Perfect powers in Arithmetic Progression

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PERFECT POWERS IN ARITHMETIC PROGRESSION

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Abstract. The conjecture of Masser-Oesterlé, popularly known as abc-conjecture have many consequences. We use an explicit version due to Baker to solve the equation
\[ n(n + d) \cdots (n + (k - 1)d) = by^l \]
in positive integer variables \(n, d, k, b, y, l\) such that \(b\) square free with the largest prime divisor of \(b\) at most \(k, k \geq 2, l \geq 2\) and \(\gcd(n, d) = 1\).

1. Introduction

Let \(n, d, k, b, y\) be positive integers such that \(b\) is square free with \(P(b) \leq k, k \geq 2, l \geq 2\) and \(\gcd(n, d) = 1\). Here \(P(m)\) denotes the largest prime divisor of \(m\) with the convention \(P(1) = 1\). We consider the equation
\[ n(n + d) \cdots (n + (k - 1)d) = by^l \]
in variables \(n, d, k, b, y, l\). If \(k = 2\), we observe that (4) has infinitely many solutions. Therefore we always suppose that \(k \geq 3\). It has been conjectured (see [Tij88], [SaSh05]) that

**Conjecture 1.1.** Equation (1) implies that \((k, l) \in \{(3, 3), (4, 2), (3, 2)\}\).

It is known that (1) has infinitely many solutions when \((k, l) \in \{(3, 2), (3, 3)\}(4, 2)\). A weaker version of Conjecture 1.1 is the following conjecture due to Erdős.

**Conjecture 1.2.** Equation (1) implies that \(k\) is bounded by an absolute constant.

For an account of results on (1), we refer to Shorey [Sho02b] and [Sho06].

The well known conjecture of Masser-Oesterle states that

**Conjecture 1.3. Oesterlé and Masser’s abc-conjecture:** For any given \(\epsilon > 0\) there exists a computable constant \(c_\epsilon\) depending only on \(\epsilon\) such that if
\[ a + b = c \]
where \(a, b\) and \(c\) are coprime positive integers, then
\[ c \leq c_\epsilon \left( \prod_{p | abc} p \right)^{1+\epsilon}. \]

Key words and phrases. abc Conjecture, Arithmetic progressions, Erdős Conjecture.
It is known as \(abc\)-conjecture; the name derives from the usage of letters \(a, b, c\) in (2). For any positive integer \(i > 1\), let \(N = N(i) = \prod_{p|i} p\) be the radical of \(i\), \(P(i)\) be the greatest prime factor of \(i\) and \(\omega(i)\) be the number of distinct prime factors of \(i\) and we put \(N(1) = 1, P(1) = 1\) and \(\omega(1) = 0\).

It has been shown in Elkies [Elk91] and Granville and Tucker [GrTu02, (13)] that \(abc\)-conjecture is equivalent to the following:

**Conjecture 1.4.** Let \(F(x, y) \in \mathbb{Z}[x, y]\) be a homogenous polynomial. Assume that \(F\) has pairwise non-proportional linear factors in its factorisation over \(\mathbb{C}\). Given \(\epsilon > 0\), there exists a computable constant \(\kappa_\epsilon\) depending only on \(F\) and \(\epsilon\) such that if \(m\) and \(n\) are coprime integers, then

\[
\prod_{p|F(m,n)} p \geq \kappa_\epsilon (\max\{|m|, |n|\})^{\deg(F)-2-\epsilon}.
\]

Shorey [Sho99] showed that \(abc\)-conjecture implies Conjecture 1.2 for \(\ell \geq 4\) using \(d \geq k^{c_1 \log \log k}\). Granville (unpublished) gave a proof of the preceding result without using the inequality \(d \geq k^{c_1 \log \log k}\). Furthermore his proof is also valid for \(\ell = 2, 3\).

**Theorem 1.** The \(abc\)–conjecture implies Conjecture (1.2).

The proof was first published in the Master’s Thesis of first author [Lai04]. We include the proof in this paper to have a published literature. This is given in Section 6. We would like to thank Professor A. Granville for allowing us to publish his proof.

An explicit version of Conjecture 2 due to Baker [Bak94] is the following:

**Conjecture 1.5.** Explicit \(abc\)-conjecture: Let \(a, b\) and \(c\) be pairwise coprime positive integers satisfying (2). Then

\[
c < \frac{6}{5} N \frac{(\log N)^\omega}{\omega!}
\]

where \(N = N(abc)\) and \(\omega = \omega(N)\).

We observe that \(N = N(abc) \geq 2\) whenever \(a, b, c\) satisfy (2). We shall refer to Conjecture 1.3 as \(abc\)–conjecture and Conjecture 1.5 as explicit \(abc\)–conjecture. Conjecture 1.5 implies the following explicit version of Conjecture 1.3 proved in [LaSh12].

**Theorem 2.** Assume Conjecture 1.5. Let \(a, b\) and \(c\) be pairwise coprime positive integers satisfying (2) and \(N = N(abc)\). Then we have

\[
c < N^{1 + \frac{3}{4}}.
\]

Further for \(0 < \epsilon \leq \frac{3}{4}\), there exists an integer \(\omega_\epsilon\) depending only \(\epsilon\) such that when \(N = N(abc) \geq N_\epsilon = \prod_{p \leq p_{\omega_\epsilon}} p\), we have

\[
c < \kappa_\epsilon N^{1 + \epsilon}
\]
where

\[ \kappa_\epsilon = \frac{6}{5\sqrt{2\pi}\max(\omega, \omega_\epsilon)} \leq \frac{6}{5\sqrt{2\pi}\omega_\epsilon} \]

with \( \omega = \omega(N) \). Here are some values of \( \epsilon, \omega_\epsilon \) and \( N_\epsilon \).

