Inertia of Loewner Matrices

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Abstract
Given positive numbers \(p_1 < p_2 < \cdots < p_n\), and a real number \(r\) let \(L_r\) be the \(n \times n\) matrix with its \(i, j\) entry equal to \((p_i^r - p_j^r)/(p_i - p_j)\). A well-known theorem of C. Loewner says that \(L_r\) is positive definite when \(0 < r < 1\). In contrast, R. Bhatia and J. Holbrook, (Indiana Univ. Math. J., 49 (2000) 1153-1173) showed that when \(1 < r < 2\), the matrix \(L_r\) has only one positive eigenvalue, and made a conjecture about the signatures of eigenvalues of \(L_r\) for other \(r\). That conjecture is proved in this paper.

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1 Introduction
Let \(f\) be a real-valued \(C^1\) function on \((0, \infty)\). Let \(p_1 < p_2 < \cdots < p_n\) be any \(n\) points in \((0, \infty)\). The \(n \times n\) matrix

\[
L_f(p_1, \ldots, p_n) = \left[ \frac{f(p_i) - f(p_j)}{p_i - p_j} \right]_{i,j=1}^n
\]

is called a \textit{Loewner matrix} associated with \(f\). It is understood that when \(i = j\), the quotient in (1) represents the limiting value \(f'(p_i)\). Of particular interest to us are the functions \(f(t) = t^r\), \(r \in \mathbb{R}\). In this case we write \(L_r\) for \(L_f(p_1, \ldots, p_n)\), where the roles of \(n\) and \(p_1, \ldots, p_n\) can be inferred from the context. Thus \(L_r\) is the \(n \times n\) matrix

\[
L_r = \left[ \frac{p_i^r - p_j^r}{p_i - p_j} \right]_{i,j=1}^n
\]

Loewner matrices are important in several contexts, of which we mention two that led to the present study. (The reader may see Section 4.1 of [12] for an
excellent discussion of both these aspects of Loewner matrices.) The function \( f \) on \((0, \infty)\) induces, via the usual functional calculus, a matrix function \( f(A) \) on the space of positive definite matrices. Let \( Df(A) \) be the Fréchet derivative of this function. This is a linear map on the space of Hermitian matrices. The Daleckii-Krein formula describes the action of this map in terms of Loewner matrices. Choose an orthonormal basis in which \( A = \text{diag}(p_1, \ldots, p_n) \). Then the formula says that for every Hermitian \( X \)

\[
Df(A)(X) = L_f(p_1, \ldots, p_n) \circ X, \tag{3}
\]

where \( A \circ B \) stands for the entrywise product \([a_{ij}b_{ij}]\) of \( A \) and \( B \).

The function \( f \) is said to be \textit{operator monotone} on \((0, \infty)\) if \( A \geq B > 0 \) implies \( f(A) \geq f(B) \). (As usual \( A \geq 0 \) means \( A \) is positive semidefinite.) A fundamental theorem due to Charles Loewner says that \( f \) is operator monotone if and only if all Loewner matrices associated with \( f \) (for every \( n \) and for every choice \( p_1, \ldots, p_n \)) are positive semidefinite. Another basic fact, again proved first by Loewner, says that \( f(t) = t^r \) is operator monotone if and only if \( 0 \leq r \leq 1 \).

See [1] Chapter V. Combining these various facts with some well-known theorems on positive linear maps [2] one can see that if \( f \) is operator monotone, then the norm of \( Df(A) \) obeys the relations

\[
\|Df(A)\| = \|Df(A)(I)\| = \|f'(A)\|, \tag{4}
\]

and is therefore readily computable. In particular, for the function \( f(t) = t^r \) if we write \( DA^r \) for \( Df(A) \), then (4) gives

\[
\|DA^r\| = \|rA^{r-1}\|, \quad \text{for } 0 \leq r \leq 1. \tag{5}
\]

This was first noted in [3], and used to derive perturbation bounds for the operator absolute value. Then in [8] Bhatia and Sinha showed that the relation (5) holds also for \(-\infty < r < 0\) and for \( 2 \leq r < \infty \) but, mysteriously, not for \( 1 < r < \sqrt{2} \). The case \( \sqrt{2} \leq r < 2 \), left open in this paper, was resolved in [4] by Bhatia and Holbrook, who showed that here again the relation (5) is valid.

