-\pi,\pi]^2} e^{\iota(k_u x+k_v y)}g_n(x,-y)dxdy\,.$$

Fix $u\in\mathbb{Z}\setminus\{0\}$, and notice that

$$\int_{[-\pi,\pi]^2} e^{\iota ux}g_n(x,-y)dxdy = \sum_{i,j\in\mathbb{Z}} d_{i,j}d_{i+u,j}\mathbf{1}(i,j,i+u\in A_n) = 0\,.$$

From the above, a simple induction on m will show that for all $\sigma \in NC_2(2m)$ and for all $k \in S(\sigma)$,

$$\prod_{(u,v)\in\sigma} \int_{[-\pi,\pi]^2} e^{\iota(k_u x + k_v y)} g_n(x,-y)\,dx\,dy \neq 0$$

implies that $k = (0,\ldots,0)$. Therefore, (3.55) boils down to

$$\int_{\mathbb{R}} x^{2m} \mu_{g_n}(dx) = (2\|g_n\|_1)^m \# NC_2(2m),\ m \geq 1\,,$$

and hence

$$\mu_{g_n} = WSL(2\|g_n\|_1)\,.$$

Equation (3.54) with an appeal to Lemma 3.3 shows (3.53) and thus completes the proof. □

3.2. **Proofs of Theorems 2.6 - 2.7.** As the first step towards proving Theorem 2.6, we start with a special case.

Proposition 3.2. *There exists a random point measure ξ which is almost surely in $\mathcal{C}_2$ such that*

$$d_2\left(\mathrm{EM}(\widetilde{W}_N/N), \xi\right) \xrightarrow{P} 0\,, \tag{3.56}$$

as $N \to \infty$, where $\widetilde{W}_N$ is as in (3.11).

Remark 3.2. *In view of the inequality*

$$P\left(d_4(\xi_1,\xi_2) > \varepsilon\right) \leq P\left(d_2(\xi_1,\xi_2) > \varepsilon^2\right)$$

for all $\varepsilon \in (0,1)$ and random measures ξ_1, ξ_2 which are almost surely in $\mathcal{C}_2$, (3.56) implies that

$$d_4\left(\mathrm{EM}(\widetilde{W}_N/N), \xi\right) \xrightarrow{P} 0\,, \tag{3.57}$$

as $N \to \infty$. Thus, the assertion of Proposition 3.2 is stronger than that of Theorem 2.6 in the special case when the spectral measure of the input process is discrete.

A few facts from the literature will be used in the proof of Proposition 3.2, which we shall now list below. The first fact is essentially a consequence of the well known result that any two norms on a finite dimensional vector space are equivalent.

Fact 3.6. *Suppose that for every $N \geq 1$, B_N is a $N \times p$ matrix, where p is a fixed finite integer. Assume that*

$$\lim_{N\to\infty} (B_N^T B_N)(i,j) = C(i,j) \text{ for all } 1 \leq i,j \leq p\,.$$

Then $C \geq 0$, and for any $p \times p$ symmetric matrix P

$$\lim_{N\to\infty} \mathrm{Tr}\left[\left\{(B_N^T B_N)^{1/2} P (B_N^T B_N)^{1/2} - C^{1/2} P C^{1/2}\right\}^2\right] = 0\,. \tag{3.58}$$

The next fact is a trivial consequence of the Sylvester's determinant theorem

Fact 3.7. *Suppose that B and P are $N_1 \times N_2$ and $N_2 \times N_2$ matrices respectively, the latter being symmetric. Then,*

$$\mathrm{EM}\left(BPB^T\right) = \mathrm{EM}\left((B^TB)^{1/2}P(B^TB)^{1/2}\right).$$

The next is a well known fact from linear algebra.

Fact 3.8. *For any symmetric matrix A, and a positive integer p,*

$$\int_{\mathbb{R}} x^p(\mathrm{EM}(A))(dx) = \mathrm{Tr}(A^p).$$

The last fact that we shall use is a combination of Corollary 5.3 on page 115 in Markus (1964) and the above fact. A detailed survey of results similar to this one can be found in Chapter 13 of Bhatia (2007).