<table>
<thead>
<tr>
<th>( \epsilon )</th>
<th>( \omega_\epsilon )</th>
<th>( N_\epsilon )</th>
</tr>
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<tr>
<td>( \frac{3}{4} )</td>
<td>14</td>
<td>( e^{37.1101} )</td>
</tr>
<tr>
<td>( \frac{7}{12} )</td>
<td>49</td>
<td>( e^{204.75} )</td>
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<td>( e^{38.5586} )</td>
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<td>( e^{63727} )</td>
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<tr>
<td>( \frac{1}{3} )</td>
<td>6460</td>
<td>( e^{13006.2} )</td>
</tr>
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Thus \( c < N^2 \) which was conjectured in Granville and Tucker [GrTu02].

As a consequence of Theorem 2, we prove

**Theorem 3.** Assume Conjecture 1.5. Then the equation

\[ n(n + d) \cdots (n + (k - 1)d) = b \ell^k \]

in integers \( n \geq 1, d > 1, k \geq 4, b \geq 1, y \geq 1, \ell > 1 \) with \( \gcd(n, d) = 1 \) and \( P(b) \leq k \) implies \( \ell \leq 7 \). Further \( k < e^{13006.2} \) when \( \ell = 7 \).

We observe that \( e^{13006.2} < e^{e^{9.52}} \). Theorem 3 is a considerable improvement of Saradha [Sar12] where it is shown that (4) with \( k \geq 8 \) implies that \( \ell \leq 29 \) and further \( k \leq 8, 32, 10^2, 10^7 \) and \( e^{e^{280}} \) according as \( \ell = 29, \ell \in \{23, 19\}, \ell = 17, 13 \) and \( \ell \in \{11, 7\} \), respectively.

### 2. Notation and Preliminaries

For an integer \( i > 0 \), let \( p_i \) denote the \( i \)--th prime. We always write \( p \) for a prime number. For a real \( x > 0 \) and \( d \in \mathbb{Z}, d \geq 1 \), let

\[ \pi_d(x) = \sum_{p \leq x, p \mid d} 1, \quad \pi(x) = \pi_1(x) = \sum_{p \leq x} 1, \quad \Theta(x) = \prod_{p \leq x} p \quad \text{and} \quad \theta(x) = \log(\Theta(x)). \]

We write \( \log_2 i \) for \( \log(\log i) \). Here we understand that \( \log_2 1 = -\infty \).

**Lemma 2.1.** We have

(i) \( \pi(x) \leq \frac{x}{\log x} \left(1 + \frac{1.2762}{\log x}\right) \) for \( x > 1 \).

(ii) \( p_i \geq i(\log i + \log_2 i - 1) \) for \( i \geq 1 \)

(iii) \( \theta(p_i) \geq i(\log i + \log_2 i - 1.076869) \) for \( i \geq 1 \)

(iv) \( \theta(x) < 1.000081x \) for \( x > 0 \)

(v) \( \sqrt{2\pi k(\frac{2}{e})^k e^{\frac{1}{12k+1}}} \leq k! \leq \sqrt{2\pi k(\frac{2}{e})^k e^{\frac{1}{12k}}} \).

The estimates (i) and (ii) are due to Dusart, see [Dus99b] and [Dus99a], respectively. The estimate (iii) is [Rob83, Theorem 6]. For estimate (iv), see [Dus99b]. The estimate (v) is [Rob55, Theorem 6].
3. Proof of Theorem 3

Let $n, d, k, b, y$ be positive integers with $n \geq 1, d > 1, k \geq 4, b \geq 1, y \geq 1, \gcd(n, d) = 1$ and $P(b) \leq k$. We consider the Diophantine equation

$$n(n + d) \cdot \cdots \cdot (n + (k - 1)d) = by^\ell. \tag{5}$$

Observe that $P(n(n + d) \cdot \cdots \cdot (n + (k - 1)d)) > k$ by a result of Shorey and Tijdeman [ShTi90] and hence $P(y) > k$ and $n + (k - 1)d \geq (k + 1)^\ell$. For every $0 \leq i < k$, we write

$$n + id = A_i x_i^\ell \text{ with } P(A_i) \leq k \text{ and } (X_i, \prod_{p \leq k} p) = 1.$$

Without loss of generality, we may assume that $k = 4$ or $k \geq 5$ is a prime which we assume throughout in this section. We observe that $(A_i, d) = 1$ for $0 \leq i < k$ and $(X_i, X_j) = 1$. Let

$$S_0 = \{A_0, A_1, \ldots, A_{k-1}\}.$$

For every prime $p \leq k$ and $p \nmid d$, let $i_p$ be such that $ord_p(A_i) = ord_p(n + id) \leq ord_p(n + i_p d)$ for $0 \leq i < k$. For a $S \subset S_0$, let

$$S' = S - \{A_{i_p} : p \leq k, p \nmid d\}.$$

Then $|S'| \geq |S| - \pi_d(k)$. By Sylvester-Erdős inequality (see [ErSe75, Lemma 2] for example), we obtain

$$\prod_{A_i \in S'} A_i |(k - 1)! \prod_{p \mid d} p^{-ord_p((k-1)!)}. \tag{6}$$

As a consequence, we have

**Lemma 3.1.** Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq 1, \beta < 1$ and $e^\beta < \alpha$. Let

$$S_1 := S_1(\alpha) := \{A_i \in S_0 : A_i \leq \alpha k\}.$$

For

$$k \geq \max\left\{\frac{\log(e \alpha \sqrt{\beta}) + \frac{k \log(\alpha k)}{\log k} \left(1 + \frac{1.2762}{\log k}\right) - \log(\alpha k)}{\log(e \alpha) + \beta \log\left(\frac{\beta}{e \alpha}\right)}, \exp\left(\frac{1 + \frac{1.2762}{\log k}}{1 - \beta}\right)\right\}, \tag{7}$$

we have $|S_1| > \beta k$.