One ingredient of the proof in [4] is their Proposition 2.1 which says that when \( 1 < r < 2 \), the \( n \times n \) matrix \( L_r \) has just one positive eigenvalue. We have remarked earlier that when \( 0 < r < 1 \), the matrix \( L_r \) is positive semidefinite and therefore, none of its eigenvalues is negative. This contrast as \( r \) moves from \((0,1)\) to \((1,2)\) is intriguing, and raises the natural question about the behaviour of eigenvalues of \( L_r \) for other values of \( r \). Bhatia and Holbrook [4] made a conjecture about this behaviour and established a small part of it: they settled the cases \( r = 1, 2, \ldots, n-1 \) apart from \( 0 < r < 1 \) and \( 1 < r < 2 \) already mentioned. The main goal of this paper is to prove this conjecture in full. This is our Theorem 1.1.

Let \( A \) be an \( n \times n \) Hermitian matrix. The \textit{inertia} of \( A \) is the triple

\[
\text{In}(A) = (\pi(A), \zeta(A), \nu(A)),
\]

2
where \( \pi(A) \) is the number of positive eigenvalues of \( A \), \( \zeta(A) \) is the number of zero eigenvalues of \( A \), and \( \nu(A) \) the number of negative eigenvalues of \( A \). Theorem 1.1 describes the inertia of \( L_r \) as \( r \) varies over \( \mathbb{R} \). As noted in (10) below the inertia of \( L_{-r} \) is the opposite of the inertia of \( L_r \); i.e. \( \pi(L_{-r}) = \nu(L_r) \) and \( \nu(L_{-r}) = \pi(L_r) \). So we confine ourselves to the case \( r > 0 \).

**Theorem 1.1.** Let \( p_1 < p_2 < \cdots < p_n \) and \( r \) be any positive real numbers and let \( L_r \) be the matrix defined in (2). Then

(i) \( L_r \) is singular if and only if \( r = 1, 2, \ldots, n - 1 \).

(ii) At the points \( r = 1, 2, \ldots, n \), the inertia of \( L_r \) is given as follows:

\[
\pi(L_r) = (n-r, k),
\]

and

\[
\nu(L_r) = (n-r, k).
\]

(iii) If \( 0 < r < n \) and \( r \) is not an integer, then

\[
\lfloor r \rfloor = 2k \Rightarrow \pi(L_r) = (n-k, 0, k)
\]

and

\[
\nu(L_r) = (n-k, 0, k).
\]

(iv) If \( r > n - 1 \), then \( \pi(L_r) = \pi(L_n) \).

(v) Every nonzero eigenvalue of \( L_r \) is simple.

It is helpful to illustrate the theorem by a picture. Figure 1 is a diagram of the (scaled) eigenvalues of a \( 6 \times 6 \) matrix \( L_r \) when \( p_i \) are fixed and \( r \) varies. Some of the eigenvalues are very close to zero. To be able to distinguish between them the vertical scale has been expanded.

We have already mentioned that for \( 0 < r < 1 \), statement (iii) of Theorem 1.1 follows from Loewner’s theorem, and for \( 1 < r < 2 \) it was established in [4]. The case \( 2 < r < 3 \) was accomplished by Bhatia and Sano in [7]. We briefly explain this work.

Let \( \mathcal{H}_1 \) be the space

\[
\mathcal{H}_1 = \left\{ x = (x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i = 0 \right\}.
\]

An \( n \times n \) Hermitian matrix \( A \) is said to be **conditionally positive definite** if \( \langle x, Ax \rangle \geq 0 \) for all \( x \in \mathcal{H}_1 \), and if \( -A \) has this property, then we say that \( A \) is **conditionally negative definite**. Since \( \dim \mathcal{H}_1 = n-1 \), a nonsingular conditionally positive definite matrix which is not positive definite has inertia \((n-1, 0, 1)\).
In [7] it was shown that when $1 < r < 2$, the matrix $L_r$ is nonsingular and conditionally negative definite. It follows that $\text{In} (L_r) = (1, 0, n - 1)$, a fact established earlier in [4]. It was also shown in [7] that when $2 < r < 3$, the matrix $L_r$ is nonsingular and conditionally positive definite. From this it follows that $\text{In} (L_r) = (n - 1, 0, 1)$.

