On Stochastic Comparisons of Residual Life Time at Random Time

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Abstract

Let $X_1, X_2, \Theta$ and $\Theta'$ be independent non-negative random variables. The residual life of $X_i$ at random time $\Theta$, that is, $X_i^\Theta = X_i - \Theta | X_i > \Theta$ is considered. Some sufficient conditions which lead to the likelihood ratio ordering, the failure rate ordering, the reverse failure rate ordering and the mean residual life ordering between $X_1^\Theta$ and $X_2^\Theta$ are obtained and an application in queuing theory is explained. A set of conditions which lead to the same stochastic orderings between $X_1^\Theta$ and $X_1'^\Theta$ are also derived.

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1 Introduction

Let $X$ be a non-negative random variable with distribution function $F$. The residual lifetime of $X$ at $t$, $t > 0$, denoted by $X^t$, is a random variable whose distribution is the same as the distribution of $X - t$ given that $X > t$, that is $X^t =_{st} (X - t | X > t)$. Stochastic characteristics like survival function and mean of $X^t$ are great tools to evaluate the stochastic behavior of $X$. For more details about residual random variable the reader is referred to Guess and Proschan (1988), Shaked and Shanthikumar (2007, Chapters 1 and 2), Nanda, Bhattacharjee and Balakrishnan (2010) and Cai and Zheng (2012).

If we replace $t$ with a random variable $\Theta$, independent of $X$, then the residual lifetime of $X$ at $\Theta$, denoted by $X^\Theta$, is defined to be residual lifetime of $X$ at random time $\Theta$. Stochastic aging properties and stochastic comparisons of residual lifetimes at random time have been investigated by Yue and Cao (2000), Yue and Cao (2001), Li and Zuo (2004), Misra, Gupta and Dhariyal (2008) and Eryilmaz (2013). The aim of this paper is to obtain some new stochastic orderings results among residual lifetimes at random time in one sample as well as two sample problems and give simpler proofs of some known results in the literature.

First let us recall some definitions of stochastic orders that are used later in this paper. Assume the positive random variables $X$ and $Y$ have distribution functions $F$ and $G$, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, density functions $f$ and $g$, reverse failure rate functions $\tilde{r}_X = f/F$ and $\tilde{r}_Y = g/G$ and failure rate functions $r_X = f/\bar{F}$ and $r_Y = g/\bar{G}$, respectively. The following stochastic orders are usually used to compare the random variables $X$ and $Y$.

**Definition 1.1.** $X$ is said to be smaller than $Y$ in the

(i) likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in $x$;

(ii) failure rate order (denoted by $X \leq_{fr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x$;

(iii) reverse failure rate order (denoted by $X \leq_{rf} Y$) if $G(x)/F(x)$ is increasing in $x$;

(iv) stochastic ordering (denoted by $X \leq_{st} Y$) if $F(x) \leq G(x)$ for every $x$;

(v) mean residual life order, denoted by $X \leq_{mrl} Y$, if

$$\int_t^\infty \frac{F(x)dx}{F(t)} \leq \int_t^\infty \frac{G(x)dx}{G(t)};$$

(vi) increasing convex order (denoted by $X \leq_{icx} Y$) if

$$\int_t^\infty \bar{F}(x)dx \leq \int_t^\infty \bar{G}(x)dx.$$ 

It is well known that $X \leq_{st} Y$ is equivalent to that

$$E[\phi(X)] \leq (\geq) E[\phi(Y)]$$

for all increasing (decreasing) functions $\phi : \mathcal{R} \to \mathcal{R}$, for which the expectations exist. It is also known that (cf. Shaked and Shanthikumar (2007)),

$$X \leq_{lr} Y \Rightarrow X \leq_{fr} Y \Rightarrow X \leq_{mrl} Y \Rightarrow EX \leq EY$$
and
\[ X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{icx} Y \Rightarrow EX \leq EY. \]

We shall also use the notions of totally positive of order 2 and reverse regular of order 2. Karlin (1968) is a comprehensive reference for TP$_2$ and RR$_2$ functions.