**Proof.** Let $S = S_0$, $s_1 = |S_1|$ and $s_2 = |S' - S_1|$. Then $s_2 \geq k - \pi(k) - s_1$. We get from (6) that

$$s_1! \prod_{i=1}^{k-\pi(k)-s_1} ([\alpha k + i]) \leq \prod_{A_i \in S'} A_i \leq (k - 1)! \tag{8}$$

since elements of $S' - S_1$ are distinct and the product on the left side is taken to be 1 if $k - \pi(k) \leq s_1$. 
Suppose $s_1 \leq \beta k$. If $k - \pi(k) \leq s_1$, then using Lemma 2.1 (i), we get $(1 - \beta) \log k < 1 + \frac{1.2762}{\log k}$ which is not possible by (7). Hence $k - \pi(k) > s_1$. By using Lemma 2.1 (v), we obtain

$$(\alpha k)^{k - \pi(k)} < \frac{(k - 1)!}{s_1!} (\alpha k)^{s_1} < \begin{cases} \sqrt{2\pi(k - 1)} \left(\frac{k - 1}{e}\right)^{k - 1} e^{\frac{1}{12(k - 1)}} & \text{if } s_1 = 0 \\ \sqrt{\frac{k - 1}{s_1}} \left(\frac{\alpha k}{s_1}\right)^{s_1} \left(\frac{k - 1}{e}\right)^{k - 1} & \text{if } s_1 > 0. \end{cases}$$

We check that the expression for $s_1 = 0$ is less than that of $s_1 = 1$ since $\alpha \geq 1$. Observe that $\sqrt{\frac{k - 1}{s_1}} \left(\frac{\alpha k}{s_1}\right)^{s_1}$ is an increasing function of $s_1$ since $s_1 \leq \beta k$ and $e\beta < \alpha$. This can be verified by taking log of the above expression and differentiating it with respect to $s_1$. Therefore

$$(\alpha k)^{k - \pi(k)} < \sqrt{\frac{k - 1}{\beta k}} \left(\frac{e\alpha}{\beta}\right)^{\beta k} \left(\frac{k - 1}{e}\right)^{k - 1} < \sqrt{\frac{1}{\beta}} \left(\frac{e\alpha}{\beta}\right)^{\beta k} \left(\frac{k}{e}\right)^{k - 1}$$

implying

$$(e\alpha)^{k} \left(\frac{\beta}{e\alpha}\right)^{\beta k} < \frac{e\alpha}{\sqrt{\beta}} (\alpha k)^{\pi(k) - 1}.$$ Using Lemma 2.1 (i), we obtain

$$\log(e\alpha) + \beta \log \left(\frac{\beta}{e\alpha}\right) < \frac{1}{k} \log \left(\frac{e\alpha}{\sqrt{\beta}}\right) + \frac{\log(\alpha k)}{\log k} \left(1 + \frac{1.2762}{\log k}\right) - \frac{\log(\alpha k)}{k}.$$ The right hand side of the above inequality is a decreasing function of $k$ for $k$ given by (7). This can be verified by observing that $\frac{\log(\alpha k)}{\log k} = 1 + \frac{\log \alpha}{\log k}$ and differentiating $\frac{1.2762 + \log \alpha}{\log k} - \frac{\log(\alpha k)}{k}$ with respect to $k$. This is a contradiction for $k$ given by (7). \qed

**Corollary 3.2.** For $k > 113$, there exist $0 \leq f < g < h < k$ with $h - f \leq 8$ such that $\max(A_f, A_g, A_h) \leq 4k$.

**Proof.** By dividing $[0, k - 1]$ into subintervals of the form $[9i, 9(i + 1))$, it suffices to show $S_1(4) > 2([\frac{6}{5}]) + 1$ where $S_1$ is as defined in Lemma 3.1. Taking $\alpha = 4, \beta = \frac{1}{4}$, we obtain from Lemma 3.1 that for $k \geq 700, |S_1(4)| > \frac{k}{4} > 2([\frac{6}{5}]) + 1$. Thus we may suppose $k < 700$ and $|S_1(4)| \leq 2([\frac{6}{5}]) + 1$. For each prime $k$ with $113 < k < 700$, taking $\alpha = 4$ and $\beta k = 2([\frac{6}{5}]) + 1$ in Lemma 3.1, we get a contradiction from (8). Therefore $|S_1(4)| > 2([\frac{6}{5}]) + 1$ and the assertion follows. \qed

Given $0 \leq f < g < h \leq k - 1$, we have

$$ (h - f)A_gX_f^\ell = (h - g)A_fX_f^\ell + (g - f)A_hX_h^\ell. $$

Let $\lambda = \gcd(h - f, h - g, g - f)$ and write $h - f = \lambda w, h - g = \lambda u, g - f = \lambda v$. Rewriting $h - f = h - g + g - f$ as

$$ w = u + v \text{ with } \gcd(u, v) = 1,$$
(9) can be written as
\[ wA_gX_g^\ell = uA_fX_f^\ell + vA_hX_h^\ell. \]
Let \( G = \gcd(wA_g, uA_f, vA_h), \)
\[ r = \frac{uA_f}{G}, \quad s = \frac{vA_h}{G}, \quad t = \frac{wA_g}{G} \]
and we rewrite (10) as
\[ tX_g^\ell = rX_f^\ell + sX_h^\ell. \]
Note that \( \gcd(rX_f^\ell, sX_h^\ell) = 1. \)

From now on, we assume explicit \( abc \)-conjecture. Given \( \epsilon > 0, \) let \( N(rstX_fX_gX_h) \geq N_\epsilon \) which we assume from now on till the expression (18). By Theorem 2, we obtain
\[ tX_g^\ell < \kappa_\epsilon N(rstX_fX_gX_h)^{1+\epsilon} \]
i.e.,
\[ X_g^\ell < \kappa_\epsilon \frac{N(rst)^{1+\epsilon}(X_fX_gX_h)^{1+\epsilon}}{t}. \]
Here \( N_\epsilon = \kappa_\epsilon = 1 \) if \( \epsilon > \frac{3}{4}. \) For \( \epsilon = 3/4, \) by abuse of notation, we will be taking either \( N_\epsilon = 1, \kappa_\epsilon = 1 \) or \( N_\epsilon = e^{37.1101}, \kappa_\epsilon \leq \frac{6}{5\sqrt{28}} \) if \( N(rstX_fX_gX_h) \geq N_\frac{3}{4}. \) We will be taking \( \epsilon = \frac{3}{4} \) for \( \ell > 7 \) and \( \epsilon \in \{\frac{5}{12}, \frac{1}{3}\} \) for \( \ell = 7. \) We have from (13) that
\[ rst(X_fX_gX_h)^\ell < \kappa_\epsilon^3 N(rst)^{3(1+\epsilon)}(X_fX_gX_h)^{3(1+\epsilon)}. \]
Putting \( X^3 = X_fX_gX_h, \) we obtain
\[ X^{\ell-3(1+\epsilon)} < \kappa_\epsilon N(rst)^{\frac{3}{2}+\epsilon} = \kappa_\epsilon N(\frac{uvwA_fA_gA_h}{G^3})^{\frac{3}{2}+\epsilon}. \]