More generally, Bhatia and Sano [7] showed that $f$ on $(0, \infty)$ is operator convex if and only if all Loewner matrices $L_f$ are conditionally negative definite. This is a characterisation analogous to Loewner’s for operator monotone functions. It is well-known that $f(t) = t^r$ is operator convex for $1 \leq r \leq 2$.

The proof of Theorem 1.1 is given in Section 2. We also indicate how the proofs for the parts already given in [4] and [7] can be considerably simplified. The inertia of the matrix $[(p_i + p_j)^r]$ has been studied by Bhatia and Jain in [5]. Some ideas in our proofs are similar to the ones used there.

2 Proofs and Remarks

Let $X$ be an $n \times n$ nonsingular matrix. The transformation $A \mapsto X^*AX$ on Hermitian matrices is called a congruence. The Sylvester Law of Inertia says that

$$\text{In} (X^*AX) = \text{In} A \text{ for all } X \in GL(n). \quad (7)$$
Let $D$ be the diagonal matrix

$$D = \text{diag} (p_1, \ldots, p_n).$$

(8)

Then for every $r$

$$L_{-r} = -D^{-r}L_rD^{-r}.$$

(9)

Hence by Sylvester’s Law

$$\text{In} L_r = (i_1, i_2, i_3) \Leftrightarrow \text{In} L_{-r} = (i_3, i_2, i_1).$$

(10)

Thus all statements about $\text{In} L_r$ for $r > 0$ give information about $\text{In} L_{-r}$ as well.

Make the substitution $p_i = e^{2x_i}$, $x_i \in \mathbb{R}$. A simple calculation shows that

$$L_r = \begin{bmatrix} e^{rx_1} \sinh r(x_i - x_j) & e^{rx_j} \\ e^{rx_j} \sinh(x_i - x_j) & e^{rx_j} \end{bmatrix}. $$

In other words,

$$L_r = \Delta \tilde{L}_r \Delta,$$

(11)

where $\Delta = \text{diag} (e^{(r-1)x_1}, \ldots, e^{(r-1)x_n})$, and

$$\tilde{L}_r = \begin{bmatrix} \sinh r(x_i - x_j) \\ \sinh(x_i - x_j) \end{bmatrix}. $$

(12)

By Sylvester’s Law $\text{In} L_r = \text{In} \tilde{L}_r$. Several properties of $L_r$ can be studied via $\tilde{L}_r$, and vice versa. This has been a very effective tool in deriving operator inequalities; see, the work of Bhatia and Parthasarathy [6] and that of Hiai and Kosaki [9, 10, 11, 14].

When $n = 2$ we have

$$\tilde{L}_r = \begin{bmatrix} r \sinh r(x_1 - x_2) & \sinh r(x_1 - x_2) \\ \sinh(x_1 - x_2) & r \end{bmatrix}. $$

So $\det \tilde{L}_r = r^2 - \sinh^2 r(x_1 - x_2)/\sinh^2(x_1 - x_2)$. Thus $\det \tilde{L}_r$ is positive for $0 < r < 1$, zero for $r = 1$, and negative for $r > 1$. One eigenvalue of $\tilde{L}_r$ is always positive, and this shows that the second eigenvalue is positive, zero, or negative depending on whether $0 < r < 1$, $r = 1$, or $r > 1$, respectively. This establishes Theorem 1.1 in the simplest case $n = 2$.

An interesting corollary can be deduced at this stage. According to the two theorems of Loewner mentioned in Section 1, $f$ is operator monotone if and only if all Loewner matrices $L_f$ are positive semidefinite, and $f(t) = t^r$ is operator monotone if and only if $0 \leq r \leq 1$. Consequently, if $r > 1$, then there exists an $n$, and positive numbers $p_1, \ldots, p_n$ such that the associated Loewner matrix (2) is not positive definite. We can assert more:

**Proposition 2.1.** Let $r > 1$. Then for every $n \geq 2$, and for every choice of $p_1, \ldots, p_n$, the matrix $L_r$ defined in (2) has at least one negative eigenvalue.
Consider the \( 2 \times 2 \) top left submatrix of \( L_r \). This is a Loewner matrix. By Theorem 1.1 it has one negative eigenvalue. So, by Cauchy’s interlacing principle, the \( n \times n \) matrix \( L_r \) has at least one negative eigenvalue. \( \square \)

