On the construction of asymmetric orthogonal arrays

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Abstract: A general method of construction of asymmetric orthogonal arrays was proposed by Suen, Das and Dey (2001), which led to several new families of orthogonal arrays of strength three and four. Using this method, we construct some more asymmetric orthogonal arrays of strength greater than two.

MSC: 62K15

Keywords: Asymmetric orthogonal array; Galois field.

1 Introduction and Preliminaries

Asymmetric orthogonal arrays introduced by Rao (1973) have received considerable attention in recent years. Such arrays are useful in experimental designs as universally optimal fractions of asymmetric factorials. Asymmetric orthogonal arrays have also been found very useful in industrial experimentation for quality improvement. Construction of asymmetric orthogonal arrays of strength two has been an area of intense research and one may refer to Hedayat, Sloane and Stufken (1999) for an excellent description of these. Relatively less is known on the construction of asymmetric orthogonal arrays of strength larger than two. Apart from the methods of construction
of asymmetric orthogonal arrays of strength larger than two described in Dey and Mukerjee (1999) and Hedayat et al. (1999), further work on the construction of arrays of strength three or higher have been carried out e.g., by Suen et al. (2001), Suen and Dey (2003), Nguyen (2008) and Jiang and Yin (2013). In particular, Suen et al. (2001) proposed a general method to construct asymmetric orthogonal arrays of arbitrary strength. This method was then applied by them to obtain several families of asymmetric orthogonal arrays of strength three and four. Suen and Dey (2003) combined tools from finite projective geometry with the method of Suen et al. (2001) to construct some new families of asymmetric orthogonal arrays of strength three and four. In this paper, we apply the method of Suen et al. (2001) to obtain some more asymmetric orthogonal arrays of strength three. We also give an alternative method of construction of a family of asymmetric orthogonal arrays of strength four, which appears to be more direct than that of Suen and Dey (2003).

Recall that an orthogonal array $OA(N, n, s_1 \times \cdots \times s_n, g)$ of strength $g$, is an $N \times n$ matrix with symbols in the $i$th column from a finite set of $s_i (\geq 2)$ symbols, $1 \leq i \leq n$, such that in every $N \times g$ submatrix, all possible combinations of symbols appear equally often as a row. Orthogonal arrays with $s_1 = s_2 = \cdots = s_n = s$ (say) are called symmetric and are denoted by $OA(N, n, s, g)$; otherwise, the array is called asymmetric (or, with mixed levels).

Henceforth, the columns of an $OA(N, n, s_1 \times \cdots \times s_n, g)$ will be called factors, following the terminology in factorial experiments, and these factors will be denoted by $F_1, \ldots, F_n$. Throughout this paper, we take the integer $s \geq 2$ to be a prime or a prime power, i.e., $s = p^q$, where $p$ is a prime and $q \geq 1$ is an integer. The Galois field of order $s$ will be denoted by $GF(s)$, 0 and 1 being the identity elements of the field corresponding to the operations ‘addition’ and ‘multiplication’, respectively. Also, throughout a prime will denote transposition. We shall need the following results, the first of which is well known and the second one is due to Suen et al. (2001).

**Lemma 1.** Let $\alpha$ and $\beta$ be two elements of $GF(s)$ such that $\alpha^2 = \beta^2$. Then (i) $\alpha = \beta$ if $s$ is even, (ii) either $\alpha = \beta$ or $\alpha = -\beta$, if $s$ is odd.
Lemma 2. For a positive integer $h$, let $D$ be a $(2h + 1) \times s^h$ matrix with columns of the form $(\alpha_1^2, \ldots, \alpha_h^2, \alpha_1, \ldots, \alpha_h, 1)'$, where $(\alpha_1, \ldots, \alpha_h)$’s are all possible $h$–tuples with entries from $GF(s)$. Then any three distinct columns of $D$ are linearly independent.

If $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ are the elements of $GF(s)$, then it follows from Lemma 1 that the set $S = \{\alpha_0^2, \alpha_1^2, \ldots, \alpha_{s-1}^2\}$ contains all the elements of $GF(s)$ if $s$ is even. If $s$ is odd, then one element of $S$ is 0 and there are $(s - 1)/2$ distinct non-zero elements of $GF(s)$, each appearing twice in $S$.