Again from (12), we have
\[ rs(X_fX_h)^\ell \leq \left(\frac{rX_f^\ell + sX_h^\ell}{2}\right)^2 = \frac{t^2X_g^{2\ell}}{4} \]
implying
\[ X_fX_hX_g \leq \left(\frac{t^2}{4rs}\right)^{\frac{1}{2}} X_g^3 = \left(\frac{w^2A_g^2}{4uvA_fA_h}\right)^{\frac{1}{2}} X_g^3. \]
Therefore we have from (14) that
\[ X_g^\ell < \kappa_\epsilon \frac{N(rst)^{1+\epsilon}X_g^{3+3\epsilon}}{t} \left(\frac{t^2}{4rs}\right)^{\frac{1}{2}+\epsilon} = \kappa_\epsilon \frac{N(rst)^{1+\epsilon}X_g^{3+3\epsilon}}{(4rst)^{1+\epsilon}t^{1-\frac{3(1+\epsilon)}{\epsilon}}} \]
i.e.,
\[ X_g^{\ell-3(1+\epsilon)} < \kappa_\epsilon \frac{N(rst)^{(1+\epsilon)(1-\frac{1}{\epsilon})}}{4^{1+\epsilon}t^{1-\frac{3(1+\epsilon)}{\epsilon}}} \leq \frac{N(rs)^{(1+\epsilon)(1-\frac{1}{\epsilon})}N(t)^{\epsilon+\frac{2(1+\epsilon)}{\epsilon}}}{4^{1+\epsilon}}. \]
We also have from (17) that

\[ X_g^{\ell-3(1+\epsilon)} < \kappa_\epsilon N \left( \frac{wA_f A_h}{G^2} \right)^{(1+\epsilon)(1-\frac{1}{2})} N \left( \frac{wA_f}{G} \right)^{\epsilon + \frac{2(1+\epsilon)}{\ell}} \frac{1}{4^{1+\epsilon}} \].

\[ \text{(18)} \]

**Lemma 3.3.** Let \( \ell \geq 11 \). Let \( S_0 = \{ A_0, A_1, \ldots, A_{k-1} \} = \{ B_0, B_1, \ldots, B_{k-1} \} \) with \( B_0 \leq B_1 \leq \ldots \leq B_{k-1} \). Then

\[ B_0 \leq B_1 < B_2 \ldots < B_{k-1}. \]

In particular \( |S_0| \geq k - 1 \).

**Proof.** Suppose there exists \( 0 \leq f < g < h < k \) with \( \{f, g, h\} = \{i_1, i_2, i_3\} \) and

\[ A_{i_1} = A_{i_2} = A \text{ and } A_{i_3} \leq A. \]

By (10) and (11), we see that \( \max(A_f, A_g, A_h) \leq G \) and therefore \( r \leq u < k, s \leq v < k \) and \( t \leq w < k \). Since \( X_g > k \), we get from the first inequality of (17) with \( \epsilon = \frac{3}{4}, N_\epsilon = \kappa_\epsilon = 1 \) that

\[ k^{\ell-3(1+\epsilon)} < (rs)^{(1+\epsilon)(1-\frac{1}{2})} t^{\epsilon + \frac{2(1+\epsilon)}{\ell}} < k^{2+3\epsilon} \]

implying \( \ell < 5 + 6\epsilon = 5 + \frac{9}{2} \). This is a contradiction since \( \ell \geq 11 \). Therefore either \( A_i \)'s are distinct or if \( A_i = A_j = A \), then \( A_m > A \) for \( m \notin \{i, j\} \) implying the assertion. \( \square \)

As a consequence, we have

**Corollary 3.4.** Let \( d \) be even and \( \ell \geq 11 \). Then \( k \leq 14 \).

**Proof.** Let \( d \) be even and \( \ell \geq 11 \). Then we get from (6) with \( S = S_0 \) that

\[ \prod_{A_i \in S'} A_i \leq (k - 1)! 2^{-\text{ord}_2((k-1)!)} = \prod_{2i+1 \leq k-1} (2i + 1). \]

Observe that the right hand side of the above inequality is the product of all positive odd numbers less than \( k \). On the other hand, since \( \gcd(n, d) = 1 \), we see that all \( A_i \)'s are odd and \( |S'| \geq |S_0| - \pi(k) \geq k - 1 - \pi(k) \) by Lemma 3.3. Hence

\[ \prod_{A_i \in S'} A_i \geq \prod_{i=1}^{k-1-\pi(k)} (2i - 1). \]

Observe here again that the right hand side of the above inequality is the product of first positive \( k - 1 - \pi(k) \) odd numbers. Hence we get a contradiction if \( 2(k - 1 - \pi(k)) - 1 > k - 1 \). Assume \( k \leq 2 + 2\pi(k) \). By Lemma 2.1 (i), we get \( 1 \leq \frac{2}{k} + \frac{2}{\log k} \left( 1 + \frac{1.2762}{\log k} \right) \) which is not possible for \( k \geq 30 \). By using exact values of \( \pi(k) \), we check that \( k \leq 2 + 2\pi(k) \) is not possible for \( 15 \leq k < 30 \). Hence the assertion. \( \square \)

**Lemma 3.5.** Let \( \ell \geq 11 \). Then \( k < 400 \).
Proof. Assume that \( k \geq 400 \). By Corollary 3.4, we may suppose that \( d \) is odd. Further by Corollary 3.2, there exists \( f < g < h \) with \( h - f \leq 8 \) and \( \max(A_f, A_g, A_h) \leq 4k \).