**Definition 1.2.** (i) A non-negative function $h(x, y)$ is said to be totally positive of order 2 (TP$_2$) if
\[ h(x, y)h(x', y') \geq h(x', y)h(x, y') \]
whenever $x \leq x'$ and $y \leq y'$.

(ii) A non-negative function $h(x, y)$ is said to be reverse regular of order 2 (RR$_2$) if
\[ h(x, y)h(x', y') \leq h(x', y)h(x, y') \]
whenever $x \leq x'$ and $y \leq y'$.

Let $X_1$, $X_2$, $\Theta_1$ and $\Theta_2$ be independent non-negative random variables. Yue and Cao (2000) considered stochastic comparisons between $X_1^{\Theta_1}$ and $X_1^{\Theta_2}$, the residual lifetime of $X_1$ at two different random times $\Theta_1$ and $\Theta_2$. They proved that if $\Theta_1 \leq_{rh} \Theta_2$ and $X$ is DFR (decreasing failure rate), then $X^{\Theta_1} \leq_{st} X^{\Theta_2}$. The inequality is reversed if $X$ is IFR (increasing failure rate). Misra, Gupta and Dhariyal (2008) in their Theorem 3.1 gave a lengthy proof for extending this result from the usual stochastic ordering to the failure rate ordering. We give a simpler proof of their result (Theorem 2.2 (c)).

Yue and Cao (2000) also showed that if $\Theta_1 \leq_{rh} \Theta_2$ and $X$ is IMRL (increasing mean residual life), then
\[ E(X^{\Theta_1}) \leq E(X^{\Theta_2}). \quad (1.2) \]
The inequality in (1.2) is reversed if $X$ is DMRL (decreasing mean residual life). Li and Zuo (2004) extended the above expectation order result to the increasing convex order. That is
\[ X^{\Theta_1} \leq_{icx} X^{\Theta_2}. \quad (1.3) \]
Misra, Gupta and Dhariyal (2008) further considered this problem and extended (1.2) to the mean residual life order. That is, they showed that if $\Theta_1 \leq_{rh} \Theta_2$ and $X$ is IMRL (DMRL), then
\[ X^{\Theta_1} \leq_{mrl} (\geq_{mrl}) X^{\Theta_2}. \quad (1.4) \]
We also give a simpler proof of the above result (Theorem 2.2 (d)).

In Section 2, we make stochastic comparisons between $X_1^\Theta$ and $X_2^\Theta$, the residual lifetimes of $X_1$ and $X_2$ at the same random time $\Theta$. We provide some sufficient conditions under which $X_1^\Theta$ is comparable with $X_2^\Theta$ according to the likelihood ratio order, the failure rate order, the reverse failure rate order and the mean residual order. We also make stochastic comparisons between $X^{\Theta_1}$ and $X^{\Theta_2}$, the residual life time of $X$ at two different random times $\Theta_1$ and $\Theta_2$ according to the likelihood ratio order and the reverse failure rate order. An application in queuing theory is explained in Section 3.
2 Main Results

We need the following lemma, which might be of independent interest, to prove the main results in this section.

Lemma 2.1. Let $h_i(x, \theta), i = 1, 2$, be a non-negative real valued function on $\mathbb{R} \times X$, where $X$ is a subset of real line. If

(i) $h_2(x, \theta)/h_1(x, \theta)$ is increasing in $x$ and $\theta$ and

(ii) if either $h_1(x, \theta)$ or $h_2(x, \theta)$ is TP $2$ in $(x, \theta)$,

then

$$s_i(x) = \int_X h_i(x, \theta) l(\theta) d\theta \quad (2.1)$$

is TP $2$ in $(x, \theta)$, where $l$ is a continuous function with $\int_X l(\theta) d\theta < \infty$.

Proof. First, we prove the required result when $h_1(x, \theta)$ is TP $2$ in $(x, \theta)$.

Let $\Theta^*(x)$ denote a random variable with density function given by

$$f_{\Theta^*}(x, \theta) = \frac{h_1(x, \theta) l(\theta)}{\int_X h_1(x, \theta) l(\theta) d\theta}.$$ 

Then the assumption $(ii)$ is equivalent to the fact that for $x_1 \leq x_2$, $\Theta^*(x_1) \leq_{tr} \Theta^*(x_2)$, which in turn implies that $\Theta^*(x_1) \leq_{st} \Theta^*(x_2)$.