For the factor $F_i$ ($1 \leq i \leq n$), define the $m \times 1$ columns, $p_{i1}, \ldots, p_{iu_i}$, with elements from $GF(s)$. Then, for the $n$ factors we have in all $\sum_{i=1}^{n} u_i$ columns. Also, let $B$ be an $s^m \times m$ matrix whose rows are all possible $m$-tuples over $GF(s)$. Suen et al. (2001) proved the following result.

Theorem 1. Consider an $m \times \sum_{i=1}^{n} u_i$ matrix $C = [A_1:A_2: \cdots:A_n]$, $A_i = [p_{i1}, \ldots, p_{iu_i}]$, $1 \leq i \leq n$, such that for every choice of $g$ matrices $A_{i_1}, \ldots, A_{i_g}$ from $A_1, \ldots, A_n$, the $m \times \sum_{j=1}^{g} u_{ij}$ matrix $[A_{i_1}, \ldots, A_{i_g}]$ has full column rank over $GF(s)$. Then an $OA(s^m, n, (s^{u_1}) \times (s^{u_2}) \times \cdots \times (s^{u_n}), g)$ can be constructed.

A little elaboration of the result in Theorem 1 seems to be in order to make the construction transparent. For a fixed choice of $g$ indices $\{i_1, \ldots, i_g\} \in \{1, \ldots, n\}$, let $C_1 = [A_{i_1}, \ldots, A_{i_g}]$ and $r = \sum_{j=1}^{g} u_{ij}$. By the rank condition of Theorem 1, it follows that in the product $BC_1$, each possible $1 \times r$ vector with entries from $GF(s)$ appears $s^{m-r}$ times. Now, for each $j$, $1 \leq j \leq g$, replace the $s^{u_j}$ distinct combinations under $A_{i_j}$ by $s^{u_j}$ distinct symbols using a 1–1 correspondence. In the resultant $s^m \times g$ matrix, (i) the $i_j$th column has $s^{u_j}$ symbols ($1 \leq j \leq g$) and (ii) each of the $\prod_{j=1}^{g} s^{u_j}$ combinations of the symbols occurs equally often as a row. Hence, the desired orthogonal array with parameters as in Theorem 1 can be constructed.

2 Construction of orthogonal arrays of strength three

In this section, we construct two families of orthogonal arrays of strength three.
Theorem 2. Let \( k \geq 2 \) be an integer and \( t \) denote the largest integer not exceeding \( k/2 \).

(i) If \( s \) is an odd prime or an odd prime power, then an orthogonal array \( OA(s^{2k+1}, s^k + (k - 1)(s - 1)^t + 2, (s^2) \times s^{k+(k-1)(s-1)^t+1}, 3) \) can be constructed.

(ii) If \( s \) is a prime power of two, then an orthogonal array \( OA(s^{2k+1}, s^k + (k - 1)s^t + 2, (s^2) \times s^{k+(k-1)s^t+1}, 3) \) can be constructed.

Proof. (i) Let \( s \) be an odd prime or an odd prime power. Let \( F_1 \) have \( s^2 \) symbols and the rest of the factors have \( s \) symbols each. The matrices \( A_i, 1 \leq i \leq n \), corresponding to the different factors are chosen as below, where \( n = s^k + (k - 1)(s - 1)^t + 2 \).

\[ A_1 \] is chosen as \( A_1 = [I_2 \ 0_{2,2k-2} \ e]' \), where \( I_u \) is the identity matrix of order \( u \), \( 0_{u,v} \) is a \( u \times v \) null matrix, \( e = (0, 1)' \). The matrix \( A_2 \) is chosen as \( A_2 = [0, x, 0_{1,2k-2}, 1]' \), \( x \in GF(s) \), \( x \neq 0, 1 \).