Since \( n + (k - 1)d > k^\ell \), we observe that \( X_f > k, X_g > k, X_h > k \) implying \( X > k \).

First assume that \( N = N(r s t X_f X_g X_h) < e^{37.12} \). Then taking \( \epsilon = \frac{3}{4} \), \( N_\epsilon = 1 \) in (13), we get \( 400^{11} \leq k^{11} \leq tX^{\ell} < N^{1+\frac{3}{4}} \leq e^{37.12(1+\frac{3}{4})} \) which is a contradiction. Hence we may suppose that \( N \geq e^{37.12} \geq N_3 \).

Note that we have \( u + v = w \leq h - f \leq 8 \). We observe that \( u w v \) is even. If \( A_f A_g A_h \) is odd, then \( h - f, g - f, h - g \) are all even implying \( 1 \leq u, v, w \leq 4 \) or \( N(u w v) \leq 6 \) giving \( N(u w v A_f A_g A_h) \leq 6A_f A_g A_h \). Again if \( A_f A_g A_h \) is even, then \( N(u w v A_f A_g A_h) \leq N((u w v)' A_f A_g A_h) \leq 35A_f A_g A_h \) where \( (u w v)' \) is the odd part of \( u w v \) and \( N((u w v)' ) \leq 35 \).

Observe that \( N((u w v)' ) \) is obtained when \( w = 7, u = 2, v = 5 \) or \( w = 7, u = 5, v = 2 \). Thus we always have \( N(u w v A_f A_g A_h) \leq 35A_f A_g A_h \leq 35 \cdot (4k)^3 \) since \( \max(A_f, A_g, A_h) \leq 4k \). Therefore taking \( \epsilon = \frac{3}{4} \) in (15), we obtain using \( \ell \geq 11 \) and \( X > k \) that

\[
k^{11-3(1+\frac{3}{4})} < \frac{6}{5\sqrt{28\pi}}35^{\frac{3}{4}+\frac{3}{4}}(4k)^{3\left(\frac{3}{4}+\frac{3}{4}\right)}.
\]

This is a contradiction since \( k \geq 400 \). Hence the assertion. \( \square \)

4. PROOF OF THEOREM 3 FOR \( 4 \leq k < 400 \)

We assume that \( \ell \geq 11 \). It follows from the result of Saradha and Shorey [SaSh05, Theorem 1] that \( d > 10^{15} \). Hence we may suppose that \( d > 10^{15} \) in this section.

Lemma 4.1. Let \( r_k = [k + 1 - \pi(k) - \frac{\sum_{i \leq k} \log i}{15 \log 10}] \) and

\[
I(k) = \{ i \in [1, k] : P(n + id) > k \}.
\]

Then \( |I(k)| \geq r_k \).

Proof. Suppose not. Then \( |I(k)| \leq r_k - 1 \). Let

\[
I'(k) = \{ i \in [1, k] : P(n + id) \leq k \} = \{ i \in [1, k] : n + id = A_i \}.
\]

We have \( A_i = n + id \geq (n + d) \) for \( i \in I'(k) \). Let \( S = \{ A_i : i \in I'(k) \} \). Then \( |S| \geq k + 1 - r_k \). From (6), we get

\[
(k - 1)! \geq \prod_{A_i \in S'} A_i \geq (n + d)^{|S'|} > d^{k+1-r_k-\pi(k)}.
\]

Since \( d > 10^{15} \), we get

\[
k + 1 - \pi(k) - \frac{\sum_{i \leq k} \log i}{15 \log 10} < r_k = [k + 1 - \pi(k) - \frac{\sum_{i \leq k} \log i}{15 \log 10}].
\]

This is a contradiction. \( \square \)

Here are some values of \( (k, r_k) \).
We give the strategy here. Let $I_k = [0, k-1] \cap \mathbb{Z}$ and $a_0, b_0, z_0$ be given. Let obtain a subset $I_0 \subseteq I_k$ with the following properties:

1. $|I_0| \geq z_0 \geq 3$.
2. $P(A_i) \leq a_i$ for $i \in I_0$.
3. $I_0 \subseteq [j_0, j_0 + b_0 - 1]$ for some $j_0$.
4. $X_0 = \max_{i \in I_0} \{|X_i| > k\}$ and let $i_0 \in I_0$ be such that $X_0 = X_{i_0}$.

For any $i, j \in I_0$, taking $\{f, g, h\} = \{i, j, i_0\}$, let $N = N(rstX_fX_gX_h)$. Observe that $X_0 \geq p_{\pi(k)+1}$ and further for any $f, g, h \in I_0$, we have $N(uvw) \leq \prod_{p \leq b_0-1} p$ and $N(A_fA_gA_h) \leq \prod_{p \leq a_0} p$. We will always take $\epsilon = \frac{3}{4}$, $N_\pi = 1$ so that $\kappa_\pi = 1$ in (13) to (18).

**Case I:** Suppose there exists $i, j \in I_0$ such that $X_i = X_j = 1$. Taking $\{f, g, h\} = \{i, j, i_0\}$ and $\epsilon = \frac{3}{4}$, we obtain from (14) and $\ell \geq 11$ that

$$\frac{37}{p_{\pi(k)+1}} \leq X_0^{\frac{\ell + 3}{4}} < N(uvwA_fA_gA_h) \leq \prod_{p \leq \max\{a_0, b_0-1\}} p.$$

**Case II:** There is at most one $i \in I_0$ such that $X_i = 1$. Then $|\{i \in I_0 : X_i > k\}| \geq z_0 - 1$. We take $a_1, b_1, z_1$ and find a subset $U_0 \subset I_0$ with the following properties:

1. $|U_0| \geq z_1 \geq 3$, $\frac{a_0}{2} \leq z_1 \leq z_0$.
2. $P(A_i) \leq a_1$ for $i \in U_0$.
3. $U_0 \subseteq [i, i + b_1 - 1]$ for some $i$.