Let $x_1 \leq x_2$. Then

$$s_2(x_2)/s_1(x_2) = \frac{\int_X h_2(x_2, \theta) l(\theta) d\theta}{\int_X h_1(x_2, \theta) l(\theta) d\theta} \geq \frac{\int_X h_2(x_2, \theta) h_1(x_2, \theta) l(\theta) d\theta}{\int_X h_1(x_2, \theta) l(\theta) d\theta}$$

$$\geq \frac{\int_X h_2(x_1, \theta) h_1(x_1, \theta) l(\theta) d\theta}{\int_X h_1(x_1, \theta) l(\theta) d\theta} \geq \frac{s_2(x_1)}{s_1(x_1)}. \quad (2.2)$$

Note that inequality (2.2) follows from the assumption $(i)$ that $h_2(x, \theta)/h_1(x, \theta)$ is increasing in $\theta$ for each $x \in \mathbb{R}$ and the inequality (1.1). The inequality (2.3) follows from the assumption $(i)$ that $h_2(x, \theta)/h_1(x, \theta)$ is increasing $x$ for each $\theta \in X$.
Next assume that $h_2(x, \theta)$ is TP$_2$ in $(x, \theta)$. Let $x_1 \leq x_2$. Then

$$\frac{s_1(x_2)}{s_2(x_2)} = \frac{\int_X h_1(x_2, \theta) l(\theta) d\theta}{\int_X h_2(x_2, \theta) l(\theta) d\theta}$$

$$= \int_X h_1(x_2, \theta) \frac{h_2(x_2, \theta) l(\theta)}{\int_X h_2(x_2, \theta) l(\theta) d\theta} d\theta$$

$$\leq \int_X h_1(x_2, \theta) \frac{h_2(x_1, \theta) l(\theta)}{\int_X h_2(x_1, \theta) l(\theta) d\theta} d\theta$$

$$\leq \int_X h_1(x_1, \theta) \frac{h_2(x_1, \theta) l(\theta)}{\int_X h_2(x_1, \theta) l(\theta) d\theta} d\theta$$

$$= \frac{s_1(x_1)}{s_2(x_1)}.$$  \hspace{1cm} (2.4)

The inequalities (2.4) and (2.5) follow using arguments similar to the ones used to show inequalities (2.2) and (2.3). \hfill \Box

Let $X$ and $\Theta$ be two independent non-negative random variables with distribution functions $F$ and $H$, survival functions $F$ and $H$, density functions $f$ and $h$, respectively. The residual life time of $X$ at $\Theta$, denoted by $X \Theta$, is defined to be a random variable with a distribution function equal to that of $X - \Theta$ given that $X > \Theta$, that is $X \Theta =_{st} (X - \Theta | X > \Theta)$. Then, the density function, distribution function, survival function and mean residual life (mrl) function of $X \Theta$, are respectively given by

$$g_{X \Theta}(x) = \frac{\int_0^\infty f(x + \theta) h(\theta) d\theta}{P(X > \Theta)}, \hspace{1cm} (2.6)$$

$$G_{X \Theta}(x) = \frac{\int_0^\infty F(x + \theta) h(\theta) d\theta}{P(X > \Theta)}, \hspace{1cm} (2.7)$$

$$\overline{G}_{X \Theta}(x) = \frac{\int_0^\infty \overline{F}(x + \theta) h(\theta) d\theta}{P(X > \Theta)}, \hspace{1cm} (2.8)$$

and

$$m_{X \Theta}(x) = \frac{\int_x^\infty \overline{G}_{X \Theta}(u) du}{P(X \Theta > x)}.$$ \hspace{1cm} (2.9)

**Theorem 2.2.** Let $X_i, i = 1, 2$ be two independent random variables with $X_i, i = 1, 2$ having density function $f_i$, distribution function $F_i$, survival function $\overline{F}_i$ and mrl function $m_i$. Let $\Theta$ be a random variable with density function $h$ and distribution function $H$. $\Theta$ is independent of $X_1$ and $X_2$.