Fact 3.9. *For symmetric matrices A and B of the same size and a positive even integer p,*

$$d_p\left(\mathrm{EM}(A), \mathrm{EM}(B)\right) \le \mathrm{Tr}^{1/p}\left[(A-B)^p\right].$$

The following fact, the proof of which is an easy exercise, will also be needed.

Fact 3.10. *For all $x, y \in \mathbb{R}$, the following limits exist:*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \sin(kx)\sin(ky),$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \sin(kx)\cos(ky),$$

and

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \cos(kx)\cos(ky).$$

Furthermore, for all $x, y \in \mathbb{R}$,

$$\lim_{N\to\infty} N^{-2}\sum_{i,j=1}^N \left[\cos(ix+jy)+\cos(iy+jx)\right]^2 \tag{3.59}$$

exists, and is strictly positive.

Proof of Proposition 3.2. Recalling the definition of $\widetilde{W}_{N,n}$ from (3.12), in view of Fact 3.3, it suffices to show that there exists a random measure ξ_n which is almost surely in $\mathcal{C}_2$ such that

$$d_2\left(\mathrm{EM}(\widetilde{W}_{N,n}/N), \xi_n\right) \xrightarrow{P} 0, \tag{3.60}$$

as $N \to \infty$ for all fixed $n \geq 1$, and
(3.61)
$$\lim_{n\to\infty} \limsup_{N\to\infty} P\left[d_2\left(\mathrm{EM}(\widetilde{W}_{N,n}/N), \mathrm{EM}(\widetilde{W}_N/N)\right) > \varepsilon\right] = 0 \text{ for all } \varepsilon > 0\,.$$

Proceeding towards showing (3.60), fix $n \geq 1$. By a relabeling, it is easy to see that for all $N \geq 1$,

$$\widetilde{W}_{N,n}(i,j) = \sum_{k=1}^{4n} Y_k\left[u_k(i)v_k(j) + v_k(i)u_k(j)\right]\,,$$

where $Y_1, Y_2, \ldots, Y_{4n}$ are normal random variables which are **not necessarily independent**, and for each k, there exists $w_k \in \mathbb{R}$ such that either

$$u_k(i) = \sin(iw_k) \text{ for all } i\,,$$

or

$$u_k(i) = \cos(iw_k) \text{ for all } i\,,$$

and a similar assertion holds for v_k with w_k replaced by some z_k. For $N \geq 1$ and $1 \leq k \leq 8n$, define a $N \times 1$ vector u_{kN} by

$$u_{kN}(i) = u_k(i),\ 1 \leq i \leq N\,,$$

and similarly define the vector v_{kN}. Next define a $N \times 8n$ matrix

$$B_N := \left[\sqrt{|Y_1|}u_{1N}\ \sqrt{|Y_1|}v_{1N} \ldots \sqrt{|Y_{4n}|}u_{4n\,N}\ \sqrt{|Y_{4n}|}v_{4n\,N}\right]\,,$$

and a $8n \times 8n$ symmetric matrix P by

$$P(i,j) := \begin{cases} \mathrm{sgn}(Y_k), & \text{if } i = 2k-1 \text{ and } j = 2k \text{ for some } k\,, \\ \mathrm{sgn}(Y_k), & \text{if } i = 2k \text{ and } j = 2k-1 \text{ for some } k\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

for all $1 \leq i, j \leq 8n$. Then, it is easy to see that

$$\widetilde{W}_{N,n} = B_N P B_N^T,\ N \geq 1\,.$$

Fact 3.7 implies that

$$\mathrm{EM}\left(\widetilde{W}_{N,n}/N\right) = \mathrm{EM}\left(\frac{1}{N}(B_N^T B_N)^{1/2} P (B_N^T B_N)^{1/2}\right),\ N \geq 1\,. \tag{3.62}$$