Let $X_i = \max_{i \in U_0} \{|X_i| \geq p_{\pi(k)+z_1-1} \}$ and $i_1$ be such that $X_{i_1} = X_i$. Taking $\{f, g, h\} = \{i, j, i_1\}$ for some $i, j \in U_0$ and $\epsilon = \frac{3}{4}$, we obtain from (17) and $\ell \geq 11$ that

$$\frac{23}{p_{\pi(k)+z_1-1}} \leq X_0^{\frac{\ell + 3}{4}} < N(uvwA_fA_gA_h) \leq \prod_{p \leq \max\{a_1, b_1-1\}} p,$$

since $\ell \geq 11$. One choice is $(U_0, a_1, b_1, z_1) = (I_0, a_0, b_0, z_0)$. We state the other choice.

Let $b' = \max(a_0, b_0 - 1)$. For each $\frac{b_0}{2} - 1 < p \leq b' - 1$, we remove those $i \in I_0$ such that $p|(n + id)$. There are at most $2(\pi(b' - 1) - \pi(\frac{b_0}{2} - 1))$ such $i$. Let $I'_0$ be obtained from $I_0$ after deleting those $i$'s. Then $|I'_0| \geq z_0 - 2(\pi(b' - 1) - \pi(\frac{b_0}{2} - 1))$. Let

$$U_1 = I'_0 \cap [j_0, j_0 + b_0 + \frac{b_0}{2} - 1] \quad \text{and} \quad U_1 = I'_0 \cap [j_0 + \frac{b_0}{2}, j_0 + b_0 - 1].$$

Let $U_0 \in \{U_1, U_2\}$ for which $|U_0| = \max(|U_1|, |U_2|)$ and choose one of them if $|U_1| = |U_2|$. Then $|U_0| \geq \frac{z_0}{2} - \pi(b' - 1) + \pi(\frac{b_0}{2} - 1) = z_1$. Further $P(A_i) \leq b'_0 - 1 = a_1$ and $b_1 = \frac{b_0}{2}$. Further $X_1 = \max_{i \in U_0} \{|X_i| \geq p_{\pi(k)+z_1-1} \}$.
4.1. \( k \in \{4, 5, 7, 11\} \)

We take \( I_0 = U_0 = I_k, a_i = b_i = z_i = k \) for \( i \in \{0, 1\} \) and hence \( N(uvwA_fA_gA_h) \leq \prod_{p \leq k} p \). And the assertion follows since both (19) and (20) are contradicted.

4.2. \( k \in \{13, 17, 19, 23\} \)

We take \( I_0 = \{i \in [1, 11] : p \nmid (n + id) \text{ for } 13 \leq p \leq 23\} \). Then by \( \ell_{11} = 6 \) and Lemma 4.1 with \( k = 11 \), we see that \( |I_0| \geq z_0 = 11 - 4 > 11 - r_{11} \geq 11 - |I(11)| \). Therefore there exist an \( i \in I_0 \cap I_{11} \) and hence \( X_i > 23 \). We take \( U_0 = I_0, a_i = b_i = 11, z_i = z_0 \) for \( i \in \{0, 1\} \) and hence \( N(uvwA_fA_gA_h) \leq \prod_{p \leq 11} p \). And the assertion follows since both (19) and (20) are contradicted.

4.3. \( 29 \leq k \leq 47 \)

We take \( I_0 = \{i \in [1, 17] : p \nmid (n + id) \text{ for } 17 \leq p \leq k\} \). Then by \( \ell_{17} = 10 \) and Lemma 4.1 with \( k = 17 \), we have \(|I_0| \geq z_0 = 17 - (\pi(k) - \pi(13)) = 23 - \pi(17) \geq 23 - \pi(47) = 8 > 17 - r_{17} \geq 17 - |I(17)| \) implying that there exists \( i \in I_0 \) with \( X_i > k \). We take \( a_i = 13, b_i = 17, z_i = 23 - \pi(k) \) for \( i \in \{0, 1\} \) and hence \( N(uvwA_fA_gA_h) \leq \prod_{p \leq 13} p \). And the assertion follows since both (19) and (20) are contradicted.

4.4. \( k \geq 53 \)

Given \( m \) and \( q \) such that \( mq < k \), we consider the \( q \) intervals

\[
I_j = [(j - 1)m + 1, jm] \cap \mathbb{Z} \text{ for } 1 \leq j \leq q
\]

and let \( I' = \bigcup_{j=1}^{q} I_j \) and \( I'' = \{i \in I' : m \leq P(A_i) \leq k\} \). There is at most one \( i \in I' \) such that \( mq - 1 < P(A_i) \leq k \) and for each \( 2 \leq j \leq q \), there are at most \( j \) number of \( i \in I' \) such that \( \frac{mq-1}{j} < P(A_i) \leq \frac{mq-1}{j-1} \). Therefore

\[
|I''| \leq \pi(k) - \pi(mq - 1) + \sum_{j=2}^{q} j \left( \pi\left( \frac{mq - 1}{j} \right) - \pi\left( \frac{mq - 1}{j-1} \right) \right)
\]

\[
= \pi(k) + \sum_{j=1}^{q-1} \pi\left( \frac{mq - 1}{j} \right) - q\pi(m - 1) =: T(k, m, q).
\]

Hence there is at least one \( j \) such that \( |I_j \cap I''| \leq \left[ \frac{T(k, m, q)}{q} \right] \). We will choose \( q \) such that \( \left[ \frac{T(k, m, q)}{q} \right] < r_m \). Let \( I_0 = I_j \setminus I'' \) and let \( j_0 \) be such that \( I_0 \subseteq [(j_0 - 1)m + 1, j_0m] \). Then \( p|(n + id) \text{ imply } p < m \) or \( p > k \) whenever \( i \in I_0 \). Further \( |I_0| \geq z_0 = m - \left[ \frac{T(k, m, q)}{q} \right] \). Since \( \left[ \frac{T(k, m, q)}{q} \right] < r_m \), we get from Lemma 4.1 with \( k = m \) and \( n = (j_0 - 1)m \) that there is an \( i \in I_0 \) with \( X_i > k \). Further \( P(A_i) < m \) for all \( i \in I_0 \). Here are the choices of \( m \) and \( q \).
We have $a_0 = m - 1, b_0 = m$ and $z_0 = m - \lceil \frac{T(k,m,q)}{q} \rceil$ and we check that $z_0 \geq 3$.