(a) If $X_1 \leq_{tr} X_2$ and either $X_1$ or $X_2$ is ILR, then

$$X_1^\Theta \leq_{tr} X_2^\Theta.$$  \hspace{1cm} (2.10)

(b) If $X_1 \leq_{rh} X_2$ and either $X_1$ or $X_2$ is IRFR, then

$$X_1^\Theta \leq_{rh} X_2^\Theta.$$  \hspace{1cm} (2.11)
(c) If \( X_1 \leq_{hr} X_2 \) and either \( X_1 \) or \( X_2 \) is DFR, then
\[
X_1^{\Theta} \leq_{hr} X_2^{\Theta}.
\]

(d) If \( X_1 \leq_{mrl} X_2 \) and either \( X_1 \) or \( X_2 \) is IMRL, then
\[
X_1^{\Theta} \leq_{mrl} X_2^{\Theta}.
\]

Proof. (a) From (2.6), the density function of \( X_i^{\Theta} \) is
\[
g_{X_i^{\Theta}}(x) = \frac{\int_0^\infty f_i(x + \theta) h(\theta) d\theta}{P(X > \Theta)}, \quad i = 1, 2.
\]

In Lemma 2.1, replace \( l(\theta) \) with \( h(\theta) \) and \( h_i(x, \theta) \) with \( f_i(x + \theta) \) for \( i = 1, 2 \). The random variable \( X_i \) is ILR if and only if \( f_i(x + \theta) \) is TP2 in \( x \) and \( \theta \). On the other hand, \( X_1 \leq_{tr} X_2 \) if and only if \( f_2(u)/f_1(u) \) is increasing in \( u \) which in turn implies that \( f_2(x + \theta)/f_1(x + \theta) \) is increasing in \( x \) as well as \( \theta \). Combining these observations, the required result of part (a) follows from Lemma 2.1.

(b) \( X_i \) is IRFR if and only if \( F_i(x + \theta) \) is TP2 in \( x \) and \( \theta \). On the other hand, \( X_1 \leq_{rh} X_2 \) if and only if \( F_2(u)/F_1(u) \) is increasing in \( u \) which in turn implies that \( F_2(x + \theta)/F_1(x + \theta) \) is increasing in \( x \) as well as \( \theta \). That is, the conditions of Lemma 2.1 (b) are satisfied by replacing the function \( l(\theta) \) with \( h(\theta) \) and \( h_i(x, \theta) \) with \( F_i(x + \theta) \), \( i = 1, 2 \). This proves part (b).

(c) \( X_i \) is DFR if and only if \( F_i(x + \theta) \) is TP2 in \( x \) and \( \theta \). On the other hand, \( X_1 \leq_{hr} X_2 \) if and only if \( F_2(u)/F_1(u) \) is increasing in \( u \) which in turn implies that \( F_2(x + \theta)/F_1(x + \theta) \) is increasing in \( x \) as well as \( \theta \). That is, the conditions of Lemma 2.1 (c) are satisfied by replacing the function \( l(\theta) \) with \( h(\theta) \) and \( h_i(x, \theta) \) with \( F_i(x + \theta) \), \( i = 1, 2 \). This proves part (c).

(d) Using (2.9), the mrl function of \( X_i^{\Theta} \), \( i = 1, 2 \) can be written as
\[
m_{X_i^{\Theta}}(x) = \frac{\int_x^\infty G_{X_i^{\Theta}}(u) du}{P(X_i^{\Theta} > x)}
= \frac{\int_0^\infty \{\int_x^\infty F_i(u + \theta) du\} h(\theta) d\theta}{P(X_i^{\Theta} > x)}
= \frac{\int_0^\infty \{\int_{x+\theta}^\infty F_i(u) du\} h(\theta) d\theta}{P(X_i^{\Theta} > x)}
\]