By Fact 3.10, it follows that there exists a $8n \times 8n$ matrix C_n such that

$$\lim_{N\to\infty} \frac{1}{N}(B_N^T B_N)(i,j) = C_n(i,j) \text{ almost surely}\,,$$

for all $1 \leq i, j \leq 8n$. Facts 3.6 and 3.9 ensure that

$$\lim_{N\to\infty} d_2\left(\mathrm{EM}\left(\frac{1}{N}(B_N^T B_N)^{1/2} P (B_N^T B_N)^{1/2}\right), \mathrm{EM}\left(C_n^{1/2} P C_n^{1/2}\right)\right) = 0\,,$$

almost surely, which with the aid of (3.62) ensures (3.60), with

$$\xi_n := \mathrm{EM}(C_n^{1/2} P C_n^{1/2})\,.$$

For (3.61), it suffices to show that

$$\lim_{n\to\infty}\limsup_{N\to\infty} \mathrm{E}\left[d_2^2\left(\mathrm{EM}(\widetilde{W}_{N,n}/N), \mathrm{EM}(\widetilde{W}_N/N)\right)\right] = 0\,.$$

To that end, notice that by Fact 3.9,

$$\begin{aligned}\mathrm{E}\left[d_2^2\left(\mathrm{EM}(\widetilde{W}_{N,n}/N), \mathrm{EM}(\widetilde{W}_N/N)\right)\right] &\leq N^{-2}\mathrm{E}\,\mathrm{Tr}\left[(\widetilde{W}_{N,n}-\widetilde{W}_N)^2\right] \\ &\leq 4\sum_{k=n+1}^{\infty} a_k\,,\end{aligned}$$

the last line following from arguments analogous to those leading from (3.27) to (3.28). This establishes (3.61) which along with (3.60) and Fact 3.3 shows the existence of ξ which is almost surely in $\mathcal{C}_2$, and satisfies

$$d_2(\xi_n, \xi) \xrightarrow{P} 0 \text{ as } n\to\infty\,, \tag{3.63}$$

and (3.56). This completes the proof of Proposition 3.2. □

Proof of Theorem 2.6. By the arguments mentioned in Remark 3.2, (3.57) follows from Proposition 3.2. In view of that, to complete the proof of (2.7), all that needs to be shown is

$$d_4\left(\mathrm{EM}(\widetilde{W}_N/N), \mathrm{EM}(W_N/N)\right) \xrightarrow{P} 0 \text{ as } N\to\infty\,. \tag{3.64}$$

To that end, recall (3.5), and the definition of $W_{N,n}$ from (3.8). By the triangle inequality, it follows that for all $N, n\geq 1$,

$$\begin{aligned}&d_4\left(\mathrm{EM}(\widetilde{W}_N/N), \mathrm{EM}(W_N/N)\right)\\ \leq\;& d_4\left(\mathrm{EM}(\widetilde{W}_N/N), \mathrm{EM}((\widetilde{W}_N+W_{N,n})/N)\right)\\ &+d_4\left(\mathrm{EM}((\widetilde{W}_N+W_{N,n})/N), \mathrm{EM}(W_N/N)\right)\,.\end{aligned}$$

By Fact (3.9), it follows that

$$\begin{aligned}&\mathrm{E}\left[d_4^4\left(\mathrm{EM}(\widetilde{W}_N/N), \mathrm{EM}((\widetilde{W}_N+W_{N,n})/N)\right)\right]\\ \leq\;& \mathrm{E}\left[\mathrm{Tr}((W_{N,n}/N)^4)\right]\to 0 \text{ as } N\to\infty\,,\end{aligned} \tag{3.65}$$

for all fixed $n\geq 1$ using (3.16) with $m=2$. In order to show (3.64), it suffices to prove that

$$\lim_{n\to\infty}\limsup_{N\to\infty} P\left[d_4\left(\mathrm{EM}((\widetilde{W}_N+W_{N,n})/N), \mathrm{EM}(W_N/N)\right) > \varepsilon\right] = 0\,,$$

for all $\varepsilon \in (0,1)$. To that end fix such a ε, and notice by the arguments in Remark 3.2,

