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Remarks on absolute continuity in the context of free probability and random matrices

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REMARKS ON ABSOLUTE CONTINUITY IN THE CONTEXT OF FREE PROBABILITY AND RANDOM MATRICES

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Abstract. In this note, we show that the limiting spectral distribution of symmetric random matrices with stationary entries is absolutely continuous under some sufficient conditions. This result is applied to obtain sufficient conditions on a probability measure for its free multiplicative convolution with the semicircle law to be absolutely continuous.

1. Introduction

For two probability measures μ and ν on $\mathbb{R}$, one can associate the free additive convolution $\mu \boxplus \nu$. This is defined as the distribution of $X_\mu + Y_\nu$ where X_μ and Y_ν are self-adjoint variables affiliated to a tracial W^*- probability space and are free from each other. Similarly, for probability measures μ and ν on $[0, \infty)$, the free multiplicative convolution is denoted by $\mu \boxtimes \nu$ and represents the law of $X_\mu^{1/2} Y_\nu X_\mu^{1/2}$ where X_μ and Y_ν are free positive variables as before. In general, one can extend $\mu \boxtimes \nu$ to measures μ which are symmetric and ν which are supported on $[0, \infty)$ such that $\mu(\{0\}) \vee \nu(\{0\}) < 1$. We refer to [4] and [1] for details of these notions. The questions of absolute continuity of these convolutions, with respect to the Lebesgue measure, are important. For compactly supported and absolutely continuous measures μ and ν, [9] showed that $\mu \boxplus \nu$ is absolutely continuous. The result was extended to the non-compactly supported case when one of the measures is the semicircle law by [5]. Further regularity properties of additive convolution were studied by [2].

Some recent works study regularity properties in the context of free multiplicative convolutions, see for example [3, 10]. However, absolute continuity is much less understood. The following question is a step in that direction, namely, the multiplicative analogue of the problem addressed in [5].

Question 1. *Let μ be any probability measure on $[0, \infty)$, and let μ_s denote the semicircle law, defined in* (5)*. Under what conditions on μ, is $\mu \boxtimes \mu_s$ absolutely continuous?*

Our first result gives a sufficient condition on μ to answer Question 1.

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Theorem 1.1. *Let μ be a probability measure on $\mathbb{R}$ such that $\mu([\delta,\infty))=1$ for some $\delta>0$. Assume furthermore that μ has finite mean. Then, $\mu\boxtimes\mu_s$ is absolutely continuous.*

For stating the next question, we need to introduce a random matrix model. Let f be any non-negative integrable function on $[-\pi,\pi]^2$. Then, there exists a mean zero stationary Gaussian process $(G_{i,j}: i,j\in\mathbb{Z})$ such that

$$\mathrm{E}\left(G_{i,j}G_{i+u,j+v}\right)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}e^{\iota(ux+vy)}f(x,y)dxdy,\ \text{for all } i,j,u,v\in\mathbb{Z}\,. \tag{1}$$

For $N\geq 1$, let $\overline{G}_N$ be the $N\times N$ matrix defined by

$$\overline{G}_N(i,j):=(G_{i,j}+G_{j,i})/\sqrt{N},\ 1\leq i,j\leq N\,. \tag{2}$$

Above and elsewhere, for any matrix H, $H(i,j)$ denotes its (i,j)-th entry. It has been shown in Theorem 2.1 of [6] that there exists a (deterministic) probability measure ν_f such that

$$\mathrm{ESD}(\overline{G}_N)\to\nu_f\,, \tag{3}$$

weakly in probability, as $N\to\infty$. ESD stands for the empirical spectral distribution for a symmetric $N\times N$ random matrix H, which is a random probability measure on $\mathbb{R}$ defined by

$$\left(\mathrm{ESD}(H)\right)(\cdot):=\frac{1}{N}\sum_{j=1}^{N}\delta_{\lambda_j}(\cdot)\,,$$

where $\lambda_1\leq\ldots\leq\lambda_N$ are the eigenvalues of H, counted with multiplicity.

The second question that this paper attempts to answer is the following.

Question 2. *Under what conditions on f, is ν_f absolutely continuous?*

The answer is provided by the following result.