The Sylvester Law has a generalisation that is useful for us. Let \( n \geq r \), and let \( A \) be an \( r \times r \) Hermitian matrix and \( X \) an \( r \times n \) matrix of rank \( r \). Then

\[ \text{In } X^*AX = \text{In } A + (0,n-r,0). \] (13)

A proof of this may be found in [5]. This permits a simple transparent proof of Part (ii) of Theorem 1.1. (This part has already been proved in [4].) When \( r \) is a positive integer we have

\[ L_r = \left[ p_r^{r-1} + p_r^{r-2}p_j + \cdots + p_j^{r-1} \right] = W^*VW, \]

where \( W \) is the \( r \times n \) Vandermonde matrix

\[
W = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
p_1 & p_2 & \cdots & p_n \\
\cdot & \cdot & \cdots & \cdot \\
p_1^{r-1} & p_2^{r-1} & \cdots & p_n^{r-1}
\end{bmatrix},
\]

and \( V \) is the \( r \times r \) antidiagonal matrix with all entries 1 on its sinister diagonal and all its other entries equal to 0. If \( r = 2k \), the matrix \( V \) has \( k \) of its eigenvalues equal to 1, and the other \( k \) equal to \(-1\). If \( r = 2k-1 \), then \( k \) of its eigenvalues are equal to 1, and \( k-1 \) are equal to \(-1\). So, statement (ii) of Theorem 1.1 follows from the generalised Sylvester’s Law (13). Next we prove statement (i).

Let \( c_1, c_2, \ldots, c_n \) be real numbers, not all of which are zero. Let \( f \) be the function on \((0, \infty)\) defined as

\[ f(x) = \sum_{j=1}^{n} c_j x^r - p_j^r \frac{x^r}{x - p_j}. \] (14)

**Theorem 2.2.** Let \( r \) be a positive real number not equal to 1, 2, \ldots, \( n-1 \). Then the function \( f \) defined in (14) has at most \( n-1 \) zeros in \((0, \infty)\).

**Proof** Let \( r_1 < r_2 < \cdots < r_m \), and let \( a_1, \ldots, a_m \) be real numbers not all of which are zero. Then the function

\[ g(x) = \sum_{j=1}^{m} a_j x^{r_j}, \] (15)

has at most \( m-1 \) zeros in \((0, \infty)\). This is a well-known fact, and can be found in e.g., [16], p.46.

Now let \( f \) be the function defined in (14) and let

\[ g(x) = f(x) \prod_{j=1}^{n}(x - p_j). \] (16)
Then $g$ can be expressed in the form (15) with $m = 2n$ and
\[
\{r_1, \ldots, r_{2n}\} = \{0, 1, \ldots, n-1, r, r+1, \ldots, r+n-1\}.
\]
Further, we have $g(x) = x^r h_1(x) - h_2(x)$, where
\[
h_1(x) = \sum_{i=1}^{n} c_i \prod_{j \neq i} (x - p_j), \quad h_2(x) = \sum_{i=1}^{n} c_i p_i^r \prod_{j \neq i} (x - p_j).
\]
Both $h_1$ and $h_2$ are Lagrange interpolation polynomials of degree at most $n - 1$. Since not all $c_i$ are zero, neither of these polynomials is identically zero. So, if $r \neq 1, 2, \ldots, n - 1$, then $g$ is not the zero function.

Hence the function $g$ defined by (16) has at most $2n - 1$ zeros in $(0, \infty)$. Of these, $n$ zeros occur at $x = p_j$, $1 \leq j \leq n$. So $f$ has at most $n - 1$ zeros in $(0, \infty)$.

\[\square\]

**Corollary 2.3.** Let $r$ be a positive real number different from $1, 2, \ldots, n - 1$. Then the matrix $L_r$ defined in (2) is nonsingular.

**Proof.** The matrix $L_r$ is singular if and only if there exists a nonzero vector $c = (c_1, \ldots, c_n)$ such that $L_r(c) = 0$. In other words there exist real numbers $c_1, \ldots, c_n$, not all zero, such that
\[
\sum_{j=1}^{n} c_j \frac{p_i^r - p_j^r}{p_i - p_j} = 0
\]
for $i = 1, 2, \ldots, n$. But then the function $f(x)$ in (14) would have $n$ zeros, viz., $x = p_1, \ldots, p_n$. That is not possible.