$$\begin{aligned}
& P\left[d_4\left(\mathrm{EM}((\widetilde{W}_N + W_{N,n})/N), \mathrm{EM}(W_N/N)\right) > \varepsilon\right] \\
\leq\ & P\left[d_2\left(\mathrm{EM}((\widetilde{W}_N + W_{N,n})/N), \mathrm{EM}(W_N/N)\right) > \varepsilon^2\right] \\
\leq\ & \varepsilon^{-4}\mathrm{E}\left[d_2^2\left(\mathrm{EM}((\widetilde{W}_N + W_{N,n})/N), \mathrm{EM}(W_N/N)\right)\right] \\
\leq\ & \varepsilon^{-4}N^{-2}\mathrm{E}\left[\mathrm{Tr}\left[(\widetilde{W}_N + W_{N,n} - W_N)^2\right]\right] \\
\leq\ & C \sum_{i,j\in\mathbb{Z}:|i|\vee|j|>n} c_{i,j}^2\,,
\end{aligned}$$

for some finite constant C which is independent of N and n. In the above calculation, the second last line follows from Fact 3.9, and the last line is analogous to (3.28). This shows (3.64) which in turn proves (2.7).

In order to complete the proof of Theorem 2.6, all that needs to be shown is that the distribution of ξ is determined by ν_d. That is, however, obvious from (3.56) and the fact that the spectral measure of the stationary process $(Z_{i,j} : i,j \in \mathbb{Z})$ is ν_d. □

Remark 3.3. *The only reason that in* (2.7)*, d_4 cannot be changed to d_2 is that the limit* (3.65) *will become false if the index* 4 *is replaced by* 2*. Every other step in the above proof goes through perfectly fine for d_2.*

We next proceed towards proving Theorem 2.7. For that, we shall need the following lemma.

Lemma 3.6. *Suppose that $G_1, G_2, \ldots$ are i.i.d. $N(0,1)$ random variables and $\{\alpha_{jk} : j,k \in \mathbb{Z}\}$ are deterministic numbers such that*

$$\sum_{j,k=1}^{2n} \alpha_{jk} G_j G_k \xrightarrow{P} Z\,,$$

as $n \to \infty$, for some finite random variable Z. If $\alpha_{11} \neq 0$, then Z has a continuous distribution.

Proof. The given hypothesis implies that

$$G_1 \sum_{j=2}^{2n} \alpha_{1j} G_j + \sum_{j,k=2}^{2n} \alpha_{jk} G_j G_k \xrightarrow{P} Z - \alpha_{11} G_1^2\,,$$

as $n \to \infty$. By passing to a subsequence, we get a family of random variables $(X_n, Y_n : n \geq 1)$ which is independent of G_1 such that

$$G_1 X_n + Y_n \to Z - \alpha_{11} G_1^2, \text{ almost surely, as } n \to \infty\,.$$

From here, by conditioning on G_1 and using the independence, it follows that X_n and Y_n converge almost surely. Therefore, there exist random variables

X and Y such that G_1 is independent of (X, Y) and

$$Z = \alpha_{11} G_1^2 + G_1 X + Y \text{ a.s.}.$$

Since $\alpha_{11} \neq 0$, for all $z \in \mathbb{R}$, it holds that

$$P(Z = z) = \int_{\mathbb{R}^2} P\left[\alpha_{11} G_1^2 + G_1 x + y = z\right] P(X \in dx, Y \in dy) = 0$$

because for every fixed x and y, the integrand is zero. This completes the proof. □