Theorem 1.2. *If*

$$ess\inf_{(x,y)\in[-\pi,\pi]^2}\left[f(x,y)+f(y,x)\right]>0\,,$$

then ν_f is absolutely continuous, where "$ess\inf$" denotes the essential infimum.

While Questions 1 and 2 seem unrelated a priori, the reader will notice after seeing the proofs that they are not so. This is because random matrix theory is used as a tool for proving Theorem 1.1, Theorem 1.2 being anyway a question about a random matrix. It is shown in Proposition 2.1, which is a consequence of Theorem 1.2, that if a measure satisfies the conditions of Theorem 1.1 then, its free multiplicative convolution with a semicircle law is also the free additive convolution of another measure and a dilated semicircle law. The proofs are compiled in the following section. Many known facts are used, which are collected in Section 3 for the convenience of the reader.

2. Proofs

For the proof of the results we shall refer to various known facts which are listed in the Appendix of this article. One of the main ingredients is the following fact which follows from Proposition 22.32, page 375 of [7]. The latter result reasserts the seminal discovery in [8] that the Wigner matrix is asymptotically freely independent of a deterministic matrix which has a compactly supported limiting spectral distribution.

Fact 2.1. *Assume that for each N, A_N is a $N\times N$ Gaussian Wigner matrix scaled by $\sqrt{N}$, that is, $(A_N(i,j) : 1 \le i \le j \le N)$ are i.i.d. normal random variables with mean zero and variance $1/N$, and $A_N(j,i) = A_N(i,j)$. Suppose that B_N is a $N \times N$ random matrix, such that as $N \to \infty$,*

$$\frac{1}{N}\operatorname{Tr}(B_N^k) \xrightarrow{P} \int_{\mathbb{R}} x^k \mu(dx),\ k \ge 1\,, \tag{4}$$

for some compactly supported (deterministic) probability measure μ. Furthermore, let the families $(A_N : N \ge 1)$ and $(B_N : N \ge 1)$ be independent. Then, as $N \to \infty$,

$$\frac{1}{N}\mathrm{E}_{\mathcal{F}}\operatorname{Tr}\left[(A_N + B_N)^k\right] \xrightarrow{P} \int_{\mathbb{R}} x^k \mu \boxplus \mu_s(dx) \text{ for all } k \ge 1\,,$$

where $\mathcal{F} := \sigma(B_N : N \ge 1)$ and $\mathrm{E}_{\mathcal{F}}$ denotes the conditional expectation with respect to $\mathcal{F}$.

We first proceed towards proving Theorem 1.2. The first step in that direction is Lemma 2.1 below. However, before stating that, we define a dilated semicircle law $\mu_s(t)$ for all $t > 0$. It is a probability measure on $\mathbb{R}$ given by

$$(\mu_s(t))\,(dx) = \frac{\sqrt{4t - x^2}}{2\pi t}\mathbf{1}(|x| \le 2\sqrt{t}),\ x \in \mathbb{R}\,. \tag{5}$$

For $t = 1$, it equals the standard semicircle law, that is, $\mu_s \equiv \mu_s(1)$.

Lemma 2.1. *Let f be a non-negative trigonometric polynomial on $[-\pi, \pi]^2$ as in (8), and $\alpha > 0$. Denote*

$$(f + \alpha)(\cdot,\cdot) := f(\cdot,\cdot) + \alpha\,.$$

Then,

$$\nu_{f+\alpha} = \nu_f \boxplus \mu_s(8\pi^2\alpha)\,.$$

Proof. By Fact 3.5, $\nu_{f+\alpha}$ and ν_f have compact supports, and hence so does $\nu_f \boxplus \mu_s(8\pi^2\alpha)$. Therefore, it suffices to check that

$$\int x^k \nu_{f+\alpha}(dx) = \int x^k (\nu_f \boxplus \mu_s(8\pi^2\alpha))(dx) \text{ for all } k \ge 1\,. \tag{6}$$

