Irreducibility and Galois Groups of Generalized Laguerre Polynomials

$L_n^{(-1-n-r)}(x)$

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IRREDUCIBILITY AND GALOIS GROUPS OF GENERALIZED LAGUERRE POLYNOMIALS $L_n^{(-1-n-r)}(x)$

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Dedicated to Professor T. N. Shorey on his 70th birthday

ABSTRACT. We study the algebraic properties of Generalized Laguerre polynomials for negative integral values of a given parameter which is $L_n^{(-1-n-r)}(x) = \sum_{j=0}^{n} \binom{n-j+r}{n-j} x^j/j!$ for integers $r \geq 0, n \geq 1$. For different values of parameter $r$, this family provides polynomials which are of great interest. Hajir conjectured that for integers $r \geq 0$ and $n \geq 1$, $L_n^{(-1-n-r)}(x)$ is an irreducible polynomial whose Galois group contains $A_n$, the alternating group on $n$ symbols. Extending earlier results of Schur, Hajir, Sell, Nair and Shorey, we confirm this conjecture for all $r \leq 60$. We also prove that $L_n^{(-1-n-r)}(x)$ is an irreducible polynomial whose Galois group contains $A_n$ whenever $n > e^{r(1+\frac{\log 2}{3.25})}$.

1. INTRODUCTION

For an arbitrary real number $\alpha$ and a positive integer $n$, the Generalized Laguerre Polynomials (GLP) is a family of polynomials defined by

$$L_n^{(\alpha)}(x) = (-1)^n \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!}.$$  

The inclusion of the sign $(-1)^n$ is not standard. The corresponding monic polynomial is obtained as $L_n^{(\alpha)}(x) = n!L_n^{(\alpha)}(x).$ These classical orthogonal polynomials play an important role in various branches of analysis and mathematical physics and has been well studied. Schur [15], [16] was the first to study the algebraic properties of these polynomials by proving that $L_n^{(\alpha)}(x)$ where $\alpha \in \{0, 1, -n-1\}$ are irreducible. For an account of results obtained on GLP, we refer to Hajir [10] and Filaseta, Kidd and Trifonov [6].

In this paper, we study $\alpha$ at negative integral values via a parameter $r$. For integer $r \geq 0$, we consider

$$L_n^{(r)}(x) := L_n^{(-1-n-r)}(x)$$

$$= (-1)^n \sum_{j=0}^{n} \binom{-1-r}{n-j} \frac{(-x)^j}{j!}$$

$$= \sum_{j=0}^{n} \binom{n-j+r}{n-j} \frac{x^j}{j!}.$$  

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By a factor of a polynomial, we always mean its factor over \( \mathbb{Q} \). We observe that \( \mathcal{L}_n^{(r)}(x) := n! L_n^{(r)}(x) = \sum_{j=0}^{n} \binom{n}{j} (r+1) \ldots (r+n-j)x^j \) is a monic polynomial with integer coefficients and \( L_n^{(r)}(x) \) is irreducible if and only if \( \mathcal{L}_n^{(r)}(x) \) is irreducible. Schur [16] computed the discriminant of \( \mathcal{L}_n^{(r)}(x) \) which is

\[
\Delta_n^{(r)} = \prod_{j=2}^{n} j^j(-1 - n - r + j)^{j-1}.
\]

Let \( G_n(r) \) denote the Galois group of \( \mathcal{L}_n^{(r)}(x) \) over \( \mathbb{Q} \). Let \( S_n \) denote the symmetric group on \( n \) symbols and \( A_n \), the alternating group on \( n \) symbols. Schur [15, 16] and Coleman [2] used two different techniques to prove that \( L_n^{(0)}(x) \) is irreducible and \( G_n(0) = S_n \) for every \( n \). Hajir [8] proved that \( L_n^{(1)}(x) \) is irreducible and \( G_n(1) \) is \( A_n \) if \( n \equiv 1(\text{mod } 4) \) and is \( S_n \), otherwise. Sell [14] proved that \( L_n^{(2)}(x) \) is irreducible and \( G_n(2) \) is \( A_n \) if \( n + 1 \) is an odd square and is \( S_n \), otherwise.