\( X_i \) is IMRL if and only if \( \int_{x+\theta}^\infty F_i(u) du \) is TP2 in \( x \) and \( \theta \). On the other hand, \( X_1 \leq_{mrl} X_2 \) implies that \( \int_{x+\theta}^\infty F_2(u) du / \int_{x+\theta}^\infty F_1(u) du \) is increasing in \( x \) and \( \theta \). That is, the conditions of Lemma 2.1 (d) are satisfied by replacing the function \( l(\theta) \) with \( h(\theta) \) and \( h_i(x, \theta) \) with \( \int_{x+\theta}^\infty F_i(u) du, \ i = 1, 2 \). This proves part (d). \( \square \)
Example 2.3. Let \( X_i, i = 1, 2 \) be a gamma random variable with density function
\[
f(x; \alpha_i, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-x\beta}, \ x > 0; \ \alpha_i > 0, \ \beta > 0.
\]
If \( \alpha_1 < 1 \) and \( \alpha_1 \leq \alpha_2 \), then it is easy to see that \( X_1 \leq_{lr} X_2 \) and \( X_1 \) is ILR. Therefore it follows from Theorem 2.2 (a) that for any non-negative random variable \( \Theta \), \( X_1^\Theta \leq_{lr} X_2^\Theta \).

Example 2.4. Let \( X_1 \) be a random variable with density function
\[
f_{X_1}(x) = \left( \frac{1}{\sqrt{x}} + 1 \right) \exp(-2\sqrt{x} - x), \ x > 0
\]
and \( X_2 \) be another random variable with density function
\[
f_{X_2}(x) = \left( \frac{1}{\sqrt{x}} + \frac{1}{2} \right) \exp(-2\sqrt{x} - \frac{x}{2}), \ x > 0.
\]
It is easy to see that \( X_1 \leq_{hr} X_2 \) and both \( X_1 \) and \( X_2 \) are DFR. Therefore it follows from Theorem 2.2 (c) that for any non-negative random variable \( \Theta \), \( X_1^\Theta \leq_{hr} X_2^\Theta \). Note that in this example \( X_1 \not\leq_{lr} X_2 \).

The following three lemmas will be used below to obtain stochastic orderings between \( X_1^\Theta \) and \( X_2^\Theta \).

Lemma 2.5. (Karlin (1968), p.99) Let \( g_1 : \mathbb{R} \to \mathbb{R} \) and \( g_2 : \mathbb{R} \to \mathbb{R} \) be two continuous functions and \( f_1 \) and \( f_2 \) be two density functions. Suppose that
\[
\int_{\mathbb{R}} g_k(s) f_i(s) ds \text{ exists and is finite, } \ k = 1, 2, \ i = 1, 2
\]
and
(i) \( f_i(s) \) is TP \( (RR_2) \) in \( (i, s) \in \{1, 2\} \times \mathbb{R} \),
(ii) \( g_k(s) \) is TP \( 2 \) in \( (k, s) \in \{1, 2\} \times \mathbb{R} \),
Then \( \int g_k(s) f_i(s) ds \) is TP \( 2 \) \( (RR_2) \) in \( (i, k) \in \{1, 2\} \times \{1, 2\} \).

Lemma 2.6. (Joag-Dev, Kochar and Proschan (1995), p. 115) Let \( g_1 : \mathbb{R} \to \mathbb{R} \) and \( g_2 : \mathbb{R} \to \mathbb{R} \) be two differentiable functions with derivatives \( g_1' \) and \( g_2' \), and let \( F_1 \) and \( F_2 \) be two distribution functions with respective density functions \( f_1 \) and \( f_2 \), and respective survival functions \( \overline{F}_1 \) and \( \overline{F}_2 \). Suppose that
\[
\int_{\mathbb{R}} g_k(s) dF_i(s) \text{ exists and is finite, } \ k = 1, 2, \ i = 1, 2
\]
and
Lemma 2.7. (Khaledi and Shaked (2010), p. 2490) Let \( g_1 : \mathbb{R} \to \mathbb{R} \) and \( g_2 : \mathbb{R} \to \mathbb{R} \) be two differentiable functions with derivatives \( g_1' \) and \( g_2' \), and let \( F_1 \) and \( F_2 \) be two distribution functions with respective density functions \( f_1 \) and \( f_2 \), and respective survival functions \( F_1 \) and \( F_2 \). Suppose that
\[
\int_{\mathbb{R}} g_k(s) dF_i(s) \text{ exists and is finite, } \quad k = 1, 2, \quad i = 1, 2
\]
and
\[
\int_{\mathbb{R}} g_k(s) dF_i(s) \text{ exists and is finite, } \quad k = 1, 2, \quad i = 1, 2
\]

(i) \( F_i(s) \) is TP in \((i, s) \in \{1, 2\} \times \mathbb{R}\),
(ii) \( g_k(s) \) is TP in \((k, s) \in \{1, 2\} \times \mathbb{R}\),
(iii) \( g_1(s) \) is increasing in \( s \in \mathbb{R} \), for \( k = 1, 2 \).