The Subsection 4.3(29 $\leq k \leq 47$) is in fact obtained by considering $m = 17, q = 1$. Now we consider Cases I and II and try to get contradiction in both (19) and (20).

For these choices of $(m,q)$, we find that the Cases I are contradicted. Further taking $U_0 = I_0, a_1 = a_0 = m - 1, b_1 = b_0 = m, z_1 = z_0$, we find that Case II is also contradicted for $53 \leq k < 89$. Thus the assertion follows in the case $53 \leq k < 89$. So, we consider $k \geq 89$ and try to contradict Cases II. Recall that we have $X_i > k$ for all but at most one $i \in I_0$. Write $I_0 = U_1 \cup U_2$ where $U_1 = I_0 \cap [(j_0 - 1)m + 1, (j_0 - 1)m + m \frac{m}{2}]$ and $U_2 = I_0 \cap [(j_0 - 1)m + m \frac{m}{2} + 1, j_0m]$. Let $U_0' = U_1$ or $U_0' = U_2$ according as $|U_1| \geq \frac{m}{2}$ or $|U_2| \geq \frac{m}{2}$, respectively. Let $U_0 = \{i \in U_0' : p \nmid A_i \text{ for } \frac{m}{2} \leq p < m\}$. Then $|U_0| \geq z_1 := \frac{m}{2} - (\pi(m - 1) - \pi(\frac{m}{2})) = \frac{m - \lceil \frac{T(k,m,q)}{q} \rceil}{2} - (\pi(m - 1) - \pi(\frac{m}{2})) \geq 3$.

Further $p|(n + id)$ with $i \in U_0$ imply $p < \frac{m}{2}$ or $p > k$. Now we have Case II with $a_1 = \frac{m}{2} - 1, b_1 = \frac{m}{2}$ and find that (20) is contradicted. Hence the assertion.

5. $\ell = 7$

Let $\ell = 7$. Assume that $k \geq \exp(13006.2)$. Taking $\alpha = 3, \beta = \frac{1}{15} + \frac{2}{9}$ in Lemma 3.1, we get

$$|S_1(3)| = \{i \in [0, k - 1] : A_i \leq 3k\} > k(\frac{1}{15} + \frac{2}{9}).$$

For $i$'s such that $A_i \in S_1(3)$, we have $X_i > k$ and we arrange these $X_i$'s in increasing order as $X_{i_1} < X_{i_2} < \ldots <$. Then $X_{i_j} \geq p_{\pi(k) + j}$. Consider the set $J_0 = \{i : X_i \geq p_{\pi(k) + [\frac{k}{15}] - 2}\}$. We have

$$|J_0| > k(\frac{1}{15} + \frac{2}{9}) - \frac{k}{15} + 2 \geq 2 \left(\frac{k - 1}{9} + 1\right).$$

Hence there are $f, g, h \in J_0, f < g < h$ such that $h - f \leq 8$. Also $A_i \leq 3k$ and $X = (X_fX_gX_h)^\frac{1}{2} \geq p_{\pi(k) + [\frac{k}{15}] - 2}$.

First assume that $N = N(rstX_fX_gX_h) \geq \exp(63727) \geq N_{\frac{1}{3}}$. Observe that $uvw \leq 70$ since $2 \leq u + v = w \leq 8$, obtained at $2 + 5 = 7$. Taking $\epsilon = \frac{1}{3}$, we obtain from (15) and max($A_f, A_g, A_h) \leq 3k$ that

$$p_{\pi(k) + [\frac{k}{15}] - 2}^3 < \frac{5}{6\sqrt{2} \cdot 6458}N(uvwA_fA_gA_h) \leq \frac{5 \cdot 70 \cdot (3k)^3}{6\sqrt{12920}},$$

Since $\pi(k) > 2$ we have $\pi(k) + [\frac{k}{15}] - 2 > \frac{k}{15}$ and hence $p_{\pi(k) + [\frac{k}{15}] - 2} > \frac{k}{15}$ by Lemma 2.1 (ii). Therefore

$$\left(\log \frac{k}{15}\right)^3 < \frac{350 \cdot (3 \cdot 15)^3}{6\sqrt{12920)} \text{ or } k < 15 \cdot \exp\left(45 \cdot \left(\frac{350}{6\sqrt{12920}}\right)^{\frac{1}{3}}\right)\).$$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$53 \leq k &lt; 89$</th>
<th>$89 \leq k &lt; 179$</th>
<th>$179 \leq k &lt; 239$</th>
<th>$239 \leq k &lt; 367$</th>
<th>$367 \leq k &lt; 433$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(m,q)$</td>
<td>(17, 3)</td>
<td>(28, 3)</td>
<td>(36, 5)</td>
<td>(36, 6)</td>
<td>(36, 10)</td>
</tr>
</tbody>
</table>
which is a contradiction since \( k \geq \exp(13006.2) \).

Therefore we have \( N = N(rstXfXgXh) < \exp(63727) \). We may also assume that \( N > \exp(3895) \) otherwise taking \( \epsilon = \frac{3}{2} \) in (13), we get \( k^7 < X_g^7 < N^{1+\frac{3}{2}} \leq \exp(3895 \cdot \frac{7}{4}) \) or \( k < \exp(3895) \) which is a contradiction. Now we take \( \epsilon = \frac{5}{12} \) in (13) to get \( k^7 < X_g^7 < N^{1+\frac{5}{12}} \leq \exp(64266 \cdot \frac{17}{12}) \) or \( k < \exp(13006.2) \). Hence the assertion.