Proof of Theorem 2.7. In view of (2.6) and (3.63), it follows that

$$\int_{\mathbb{R}} x^2 \xi_n(dx) \xrightarrow{P} \int_{\mathbb{R}} x^2 \xi(dx) \text{ as } n \to \infty\,, \tag{3.66}$$

where ξ_n is as in (3.63). The content of the proof is in showing that there exists real numbers $\{\alpha_{ijkl} : 1 \leq i, j \leq 2,\, k, l \geq 1\}$ such that

$$\int_{\mathbb{R}} x^2 \xi_n(dx) = \sum_{i,j=1}^{2} \sum_{k,l=1}^{n} \alpha_{ijkl} V_{ik} V_{jl} \text{ for all } n \geq 1\,, \tag{3.67}$$

where V_{ik} is as in (3.4). Furthermore, since a premise of the result is that ν_d is non-null, we assume without loss of generality that $a_1 > 0$. Based on that, it will be shown that

$$\alpha_{1111} > 0\,. \tag{3.68}$$

Lemma 3.6 along with (3.66) - (3.68) will establish that $\int_{\mathbb{R}} x^2 \xi(dx)$ has a continuous distribution, and thus the claim of Theorem 2.7 will follow.

To that end, notice that by (2.6), (3.60) and Fact 3.8, it follows that

$$N^{-2} \operatorname{Tr}\left[\widetilde{W}_{N,n}^2\right] \xrightarrow{P} \int_{\mathbb{R}} x^2 \xi_n(dx) \text{ as } N \to \infty\,, \tag{3.69}$$

for all fixed $n \geq 1$, where $\widetilde{W}_{N,n}$ is as in (3.12). It is easy to see that

$$\widetilde{W}_{N,n} = \sum_{i=1}^{2} \sum_{k=1}^{n} V_{ik} A_{ikN}\,,$$

where A_{ikN} are $N \times N$ deterministic matrices defined by

$$A_{ikN}(u, v) := \begin{cases} \sqrt{a_k}\left[\cos(ux_k + vy_k) + \cos(vx_k + uy_k)\right], & i = 1\,, \\ \sqrt{a_k}\left[\sin(ux_k + vy_k) + \sin(vx_k + uy_k)\right], & i = 2\,, \end{cases}$$

for all $1 \leq u, v \leq N$. Therefore,

$$N^{-2} \operatorname{Tr}\left[\widetilde{W}_{N,n}^2\right] = \sum_{i,j=1}^{2} \sum_{k,l=1}^{n} V_{ik} V_{jl} N^{-2} \operatorname{Tr}\left(A_{ikN} A_{jlN}\right) \text{ for all } N, n \geq 1\,.$$

Since by (3.69) for every fixed n the right hand side converges in probability, and the random variables $\{V_{ik}V_{jl} : 1 \le i,j \le 2, 1 \le k,l \le n\}$ are uncorrelated, it follows that

$$\alpha_{ijkl} := \lim_{N\to\infty} N^{-2}\,\mathrm{Tr}\,(A_{ikN}A_{jlN}) \text{ exists for all } 1 \le i,j \le 2,\ k,l \ge 1\,,$$

and that (3.67) holds. Finally, notice that (3.68) follows from (3.59) in Fact 3.10. This completes the proof of Theorem 2.7. □

4. A corollary and examples

In this section, a corollary and a few numerical examples that follow from the results of the previous sections are discussed. The first one is a corollary of Theorem 2.5, followed by a numerical example of the same result.

Corollary 1. *Assume that $(G_n : n \in \mathbb{Z})$ is a one-dimensional stationary Gaussian process with zero mean and positive variance, and whose spectral measure is absolutely continuous. Let $((G_{in} : n \in \mathbb{Z}) : i \in \mathbb{Z})$ be a family of i.i.d. copies of $(G_n : n \in \mathbb{Z})$. Define*

$$X_{j,k} := G_{j-k,k},\ j,k \in \mathbb{Z}\,.$$

Then, $(X_{j,k} : j,k \in \mathbb{Z})$ is a stationary Gaussian process, and

$$\mu_f = WSL(2\mathrm{Var}(G_0))\,.$$

Proof. The hypotheses imply the existence of a non-negative function h on $(-\pi,\pi]$ such that

$$\mathrm{E}\,[G_0G_v] = \int_{-\pi}^{\pi} e^{\iota v x}h(x)dx,\ v \in \mathbb{Z}\,.$$