Then \( \int g_k(s) f_i(s) \, ds \) is TP in \((i, k) \in \{1, 2\} \times \{1, 2\}\).

Theorem 2.8. Let \( \Theta_i, \ i = 1, 2 \) be two independent random variables with \( \Theta_i, \ i = 1, 2 \) having density function \( h_i \), distribution function \( H_i \), survival function \( \overline{H}_i \) and mrl function \( \mu_i \). Let also \( X \) be a random variable with density function \( f \), distribution function \( F \), survival function \( \overline{F} \). \( X \) is independent of \( \Theta_1 \) and \( \Theta_2 \).

(a) If \( X \) is ILR (DLR) and \( \Theta_1 \leq_{lr} \Theta_2 \), then
\[
X^{\Theta_1} \leq_{lr} (\geq_{lr}) X^{\Theta_2}.
\]

(b) \( X \) is IRFR and \( \Theta_1 \leq_{hr} \Theta_2 \), then
\[
X^{\Theta_1} \leq_{hr} X^{\Theta_2}.
\]

(c) (Theorem 3.1 of Misra et al. (2008)) If \( X \) is IFR and \( \Theta_1 \leq_{rh} \Theta_2 \), then
\[
X^{\Theta_1} \geq_{hr} X^{\Theta_2}.
\]

(d) (Theorem 3.2 of Misra et al. (2008)) If \( X \) is DMRL and \( \Theta_1 \leq_{rh} \Theta_2 \), then
\[
X^{\Theta_1} \geq_{mrl} X^{\Theta_2}.
\]
Proof. (a) From (2.6), the density function of \( X^{\Theta_i} \) is

\[
g_{X^{\Theta_i}}(x) = \frac{\int_0^\infty f(x + \theta) h_i(\theta) d\theta}{P(X > \Theta_i)}, \quad i = 1, 2.
\]

\( f(x + \theta) \) is \( TP_2 \) (\( RR_2 \)) in \((x, \theta)\), since \( X \) is \( ILR \) (\( DLR \)). \( h_i(\theta) \) is \( TP_2 \) in \((i, \theta)\), since \( \Theta_1 \leq_{tr} \Theta_2 \). Using these results, it follows from Lemma 2.5 that the function

\[
\int_0^\infty f(x + \theta) h_i(\theta) d\theta
\]
is \( TP_2 \) (\( RR_2 \)) in \((i, x)\) which proves the required results of part (a).

(b) From (2.7), the distribution function of \( X^{\Theta_i} \), \( i = 1, 2 \), is

\[
G_{X^{\Theta_i}}(x) = \frac{\int_0^\infty F(x + \theta) h_i(\theta) d\theta}{P(X > \Theta_i)}.
\]

\( F(x + \theta) \) is \( TP_2 \) in \((x, \theta)\), since \( X \) is \( IRFR \). \( H_i(\theta) \) is \( TP_2 \) in \((i, \theta)\), since \( \Theta_1 \leq_{hr} \Theta_2 \). Hence, it follows from Lemma 2.6 that

\[
\int_0^\infty F(x + \theta) h_i(\theta) d\theta
\]
is \( TP_2 \) in \((i, x)\) which is the required result of part (b).