6. \textit{abc-conjecture implies Erdős conjecture: Proof of Theorem 1}

Assume (5). We show that \( k \) is bounded by a computable absolute constant. Let \( k \geq k_0 \) where \( k_0 \) is a sufficiently large computable absolute constant. Let \( \epsilon > 0 \). Let \( c_1, c_2, \cdots \) be positive computable constants depending only on \( \epsilon \). Let \( I = \{i_p | p \leq k \mbox{ and } p \mid d\} \) where \( i_p \) be as defined in beginning of Section 3. Let \( S = \{A_i : \left[\frac{k}{2}\right] \leq i < k \mbox{ or } i \in I\} \) and

\[
\Phi = \prod_{i \geq \left[\frac{k}{2}\right] \atop v \notin I} A_i.
\]

By Sylvester-Erdős inequality (see [ErSc75, Lemma 2] for example), we get

\[
\text{ord}_p(\Phi) \leq \text{ord}_p \left( \prod_{i \geq \left[\frac{k}{2}\right] \atop v \notin I} (i - i_p) \right) \leq \begin{cases} \text{ord}_p \left( (k - \left[\frac{k}{2}\right]) - 1 - (i_p - \left[\frac{k}{2}\right])! \right) & \text{if } i_p \geq \left[\frac{k}{2}\right], \\ \text{ord}_p \left( (k - \left[\frac{k}{2}\right]) (k - \left[\frac{k}{2}\right])! \right) & \text{otherwise}. \end{cases}
\]

Since \( \text{ord}_p(r!) \leq \text{ord}_p((r + s)!) \) and \( k - \left[\frac{k}{2}\right] = \left[\frac{k+1}{2}\right] \), we see that

\[
p^{\text{ord}_p(\Phi)} \leq p^{\text{ord}_p((k - \left[\frac{k}{2}\right] + 1)!)} = p^{\text{ord}_p((\frac{k+1}{2})!)}.
\]

Using the fact that \( p^{\text{ord}_p((k)!)} \leq x \) for any \( x \geq k \), we get

\[
\Phi \leq (k - 1)^{\pi_d(k)} \left( \frac{k + 1}{2} \right)! \leq k^\frac{k}{2} e^{c_1 k}
\]

by using Lemma 2.1 \((i),(v)\).

Let \( D \) be a fixed positive integer and let

\[
J = \left\{ \frac{k - 1}{2D} \leq j \leq \frac{k - 1}{D} - 1 | \{Dj + 1, Dj + 2, \cdots, Dj + D\} \cap I = \phi \right\}.
\]

We shall choose \( D = 20 \). Let \( j, j' \in J \) be such that \( j \neq j' \). Then \( Dj + i \neq Dj' + i' \) for \( 1 \leq i, i' \leq D \) otherwise \( Dj - j' = (i - i') \) and \( |i' - i| < D \). Further we also see that \( \left[\frac{k}{2}\right] \leq Dj + i \leq k - 1 \) for \( 1 \leq i \leq D \) and consequently \( |J| \geq \frac{k - 1}{2D} - \pi(k) \). For each \( j \in J \), let \( \Phi_j = \prod_{i=1}^{D} A_{Dj+i} \). Then \( \prod_{j \in J} \Phi_j \) divides \( \Phi \) implying

\[
\prod_{j \in J} \Phi_j \leq \Phi \leq k^\frac{k}{2} e^{c_1 k}.
\]
Thus there exists $j_0 \in J$ such that
\[
\Phi_{j_0} \leq \left( k^2 e^{c_1 k} \right)^{\frac{1}{\ell}} \leq \left( k^2 e^{c_1 k} \right)^{\frac{k-1}{\ell^{2+\epsilon}(k)}} \leq c_2^D k^D.
\]
Let
\[H := \prod_{i=1}^{D} (n + (Dj_0 + i)d).\]
Since $A_{Dj_0+i} X_{Dj_0+i} \leq n + (k-1)d$, we have $X_{Dj_0+i} \leq \left( \frac{n+(k-1)d}{A_{Dj_0+i}} \right)^{\frac{1}{\ell}}$. Thus
\[\prod_{p|H, p \leq k} p \leq \prod_{i=1}^{D} X_{Dj_0+i} \leq (n + (k-1)d)^{\frac{D}{\ell}} (\Phi_{j_0})^{-\frac{1}{\ell}} \]
Therefore
\[\prod_{p|H} p = \left( \prod_{p|H, p \leq k} p \right) \left( \prod_{p|H, p > k} p \right) \leq \Phi_{j_0} (n + (k-1)d)^{\frac{D}{\ell}} (\Phi_{j_0})^{-\frac{1}{\ell}} \leq c_2^{D(1-\frac{1}{\ell})} k^{D(1-\frac{1}{\ell})} (n + (k-1)d)^{\frac{D}{\ell}}.\]
On the other hand, we have $H = F(n + D j_0 d, d)$ where
\[F(x, y) = \prod_{i=1}^{D} (x + iy)\]
is a binary form in $x$ and $y$ of degree $D$ such that $F$ has distinct linear factors. From Conjecture 1.4, we have
\[\prod_{p|H} p \geq c_3 (n + D j_0 d)^{D-2-\epsilon}.\]
Comparing the lower and upper bounds of $\prod_{p|H} p$ and using $n + D j_0 d > \frac{n+(k-1)d}{2}$, we get
\[k > c_4 (n + (k-1)d)^{1 - \frac{2+\epsilon}{D(1-\frac{1}{\ell})}}.\]
We now use $n + (k-1)d > k^\ell$ to derive that
\[c_5 > k^{\ell(1 - \frac{2+\epsilon}{D(1-\frac{1}{\ell})})^{-1}}.\]
Taking $\epsilon = \frac{1}{2}$ and putting $D = 20$, we get
\[c_6 > k^{\ell-1 - \frac{1}{\ell} - 1} \geq k^{\frac{1}{2}}\]
since $\ell \geq 2$. This is a contradiction since $k \geq k_0$ and $k_0$ is sufficiently large. □
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