Suppose \( \gamma_1, \ldots, \gamma_t \) are non-zero elements of \( GF(s) \). For \( 3 \leq i \leq (s - 1)^t + 2 \), if \( k \) is even, then \( A_i \) is chosen to be of the form \( A_i = [\gamma_1^2, \ldots, \gamma_t^2, \gamma_t, \ldots, \gamma_1, 0, 1, 0_{1,k-1}]' \). For \( (s - 1)^t + 3 \leq i \leq 2(s - 1)^t + 2 \), let \( A_i \) be of the form

\[ A_i = [\gamma_1^2, \ldots, \gamma_t^2, \gamma_t, \ldots, \gamma_1, 0, 0, 1, 0_{1,k-2}]' \]
i.e., \( 1 \) appears in the \((k + 3)\)th position. The other \( A_i \) matrices for this case are obtained by putting 1 in the \((k + 4)\)th, \((k + 5)\)th, \ldots, \((2k)\)th position, to get a total of \((k - 1)(s - 1)^t\) columns of such a form. If \( k \) is odd, \( A_i \) is of the form \( A_i = [0, \gamma_1^2, \ldots, \gamma_t^2, \gamma_t, \ldots, \gamma_1, 0, 1, 0_{1,k-1}]', \ldots,[0, \gamma_1^2, \ldots, \gamma_t^2, \gamma_t, \ldots, \gamma_1, 0, 0_{1,k-2}, 1, 0]' \). Finally, the last \( s^k \) columns have the form \([\alpha_1^2, \ldots, \alpha_t^2, 1, \alpha_k, \ldots, \alpha_1]' \), where \( \alpha_i \in GF(s) \).

(ii) Let \( s \) be an even prime power. In this case, the matrix \( A_2 \) is chosen as \( A_2 = [0_{1,2k}, 1]' \). \( \gamma_1, \ldots, \gamma_t \) can be any element of \( GF(s) \) and thus, each \( \gamma_i \) has \( s \) different choices. If \( k \) is even, the total number of columns of the types

\[ [\gamma_t^2, \ldots, \gamma_1^2, \gamma_t, \ldots, \gamma_1, 0, 1, 0_{1,k-1}]', \ldots, [\gamma_t^2, \ldots, \gamma_1^2, \gamma_t, \ldots, \gamma_1, 0, 0_{1,k-2}, 1, 0]' \]
is \((k - 1)s^t\). If \( k \) is odd, the total number of columns of the types

\[ [0, \gamma_t^2, \ldots, \gamma_1^2, \gamma_t, \ldots, \gamma_1, 0, 1, 0_{1,k-1}]', \ldots, [0, \gamma_t^2, \ldots, \gamma_1^2, \gamma_t, \ldots, \gamma_1, 0, 0_{1,k-2}, 1, 0]' \]
is \((k - 1)s^t\). The other columns of the matrices \(A_i\) are chosen as in case (i) above.

One can then verify that the rank condition of Theorem 1 is met with the above choices of the matrices \(A_i\). We give below the proof for some of the non-trivial cases. To save space, we consider only the case when \(s\) is an odd prime or an odd prime power; the case when \(s\) is a power of two can be handled in a similar fashion.

For convenience, denote the columns \(A_i\) by 1, \(A_2\) by 2, those of \(A_i\), \(i \in \{3,\ldots,(k-1)(s-1)^t+2\}\) by \(a\), and those of \(A_i\), \(i \in \{(k-1)(s-1)^t+3,\ldots,s^k+(k-1)(s-1)^t+2\}\) by \(b\), if \(k\) is even. If \(k\) is odd, denote those of \(A_i\), \(i \in \{(k-1)(s-1)^t+3,\ldots,s^k+(k-1)s^t+2\}\) by \(a\), and those of \(A_i\), \(i \in \{(k-1)s^t+3,\ldots,s^k+(k-1)s^t+2\}\) by \(b\). The proofs below are for the case when \(k\) is even. There are many different positions of 1 in the form of 
\[
[0, \gamma_1^2, \ldots, \gamma_t^2, \gamma_t, \ldots, \gamma_1, 0, 1, 0, 1, 0, \ldots, 1, 0, 1, 0, 1, 0^t, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_t, \gamma_t, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0^t, \ldots, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0^t, \ldots, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_t, \gamma_t, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_t, \gamma_t, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_t, \gamma_t, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_t, \gamma_t, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_t, \gamma_t, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_t, \gamma_t, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_t, \gamma_t, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_t, \gamma_t, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_t, \gamma_t, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, \gamma_1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0, 0, \ldots, 0, 1, 0\}^t.\]

For convenience, we can select any three of them to prove the result. Without loss of generality, we choose the last three of them.