(c) From (2.8), the survival function of \( X^{\Theta_i} \), \( i = 1, 2 \), is

\[
\overline{G}_{X^{\Theta_i}}(x) = \frac{\int_0^\infty \overline{F}(x + \theta) h_i(\theta) d\theta}{P(X > \Theta_i)}.
\]

\( \overline{F}(x + \theta) \) is \( RR_2 \) in \((x, \theta)\), since \( X \) is \( IFR \). \( H_i(\theta) \) is \( TP_2 \) in \((i, \theta)\), since \( \Theta_1 \leq_{hr} \Theta_2 \). The function \( \overline{F}(x + \theta) \) is decreasing in \( \theta \). Therefore, it follows from Lemma 2.7 that the function

\[
\int_0^\infty \overline{F}(x + \theta) h_i(\theta) d\theta
\]
is \( RR_2 \) in \((i, x)\) which proves part (c).

(d) The \( mrl \) function of \( X^{\Theta_i} \), \( i = 1, 2 \), is

\[
m_{X^{\Theta_i}}(x) = \frac{\int_0^\infty \{\int_{x+\theta}^\infty \overline{F}(u) du\} h_i(\theta) d\theta}{P(X^{\Theta_i} > x)}
\]

\( X \) is \( DMRL \) is equivalent to \( \int_{x+\theta}^\infty \overline{F}(u) du \) is \( RR_2 \) in \((x, \theta)\). \( H_i(\theta) \) is \( TP_2 \) in \((i, \theta)\), since \( \Theta_1 \leq_{hr} \Theta_2 \). The function is decreasing in \( \theta \). Combining these results, it follows from Lemma 2.7 that \( m_{X^{\Theta_i}}(x) \) is \( RR_2 \) in \((i, x)\) which proves part (d). }
3 An Application in Queuing Theory

In a GI/G/1 queue, let $T_n$ with distribution $F_T$, denote the time between $n$th and $(n + 1)$th arrival, $W_n$ with distribution $F_W$, denote the waiting time in the queue for the $n$th customer, $S_n$ with distribution $F_S$, denote service time of the $n$th customer and $I$ with distribution $H$, denote the length of idle period between busy periods. It is well known that $W_{n+1} = \max\{0, W_n + S_n - T_n\}$ and $I =_{st} (T - (W + S))|W + S > T)$, where $st$ stands for equal in distribution (cf. Marshall (1968)).

Suppose that distributions of $T$, $W$ and $S$ are not completely known and only we know that the hazard rate of $T$ is bounded by some positive known constants. That is suppose that for $0 < \lambda_1 < \lambda_2$,

$$\lambda_1 \leq r_T(t) \leq \lambda_2,$$

then it follows from Theorem 2.2 (c) that

$$\lambda_1 \leq r_I(t) \leq \lambda_2. \quad (3.2)$$

To prove this observation, let $E_{\lambda_i}, i = 1, 2$, be an exponential random variable with hazard rate $\lambda_i$ independent of $S$ and $W$. Then (3.1) is equivalent to that

$$E_{\lambda_2} \leq_{hr} T \leq_{hr} E_{\lambda_1}. \quad (3.3)$$

On the other hand, from (2.6), it is easy to see that $I_i =_{st} E_{\lambda_i}, i = 1, 2$, where $I_i$ is the length of idle period between busy periods of a queue with inter-arrival time $E_{\lambda_i}$ and an arbitrary servicing time. Using this observation, (3.3), the fact that exponential random variable is DFR and Theorem 2.2 (c), we obtain that

$$I_2 =_{st} E_{\lambda_2} =_{st} E_{\lambda_2}^{S+W} \leq_{hr} T^{S+W} \leq_{hr} E_{\lambda_1}^{S+W} =_{st} E_{\lambda_1} =_{st} I_1,$$

which is equivalent to (3.2). Inequalities (3.2) gives a lower bound and upper bound on the hazard rate of $r_I$ without any IFR assumption on $T$. Therefore it is comparable to the ones given in Theorem 6 of Marshall (1968) and it is a generalization of Theorem 4 in Marshall (1968) which is discussed next.

If instead of (3.1) it is known that the mean residual life function of $T$ is bounded with some known constants, that is

$$\gamma_1 \leq m_T(t) \leq \gamma_2,$$

then, using similar kind of arguments, it follows from Theorem 2.2 (d) that

$$\gamma_1 \leq m_I(t) \leq \gamma_2. \quad (3.5)$$

Inequalities (3.5) were directly proved in Marshall (1968).

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References