We have proved Part (i) of Theorem 1.1. Part (iv) follows from this. If the inertia of $L_r$ were to change at some point $r_0 > n - 1$, then one of the eigenvalues has to change sign at $r_0$. This is ruled out as $L_r$ is nonsingular for all $r > n - 1$.

Our argument shows that if $p_1 < p_2 < \cdots < p_n$ and $q_1 < q_2 < \cdots < q_n$ are two $n$-tuples of positive real numbers, then the matrix $[\frac{p_i^r - q_j^r}{p_i - q_j}]$ is nonsingular for every positive $r$ different from $1, 2, \ldots, n - 1$. Using the intermediate value theorem, we see that the determinants of all such matrices must have the same sign, independent of $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$.

An $n \times n$ real matrix $A$ is said to be **strictly sign-regular** (SSR for short) if for every $1 \leq k \leq n$, all $k \times k$ sub-determinants of $A$ are nonzero and have the same sign. If this is true for every $1 \leq k \leq r$ for some $r < n$, then we say that $A$ is in the class SSR$_r$. Sign-regular matrices and kernels are studied extensively in [15].

Let $L_r$ be an $n \times n$ Loewner matrix. Let $r \neq 1, 2, \ldots, n - 1$. We have observed that all $k \times k$ sub-determinants of $L_r$ are nonzero and have the same sign. Thus $L_r$ is an SSR matrix. If $r = 1, 2, \ldots, n - 1$, then the same argument shows that for $k \leq r$ all $k \times k$ sub-determinants of $L_r$ are nonzero and have the same sign. In other words, $L_r$ is an SSR$_r$ matrix.
Let $A$ be any matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ arranged so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. The Perron theorem tells us that if $A$ is entrywise positive, then $\lambda_1 > 0$ and $\lambda_1$ is a simple eigenvalue of $A$. (See [13], p. 526). Now suppose $A$ is an SSR matrix. For $1 \leq k \leq n$, the entries of the $k$th exterior power $\Lambda^k A$ are the $k \times k$ subdeterminants of $A$, and hence they must have the same sign. The eigenvalue of $\Lambda^k A$ with the largest modulus is $\lambda_1 \lambda_2 \cdots \lambda_k$. By Perron’s theorem this eigenvalue must be simple. Since this is true for all $1 \leq k \leq n$, every eigenvalue of an SSR matrix is simple. This proves Part (v) of Theorem 1.1.

We now turn to proving Part (iii). Using the identity
\[
\frac{p^r_i - p^r_j}{p_i - p_j} = \frac{p^{r-1}_i (p_i - p_j) + p_i (p_i^{r-2} - p_j^{r-2}) p_j + (p_i - p_j) p_j^{r-1}}{p_i - p_j},
\]
we see that for every $r \in \mathbb{R}$,
\[
L_r = D^{r-1} E + DL_{r-2} D + ED^{r-1},
\]
where $D$ is the diagonal matrix in (8) and $E$ is the $n \times n$ matrix with all its entries equal to one.

By Loewner’s Theorem $L_r$ is positive definite for $0 < r < 1$, and because of (10) it is negative definite for $-1 < r < 0$. Now suppose $1 < r < 2$. Let $x$ be any nonzero vector in the space $\mathcal{H}_1$ defined in (6). Note that this $(n-1)$-dimensional space is the kernel of the matrix $E$. Using (17) we have
\[
\langle x, L_r x \rangle = \langle x, D^{r-1} E x \rangle + \langle x, D L_{r-2} D x \rangle + \langle x, E D^{r-1} x \rangle.
\]
The first and the third term on the right hand side are zero because $E x = 0$. So,
\[
\langle x, L_r x \rangle = \langle y, L_{r-2} y \rangle,
\]
where $y = D x$. The last inner product is negative because $L_{r-2} < 0$. Thus $\langle x, L_r x \rangle < 0$ for all $x \in \mathcal{H}_1$. In other words, $L_r$ is conditionally negative definite if $1 < r < 2$. The same argument shows that $L_r$ is conditionally positive definite if $2 < r < 3$ (because in this case $L_{r-2}$ is positive definite). This was proved in [7] by more elaborate arguments. In particular, we have
\[
\text{In } L_r = (1,0,n-1), \text{ if } 1 < r < 2,
\]
and
\[
\text{In } L_r = (n-1,0,1), \text{ if } 2 < r < 3.
\]
We note here that if $n = 3$, then because of Part (iv) already proved we have In $L_r = (2,0,1)$ for all $r > 2$. So the theorem is completely proved for $n = 3$.