(1aa)_1: Here, we have
\[
[A_i, A_j, A_k] = \begin{bmatrix}
1 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 1 & \beta_t^2 & \ldots & \beta_t & \ldots & \beta_1 & 0 & \ldots & 1 & 0 \\
\gamma_t^2 & \ldots & \gamma_t & \ldots & \gamma_1 & 0 & \ldots & 1 & 0
\end{bmatrix}^t.
\]

Since there exists at least one \(\beta_i \neq \gamma_i\), for \(i = 1, \ldots, t\), the determinant of the submatrix
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\beta_t^2 & \beta_t & 1 & 0 \\
\gamma_t^2 & \gamma_t & 1 & 0
\end{bmatrix}
\]
is \(\beta_i - \gamma_i \neq 0\).

(1aa)_2: Here, we have
\[
[A_i, A_j, A_k] = \begin{bmatrix}
1 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\beta_t^2 & \beta_t & \ldots & \beta_t & \ldots & \beta_1 & 0 & \ldots & 1 & 0 & \ldots & 1 & 0 & \ldots & 1 & 0 & \ldots & 1 & 0 \\
\gamma_t^2 & \ldots & \gamma_t & \ldots & \gamma_1 & 0 & \ldots & 1 & 0 & \ldots & 1 & 0 & \ldots & 1 & 0 & \ldots & 1 & 0 & \ldots & 1 & 0
\end{bmatrix}^t.
\]
The determinant of the submatrix
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\beta_t^2 & 0 & 1 & 0 \\
\gamma_t^2 & 1 & 0 & 0
\end{bmatrix}
\]
is 1.

(2aa)$_1$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} 0 & x & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}'.$$

Since there exists at least one $\beta_i \neq \gamma_i$, for $i = 1, \ldots, t$, the determinant of the submatrix

$$\begin{bmatrix} 0 & 0 & 1 \\ \beta_i & 1 & 0 \\ \gamma_i & 1 & 0 \end{bmatrix}'$$

is $\beta_i - \gamma_i \neq 0$.

(2aa)$_2$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} 0 & x & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix}'.$$

The determinant of the submatrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}'$$

equals 1.

(12a): Here

$$[A_i, A_j, A_k] = \begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 0 & 1 \\ 0 & x & \ldots & 0 & \ldots & 0 & 1 \\ \gamma_i^2 & \gamma_i^{2-1} & \ldots & \gamma_i & \ldots & \gamma_i & 1 \end{bmatrix}'.$$

The determinant of the submatrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & x & 0 & 1 \\ \gamma_i^2 & \gamma_i^{2-1} & \gamma_i & 0 \end{bmatrix}'$$

is $x\gamma_i \neq 0$.

(12b): Here

$$[A_i, A_j, A_k] = \begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 1 \\ 0 & x & \ldots & 0 & \ldots & 1 \\ \alpha_i^2 & \alpha_i^{2-1} & \ldots & 1 & \ldots & \alpha_i \end{bmatrix}'.$$
The determinant of the submatrix

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & x & 0 & 1 \\
\alpha_k^2 & \alpha_{k-1}^2 & 1 & 0
\end{bmatrix}
\]

is \( x \neq 0 \).

(1ab): Here

\[
[A_i, A_j, A_k] = \begin{bmatrix}
1 & \ldots & 0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & \ldots & 0 & 1 \\
\gamma_i^2 & \ldots & 0 & \ldots & 1 & 0 \\
\alpha_k^2 & \ldots & 1 & \ldots & \alpha_2 & \alpha_1
\end{bmatrix}
\]

The determinant of the submatrix

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\gamma_i^2 & 0 & 1 & 0 \\
\alpha_k^2 & 1 & \alpha_2 & \alpha_1
\end{bmatrix}
\]

is 1.