Clearly, for all $j,k,u,v \in \mathbb{Z}$,

$$\mathrm{E}\,[X_{j,k}X_{j+u,k+v}] \quad = \quad \mathrm{E}\,[G_0G_v]\,\mathbf{1}(u=v)\,,$$

which shows the stationarity. Extend h to whole of $\mathbb{R}$ by the identity $h(\cdot) \equiv h(\cdot + 2\pi)$. Notice that

$$\begin{aligned} &\int_{[-\pi,\pi]^2} e^{\iota(ux+vy)}h\,(x+y)\,dxdy \\ = \quad &\int_{-\pi}^{\pi} e^{\iota(u-v)x}\left[\int_{z-\pi}^{z+\pi} e^{\iota v z}h(z)dz\right]dx \\ = \quad &2\pi\mathrm{E}\,[G_0G_v]\,\mathbf{1}(u=v)\,. \end{aligned}$$

Thus,

$$f(x,y) := (2\pi)^{-1}h(x+y),\ -\pi \le x,y \le \pi\,,$$

is the spectral density for $(X_{j,k})$. Furthermore, for integers $j \ne k$,

$$\int_{[-\pi,\pi]^2} e^{-\iota(jx+ky)}\sqrt{f(x,y)}dxdy = 0\,,$$

and therefore, the hypothesis of Theorem 2.5 is satisfied with $A_n := \{-n,\dots,n\}$. This completes the proof. □

Remark 4.1. *The above corollary is false without the assumption that the process* (G_n) *has a spectral density. For example, if* $G_m = G_n$ *for all* m, n*, then the matrix* W_N *becomes a Toeplitz matrix. Bryc et al. (2006) have shown that the LSD has unbounded support in this case.*

Example 1. *Let* $(G_n : n \in \mathbb{Z})$ *be a zero mean stationary Gaussian process with spectral density* $|x|^{-1/2}$*, and let* $X_{j,k}$ *be as in the above corollary. Then, it follows that*

$$\mu_f = WSL(8\sqrt{\pi})\,,$$

where

$$f(x,y) := (2\pi)^{-1}h(x+y),\ -\pi \le x, y \le \pi\,,$$

with $h(\cdot)$ *defined on* $\mathbb{R}$ *by the identities* $h(\cdot) = h(\cdot + 2\pi)$ *and* $h(z) = |z|^{-1/2}$ *for* $-\pi < z \le \pi$*. It is easy to see that* $\|f\|_2 = \infty$*, thus showing that the converses of both parts of Theorem 2.3 are false.*

Next, we shall see two numerical examples where Theorem 2.4 hold.

Example 2. *Let*

$$f(x,y) = \mathbf{1}(-\pi/2 \le x, y \le \pi/2),\ -\pi \le x, y \le \pi\,.$$

By Theorem 2.4, it follows that

$$\mu_f = \eta_r \boxtimes WSL(1)\,,$$

where η_r *is the law of* $2^{3/2}\pi\mathbf{1}(|U| \le \pi/2)$*,* U *being an Uniform* $(-\pi, \pi)$ *random variable. A calculation of the moments of the right hand side using Theorem 14.4 in Nica and Speicher (2006) will show that* μ_f *is the law of* $2\pi BW$ *where* B *and* W *are independent (in the classical sense) random variables distributed as Bernoulli* $(1/2)$ *(that is, takes values* 0 *and* 1*) and* $WSL(1)$ *respectively. This is an example where the LSD is not a continuous probability measure.*

Example 3. *Let*

$$f(x,y) = |xy|^{-1/2},\ -\pi \le x, y \le \pi\,.$$

By Theorem 2.4, it follows that

$$\mu_f = \eta_r \boxtimes WSL(1)\,,$$

where η_r *is the law of* $2^{3/2}\pi|U|^{-1/2}$*,* U *being an Uniform* $(-\pi, \pi)$ *random variable. Since the second moment of* η_r *is infinite, it follows that*

$$\int_{\mathbb{R}} x^4 \mu_f(dx) = \infty\,.$$

Acknowledgement

The authors are grateful to Manjunath Krishnapur for helpful discussions. They are also indebted to Octavio E. Arizmendi for pointing out a minor error in a previous version.