Let $n > 3$ and suppose $3 < r < 4$. Now consider the space
\[
\mathcal{H}_2 = \{ x : \sum x_i = 0, \sum p_i x_i = 0 \} = \{ x : E x = 0, E Dx = 0 \}.
\]
This space is of dimension \( n - 2 \), being the orthogonal complement of the span of the vectors \( e = (1,1,\ldots,1) \) and \( p = (p_1,p_2,\ldots,p_n) \). Let \( x \in \mathcal{H}_2 \). Again using the relation (17) we see that
\[
\langle x, L_r x \rangle = \langle y, L_{r-2} y \rangle,
\]
where \( y = Dx \). Since \( EDx = 0 \), \( y \) is in \( \mathcal{H}_1 \), and since \( 1 < r - 2 < 2 \), we have \( \langle x, L_r x \rangle < 0 \). This is true for all \( x \in \mathcal{H}_2 \). So, by the minmax principle \( L_r \) has at least \( n - 2 \) negative eigenvalues. The case \( n = 3 \) of the theorem already proved shows that \( L_r \) has a \( 3 \times 3 \) principal submatrix with two positive eigenvalues. So, by Cauchy’s interlacing principle, \( L_r \) has at least two positive eigenvalues. Thus \( L_r \) has exactly two positive and \( n - 2 \) negative eigenvalues. In other words,
\[
\text{In } L_r = (2,0,n-2) \text{ for } 3 < r < 4.
\]
(20)

At this stage note that the Theorem is completely proved for \( n = 4 \). Now let \( n > 4 \), and consider the case \( 4 < r < 5 \). Arguing as before \( \langle x, L_r x \rangle > 0 \) for all \( x \in \mathcal{H}_2 \). So \( L_r \) has at least \( n - 2 \) positive eigenvalues. It also has a \( 4 \times 4 \) principal submatrix with two negative eigenvalues. Hence
\[
\text{In } L_r = (n-2,0,2) \text{ for } 4 < r < 5.
\]
(21)

The argument can be continued, introducing the space
\[
\mathcal{H}_3 = \left\{ x : \sum x_i = 0, \sum p_i x_i = 0, \sum p_i^2 x_i = 0 \right\}
\]
\[
= \left\{ x : Ex = 0, EDx = 0, ED^2x = 0 \right\}
\]
at the next stage. Using this we can prove statement (iii) for \( 5 < r < 6 \) and \( 6 < r < 7 \). It is clear now how to complete the proof.

All parts of Theorem 1.1 have now been established. \( \square \)

We end this section with a few questions.

1. Let \( f(z) \) be the complex function defined as
\[
f(z) = \det \left[ \frac{p_i^2 - p_j^2}{p_i - p_j} \right].
\]

Our analysis has shown that \( f \) has zeros at \( z = 0, \pm 1, \pm 2, \ldots, \pm n - 1 \); these zeros have multiplicities \( n, n-1, \ldots, 1 \), respectively; and these are the only real zeros of \( f \). It might be of interest to find what other zeros \( f \) has in the complex plane.

2. When \( n = 3 \), calculations show that
\[
\det L_3 = -(p_1 - p_2)^2(p_1 - p_3)^2(p_2 - p_3)^2,
\]
and
\[
\det L_4 = -2(p_1 - p_2)^2(p_1 - p_3)^2(p_2 - p_3)^2 \\
\{(p_1 + p_2 + p_3)(p_1 p_2 + p_1 p_3 + p_2 p_3) + p_1 p_2 p_3\}.
\]

It might be of interest to find formulas for the determinants of the matrices \(L_m\) for integers \(m\).

3. Two of the authors have studied the matrix \(P_r = [(p_i + p_j)^r]\) in [5]. It turns out that \(\text{In } P_r = \text{In } L_{r+1}\) for all \(r > 0\). Why should this be so, and are there other interesting connections between these two matrix families?

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References