Let $(G_{i,j} : i, j \in \mathbb{Z})$ be a mean zero stationary Gaussian process satisfying (1). Let $(H_{i,j} : i, j \in \mathbb{Z})$ be a family of i.i.d. $N(0, 4\pi^2\alpha)$ random variables, independent of $(G_{i,j} : i, j \in \mathbb{Z})$. For $N \geq 1$, let $\overline{G}_N$ be as in (2), and further define the $N \times N$ matrices W_N and Z_N by

$$\begin{aligned} W_N(i,j) &:= (H_{i,j} + H_{j,i})/\sqrt{N},\ 1 \leq i, j \leq N, \\ Z_N &:= \overline{G}_N + W_N\,. \end{aligned}$$

Fact 3.5 implies that

$$\frac{1}{N}\operatorname{Tr}\left(\overline{G}_N^k\right) \xrightarrow{P} \int x^k \nu_f(dx),\ k \geq 1\,,$$

as $N \to \infty$. This, along with Fact 2.1 and the observation that the upper triangular entries of W_N are i.i.d. $N(0, 8\pi^2\alpha/N)$, implies that

$$\frac{1}{N}\mathrm{E}_{\mathcal{F}}\operatorname{Tr}\left(Z_N^k\right) \xrightarrow{P} \int x^k(\nu_f \boxplus \mu_s(8\pi^2\alpha))(dx),\ k \geq 1\,, \tag{7}$$

where $\mathcal{F} := \sigma(G_{i,j} : i, j \in \mathbb{Z})$.

It is easy to see that $(G_{i,j} + H_{i,j} : i, j \in \mathbb{Z})$ is a stationary mean zero Gaussian process whose spectral density is $f + \alpha$, and hence Fact 3.5 implies that

$$\mathrm{E}\left[\frac{1}{N}E_{\mathcal{F}}\operatorname{Tr}\left(Z_N^k\right)\right] = \mathrm{E}\left[\frac{1}{N}\operatorname{Tr}\left(Z_N^k\right)\right] \to \int x^k \nu_{f+\alpha}(dx)\,,$$

and

$$\mathrm{Var}\left[\frac{1}{N}E_{\mathcal{F}}\operatorname{Tr}\left(Z_N^k\right)\right] \leq \mathrm{Var}\left[\frac{1}{N}\operatorname{Tr}\left(Z_N^k\right)\right] \to 0\,,$$

as $N \to \infty$, for all $k \geq 1$. Combining the above two limits and comparing with (7) yields (6), and completes the proof. □

Proof of Theorem 1.2. Define

$$g(x,y) := \frac{1}{2}\left[f(x,y) + f(y,x)\right],\ -\pi \leq x, y \leq \pi\,.$$

In view of Fact 3.4, it suffices to show that ν_g is absolutely continuous. The hypothesis implies that there exists $\alpha > 0$ such that $g \geq \alpha$ almost everywhere on $[-\pi, \pi]^2$. Define

$$h(\cdot,\cdot) := g(\cdot,\cdot) - \alpha\,.$$

Since the Fourier coefficients of f are real, f is an even function, and hence so is h. Therefore, by considering the Fourier series of $\sqrt{h}$, one can construct non-negative trigonometric polynomials h_n such that

$$h_n \to h \text{ in } L^1\,.$$

Lemma 2.1 implies that

$$\begin{aligned} \nu_{h_n+\alpha} &= \nu_{h_n} \boxplus \mu_s(8\pi^2\alpha) \\ &\xrightarrow{w} \nu_h \boxplus \mu_s(8\pi^2\alpha)\,, \end{aligned}$$

as $n \to \infty$, the second line following from Fact 3.3 combined with Proposition 4.13 of [4]. Applying Fact 3.3 directly to $\nu_{h_n+\alpha}$ and combining with Fact 3.4 yields

$$\nu_f = \nu_g = \nu_h \boxplus \mu_s(8\pi^2\alpha)\,.$$

Fact 3.1 completes the proof. □

For proving Theorem 1.1, we prove the following result which is of some independent interest. This along with Fact 3.1 establishes Theorem 1.1.