References

R. Adamczak. On the Marchenko-Pastur and circular laws for some classes of random matrices with dependent entries. *Electron. J. Probab.*, 16, 2011.

G. W. Anderson and O. Zeitouni. A law of large numbers for finite-range dependent random matrices. *Comm. Pure Appl. Math.*, 61(8):1118–1154, 2008.

G. W. Anderson, A. Guionnet, and O. Zeitouni. *An Introduction to Random Matrices*. Cambridge University Press, 2010.

O. E. Arizmendi and V. Pérez-Abreu. The S-transform of symmetric probability measures with unbounded support. *Proceedings of the American Mathematical Society*, 137(9):3057–3066, 2009.

Z. Bai and J. W. Silverstein. *Spectral analysis of large dimensional random matrices*. Springer Series in Statistics, New York, second edition, 2010.

H. Bercovici and D. Voiculescu. Free convolution of measures with unbounded support. *Indiana University Mathematics Journal*, 42:733–773, 1993.

R. Bhatia. *Perturbation Bounds for Matrix Eigenvalues*. Society for Industrial and Applied Mathematics, Philadelphia, 2007.

P. Billingsley. *Probability and Measure*. Wiley, New York, 3rd edition, 1995.

W. Bryc, A. Dembo, and T. Jiang. Spectral measure of large random Hankel, Markov and Toeplitz matrices. *Ann. Probab.*, 34(1):1–38, 2006.

S. Chatterjee. A generalization of the Lindeberg principle. *Ann. Probab.*, 34 (6):2061–2076, 2006. ISSN 0091-1798. doi: 10.1214/009117906000000575. URL http://dx.doi.org/10.1214/009117906000000575.

I. P. Cornfeld, S. V. Fomin, and Y. G. Sinai. *Ergodic Theory*. Springer-Verlag, 1982.

F. Götze and A. N. Tikhomirov. Limit theorems for spectra of random matrices with martingale structure. In *Stein's method and applications*, volume 5 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 181–193. Singapore Univ. Press, Singapore, 2005.

W. Hachem, P. Loubaton, and J. Najim. The empirical eigenvalue distribution of a gram matrix: from independence to stationarity. *Markov Process. Related Fields*, 11(4):629–648, 2005. ISSN 1024-2953.

K. Hofmann-Credner and M. Stolz. Wigner theorems for random matrices with dependent entries: ensembles associated to symmetric spaces and sample covariance matrices. *Electron. Commun. Probab.*, 13:401–414, 2008.

S. G. Krantz. *A panorama of harmonic analysis*. The Mathematical Society of America, Washington D.C., 1999.

A. S. Markus. The eigen and singular values of the sum and product of linear operators. *Russian Math. Surveys*, 19:92–120, 1964.

A. Naumov. Elliptic law for real random matrices. Arxiv:math/1201.1639, 2012.

H. H. Nguyen and S. O. Rourke. The elliptic law. Arxiv:math/1208.5883, 2012.

A. Nica and R. Speicher. *Lectures on the Combinatorics of Free Probability.* Cambridge University Press, New York, 2006.

O. Pfaffel and E. Schlemm. Limiting spectral distribution of a new random matrix model with dependence across rows and columns. *Linear Algebra Appl.*, 436(9):2966–2979, 2012.

R. Rashidi Far, T. Oraby, W. Bryc, and R. Speicher. On slow-fading MIMO systems with nonseparable correlation. *IEEE Trans. Inform. Theory*, 54 (2):544–553, 2008.

S. Resnick. *Heavy-Tail Phenomena : Probabilistic and Statistical Modeling.* Springer, New York, 2007.

G. Samorodnitsky. Long range dependence. *Foundations and Trends in Stochastic Systems*, 1(3):163–257, 2006.

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, New Delhi
E-mail address: arijit@isid.ac.in

Institut für Mathematik,Universität Zürich, Zürich
E-mail address: rajatmaths@gmail.com

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, New Delhi
E-mail address: deepayan@isid.ac.in