(2ab): Here

\[
[A_i, A_j, A_k] = \begin{bmatrix}
0 & x & \ldots & 0 & \ldots & 0 & 1 \\
\gamma_i^2 & \gamma_{i-1}^2 & \ldots & 0 & \ldots & 1 & 0 \\
\alpha_k^2 & \alpha_{k-1}^2 & \ldots & 1 & \ldots & \alpha_2 & \alpha_1
\end{bmatrix}
\]

The determinant of the submatrix

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & \alpha_2 & \alpha_1
\end{bmatrix}
\]

is 1.

(aab)_1: Here

\[
[A_i, A_j, A_k] = \begin{bmatrix}
\beta_i^2 & \ldots & 0 & \ldots & 1 & 0 \\
\gamma_i^2 & \ldots & 0 & \ldots & 1 & 0 \\
\alpha_k^2 & \ldots & 1 & \ldots & \alpha_2 & \alpha_1
\end{bmatrix}
\]

Since there exists at least one \( \beta_i \neq \gamma_i \), for \( i = 1, \ldots, t \), the determinant of the submatrix

\[
\begin{bmatrix}
\beta_i & 0 & 1 \\
\gamma_i & 0 & 1 \\
\alpha_j^2 & 1 & \alpha_2
\end{bmatrix}
\]

is \( \gamma_i - \beta_i \neq 0 \).
$(aab)_2$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} \beta_i^2 & \cdots & 0 & \cdots & 0 & 1 & 0 \\ \gamma_i^2 & \cdots & 0 & \cdots & 1 & 0 & 0 \\ \alpha_k^2 & \cdots & 1 & \cdots & \alpha_3 & \alpha_2 & \alpha_1 \end{bmatrix}'.$$

the determinant of the submatrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & \alpha_3 & \alpha_2 \end{bmatrix}'.$$

is 1.

$(abb)$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} \gamma_i^2 & \cdots & 0 & \cdots & 1 & 0 \\ \alpha_i^2 & \cdots & 1 & \cdots & \alpha_2 & \alpha_1 \\ \beta_i^2 & \cdots & 1 & \cdots & \beta_2 & \beta_1 \end{bmatrix}'.$$

If $\alpha_i \neq \beta_i$ for $i \neq 2$, the determinant of the submatrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & \alpha_i & \alpha_2 \\ 1 & \beta_i & \beta_2 \end{bmatrix}'.$$

is $\beta_i - \alpha_i \neq 0$. If $\alpha_i = \beta_i$ for $i \neq 2$, then $\beta_2 \neq \gamma_2$, the determinant of the submatrix

$$\begin{bmatrix} \gamma_1 & 0 & 1 \\ \alpha_i^2 & 1 & \alpha_2 \\ \beta_i^2 & 1 & \beta_2 \end{bmatrix}'.$$

is $\gamma_1(\beta_2 - \alpha_2) \neq 0$.

$(bbb)$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} \alpha_k^2 & \cdots & 1 & \cdots & \alpha_1 \\ \beta_k^2 & \cdots & 1 & \cdots & \beta_1 \\ \gamma_k^2 & \cdots & 1 & \cdots & \gamma_1 \end{bmatrix}'.$$

by Lemma 2, the rank of the matrix is three.

$(1ab)$: Here

$$[A_i, A_j, A_k] = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 1 & 0 \\ \gamma_i^2 & \cdots & 0 & \cdots & 1 & 0 \\ \alpha_k^2 & \cdots & 1 & \cdots & \alpha_2 & \alpha_1 \end{bmatrix}'.$$

The determinant of the submatrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \gamma_i^2 & 0 & 1 & 0 \\ \alpha_k^2 & 1 & \alpha_2 & \alpha_1 \end{bmatrix}'.$$
is 1.

\((aaa)_1\): Here

\[
[A_i, A_j, A_k] = \begin{bmatrix}
\alpha_t^2 & \ldots & \alpha_t^1 & \alpha_t & 0 & \ldots & 0 & 1 & 0 \\
\beta_t^2 & \ldots & \beta_t^1 & \beta_t & 0 & \ldots & 0 & 1 & 0 \\
\gamma_t^2 & \ldots & \gamma_t^1 & \gamma_t & 0 & \ldots & 0 & 1 & 0
\end{bmatrix}.
\]

by Lemma 2, the rank of matrix is three.