The irreducibility of \( L_n^{(n)}(x) \), also known as Bessel polynomials, was conjectured for all \( n \) by Grosswald [7] and assuming his conjecture he proved that the Galois group is \( S_n \) for every \( n \). The irreducibility of all Bessel polynomials was proved, first for all but finitely many \( n \) by Filaseta [4] and later for all \( n \) by Filaseta and Trifonov [5].

Hajir [10] conjectured that for integers \( r \geq 0 \), \( n \geq 1 \), \( L_n^{(r)}(x) \) is irreducible and \( G_n(r) \) contains \( A_n \). It was also proved in [10] that if \( r \) is a fixed integer in the range \( 0 \leq r \leq 8 \), then for all \( n \geq 1 \), \( L_n^{(r)}(x) \) is irreducible and has Galois group containing \( A_n \). This was extended by Nair and Shorey [13] who proved the following.

**Theorem A.** For \( n \geq 1 \),

(i) \( L_n^{(r)}(x) \) is irreducible for \( 3 \leq r \leq 22 \).

(ii) For \( 9 \leq r \leq 22 \), \( G_n(r) = S_n \) unless \( (n, r) \in \{(8, 9), (12, 13), (13, 16), (16, 17), (17, 18), (20, 21)\} \) in which case \( G_n(r) = A_n \). For \( 3 \leq r \leq 8 \), \( G_n(r) = S_n \) unless \( (n, r) \in \{(2, 3), (24, 4), (4, 5), (6, 7), (7, 8), (9, 8), (2, 8)\} \) or \( r = 3; n \equiv 1(\text{mod } 24) \) and \( n+2 \) is a square.

\( r = 4; n + 2 \) is a rational part of \( (2 + \sqrt{3})^{2k+1} \) where \( k \geq 0 \) is an integer.

\( r = 5; n + 3 \) is a rational part of \( (4 + \sqrt{15})^{2k+1} \) where \( k \geq 0 \) is an integer in which case \( G_n(r) = A_n \).

We further extend this work to confirm the conjecture of Hajir for all \( r \leq 60 \). We prove

**Theorem 1.1.** For \( n \geq 1 \) and \( 23 \leq r \leq 60 \), we have

(i) \( L_n^{(r)}(x) \) is irreducible.

(ii) \( G_n(r) = S_n \) unless \( (n, r) \in \{(4, 24), (5, 28), (24, 25), (25, 24), (28, 23), (28, 29), (32, 33), (33, 36), (36, 37), (40, 41), (44, 45), (48, 49), (48, 51), (49, 48), (49, 50), (52, 53), (56, 57)\} \) in which case \( G_n(r) = A_n \).

The proof of Theorem 1.1 is given in Sections 4 and 5. We see that Theorem 1.1 considerably extends earlier results of [10] and [13]. The new ingredients in the proof are Lemma 3.1 which arise from clever and important observations on prime divisors of \( n \) and \( \binom{n+r}{r} \) and Lemmas 3.5-3.7 which arise from an application of p-adic Newton polygons. These results are general in nature and make our computations much less. In fact, for checking irreducibility of \( L_n^{(r)}(x) \), we need to exclude factors of degrees up
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to 3 which can be handled easily. The observations also imply the following result which improves the bound for $n$ given by Hajir [10] and Nair and Shorey [13].

**Theorem 1.2.** $L_n^{(r)}(x)$ is irreducible and $G_n(r)$ contains $A_n$ if

$$n > e^{r \left(1 + \frac{1.2762}{\log r}\right)}.$$  

We prove Theorem 1.2 in Section 6.

The computations in this paper are carried out with SAGE except for computing a few Galois groups in Section 5 for which MAGMA online is used.

2. Preliminaries

Henceforth, we always use $p$ for a prime and $n$, $r$ for integers with $r \geq 0$, $n \geq 1$ unless otherwise specified.