Proposition 2.1. *If μ satisfies the hypothesis of Theorem 1.1, then there exists a probability measure η such that*

$$\mu \boxtimes \mu_s = \eta \boxplus \mu_s(\delta^2)\,.$$

Proof. Define

$$r(x) := \frac{1}{2^{3/2}\pi} \inf\left\{y \in \mathbb{R} : \frac{x+\pi}{2\pi} \le \mu(-\infty, y]\right\},\ -\pi < x < \pi\,.$$

Let U be an Uniform$(-\pi, \pi)$ random variable. Clearly,

$$P(2^{3/2}\pi r(U) \in \cdot) = \mu(\cdot)\,,$$

which implies that

$$\int_{-\pi}^{\pi} r(x)dx = 2\pi \mathrm{E}[r(U)] = 2^{-1/2}\int_0^\infty x\mu(dx) < \infty\,.$$

The hypothesis that $\mu(-\infty, \delta) = 0$ implies that

$$r(x) \ge \frac{\delta}{2^{3/2}\pi},\ -\pi < x < \pi\,.$$

Defining

$$f(x, y) := r(x)r(y)\,,$$

it follows that f is a non-negative integrable function bounded below by α, where

$$\alpha := \frac{\delta^2}{8\pi^2}\,.$$

Fact 3.2 implies that

$$\begin{aligned} \mu \boxtimes \mu_s &= \nu_f \\ &= \nu_{f-\alpha} \boxplus \mu_s(8\pi^2\alpha)\,, \end{aligned}$$

the second equality following from Lemma 2.1. Setting $\eta := \nu_{f-\alpha}$, this completes the proof. □

Proof of Theorem 1.1. Follows from Proposition 2.1 and Fact 3.1. □

3. Appendix

In this section, we collect the various facts that have been used in the proofs in Section 2.

The following fact is Corollary 2 of [5].

Fact 3.1. *For any probability measure μ, $\mu \boxplus \mu_s$ is absolutely continuous with respect to the Lebesgue measure.*

The remaining facts are all quoted from [6].

Fact 3.2 (Theorem 2.4, [6])**.** *Let r be a non-negative integrable function defined on $[-\pi, \pi]$, and*

$$f(x, y) := r(x)r(y), \; -\pi \le x, y \le \pi \, .$$

Then,

$$\nu_f = \mu_r \boxtimes \mu_s \, ,$$

where ν_f is as in (3), μ_r is the law of $2^{3/2}\pi r(U)$, and U is a Uniform$(-\pi, \pi)$ random variable.

Fact 3.3 (Lemma 3.3, [6])**.** *Suppose that for all $1 \le n \le \infty$, g_n is a non-negative, integrable and even function on $[-\pi, \pi]^2$. By even, it is meant that $g_n(-x, -y) = g_n(x, y)$ for all x, y. If*

$$g_n \to g_\infty \text{ in } L^1 \text{ as } n \to \infty \, ,$$

then

$$\nu_{g_n} \xrightarrow{w} \nu_{g_\infty} \, .$$

Fact 3.4 (Lemma 3.5, [6])**.** *If f is a non-negative integrable function on $[-\pi, \pi]^2$, and*

$$g(x, y) := \frac{1}{2} \left[f(x, y) + f(y, x) \right] \, ,$$

then

$$\nu_f = \nu_g \, .$$

The following fact has been proved in the course of proving Proposition 3.1 of [6]; see (3.16) and (3.17).

Fact 3.5. *Let f be a non-negative trigonometric polynomial on $[-\pi, \pi]^2$, that is,*

$$f(x, y) = \sum_{j,k=-n}^{n} a_{j,k} e^{\iota(jx+ky)} \ge 0 \, , \tag{8}$$

for some finite n and real numbers $a_{j,k}$. Let the matrix $\overline{G}_N$ be constructed as in (2) using the random variables $(G_{i,j})$ which are as in (1). Then, ν_f

has compact support, and for all $k \geq 1$,

$$\lim_{N\to\infty} \mathrm{E}\left[\frac{1}{N}\,\mathrm{Tr}(\overline{G}_N^k)\right] = \int_{\mathbb{R}} x^k \nu_f(dx)\,,$$
$$\textit{and} \lim_{N\to\infty} \mathrm{Var}\left[\frac{1}{N}\,\mathrm{Tr}(\overline{G}_N^k)\right] = 0\,.$$

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