\((aaa)_2\): Here

\[
[A_i, A_j, A_k] = \begin{bmatrix}
\alpha_t^2 & \ldots & \alpha_t^1 & \alpha_t & 0 & \ldots & 0 & 1 & 0 \\
\beta_t^2 & \ldots & \beta_t^1 & \beta_t & 0 & \ldots & 1 & 0 & 0 \\
\gamma_t^2 & \ldots & \gamma_t^1 & \gamma_t & 0 & \ldots & 1 & 0 & 0
\end{bmatrix}.
\]

Since \(\beta_i \neq \gamma_i\) for at least one \(i\), the determinant of the submatrix

\[
\begin{bmatrix}
\alpha_i & 0 & 1 \\
\beta_i & 1 & 0 \\
\gamma_i & 1 & 0
\end{bmatrix}
\]

is nonzero. \(\Box\)

**Remark.** As before, let \(s\) be a prime or a prime power and let \(i, k\) be integers such that \(1 \leq i \leq k\). Suen and Dey (2003) constructed the following families of orthogonal arrays:

(i) \(OA(s^{2k+i}, s^k + 1, (s^k)^2 \times (s^i)^{s^k-1}, 3)\), if \(s\) is odd and,

(ii) \(OA(s^{2k+i}, s^k + 2, (s^k)^2 \times (s^i)^s, 3)\), if \(s\) is even.

Setting \(i = 1\), one gets the arrays

(a) \(OA(s^{2k+1}, s^k + 1, (s^k)^2 \times s^{s^k-1}, 3)\), if \(s\) is odd and,

(b) \(OA(s^{2k+1}, s^k + 2, (s^k)^2 \times s^s, 3)\), if \(s\) is even.

First, replace one of the \(s^k\)–symbol columns by \(k\) columns having \(s\) symbols each and then replace symbols in the other \(s^k\)–symbol column by all possible combinations arising out of one \(s^2\)–symbol column and \(k - 2\) columns each having \(s\) symbols. By this process, one obtains the following arrays corresponding to the arrays in (a) and (b) respectively:

(c) \(OA(s^{2k+1}, s^k + 2k - 2, (s^2) \times s^{s^k+2k-3}, 3)\), if \(s\) is odd, and,
(d) $OA(s^{2k+1}, s^k + 2k - 1, (s^2) \times s^{k+2k-2}, 3)$, if $s$ is even.

It is easy to verify that the arrays given by Theorem 2 have more $s$–symbol columns than in the arrays (c) and (d) above. Thus, the arrays in Theorem 2 appear to be superior to the ones in (c) and (d) in terms of having more $s$–symbol columns.

In closing this paper, we make an observation about a family of asymmetric orthogonal arrays of strength four. Suen and Dey (2003) constructed a family of orthogonal arrays $OA(s^5, s + 3, (s^2) \times s^{s+2}, 4)$, where $s$ is a power of two. We give below an alternative method of obtaining the same family, which appears to be more direct than that of Suen and Dey (2003).

Let $F_1$ have $s^2$ symbols and the rest of the factors have $s$ symbols each. The matrices $A_i$, $1 \leq i \leq s + 3$ are chosen as follows.

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}', \ A_2 = [1 \ 0 \ 0 \ 0 \ 1]', \ A_3 = [1 \ 0 \ 0 \ 1 \ 0]',$$

and for $4 \leq j \leq s + 3$, $A_j = [0, \alpha_j^3, 1, \alpha_j, \alpha_j^2]'$, where $\alpha_4, \ldots, \alpha_{s+3}$ are distinct elements of $GF(s)$. It can be verified that with these choices of the $A_i$ matrices, the rank condition of Theorem 1 is met. We omit the details.

**Acknowledgement**

The work of A. Dey was supported by the National Academy of Sciences, India under the Senior Scientist program of the Academy. The support is gratefully acknowledged. The authors thank Dr. Liu Yanjun for useful comments on an earlier draft.

**References**