**Definition 1.** The $p$-adic valuation of an integer $m$ with respect to $p$, denoted by $\nu_p(m)$, is defined as

$$\nu_p(m) = \begin{cases} 
\max\{k : p^k \mid m\} & \text{if } m \neq 0, \\
\infty & \text{if } m = 0.
\end{cases}$$

**Definition 2.** Let $m$ be a positive integer. Let $m = m_0 + m_1 p + \cdots + m_t p^t$ with $m_t \neq 0$ be the $p$-adic representation of $m$. We define $\sigma_p(m) := m_0 + m_1 + \cdots + m_t$.

For integers $m \geq 1$ and $t \geq 0$, we have

$$\nu_p(m!) = \frac{m - \sigma_p(m)}{p - 1},$$

and

$$\nu_p\left(\binom{m}{t}\right) = \frac{\sigma_p(t) + \sigma_p(m-t) - \sigma_p(m)}{p - 1}.$$  

These are well known results of Legendre [12].

**Definition 3.** Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ with $a_o a_n \neq 0$. We consider the set

$$S = \{(0, \nu_p(a_n)), (1, \nu_p(a_{n-1})), \ldots, (n, \nu_p(a_0))\}$$

consisting of points in the extended plane $\mathbb{R}^2 \cup \{\infty\}$. The polygonal path formed by the lower edges along the convex hull of $S$ is called the Newton polygon associated to $f(x)$ with respect to prime $p$ and is denoted by $NP_p(f)$.

It can be observed that the left-most edge has one end-point being $(0, \nu_p(a_n))$ and the right-most edge has $(n, \nu_p(a_0))$ as an end point. The end points of every edge belong to the set $S$. Thus every point in $S$ lies either on or above the line obtained by extending such an edge. In particular, if $(i, \nu_p(a_{n-i}))$ and $(j, \nu_p(a_{n-j}))$ are the two end-points of such an edge, then every point $(u, \nu_p(a_{n-u}))$ with $i < u < j$ lies on or above the line passing through $(i, \nu_p(a_{n-i}))$ and $(j, \nu_p(a_{n-j}))$. Also the slopes of the edges are always increasing when calculated from the left-most edge to the right-most edge.

We need the following result due to Filaseta [4, Lemma 2] which is an application of Newton polygons.
Lemma 2.1. Let $k$ and $l$ be integers with $k > l \geq 0$. Suppose $g(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{Z}[x]$ and $p$ is a prime such that $p \mid b_n$, $p \mid b_j$ for all $j \in \{0, 1, \ldots, n-l-1 \}$ and the right-most edge of the Newton polygon for $g(x)$ with respect to $p$ has slope $< \frac{1}{k}$. Then for any integers $a_0, a_1, \ldots, a_n$ with $|a_0| = |a_n| = 1$, the polynomial $f(x) = \sum_{j=0}^{n} a_j b_j x^j$ cannot have a factor with degree in the interval $[l+1, k]$.

In this paper, we use Lemma 2.1 with $a_0 = a_1 = \cdots = a_n = 1$ always.

Definition 4. Given $f \in \mathbb{Q}[x]$, we define the Newton Index of $f$, denoted by $\mathcal{N}_f$, to be the least common multiple of the denominators (in lowest terms) of all slopes of $NP_p(f)$ as $p$ ranges over all primes.

The following results by Hajir [9, Theorem 2.2] are used for calculating the Galois groups of polynomials.

Lemma 2.2. Given an irreducible polynomial $f \in \mathbb{Q}[x]$, $\mathcal{N}_f$ divides the order of the Galois group of $f$. Moreover, if $\mathcal{N}_f$ has a prime divisor $q$ in the range $\frac{n}{2} < q < n-2$, where $n$ is the degree of $f$, then the Galois group of $f$ contains $A_n$.

As a consequence of Lemma 2.2, Hajir [10, Theorem 5.4] proved the following result.

Lemma 2.3. Let $L_n^{(r)}(x)$ be irreducible.

(i) If there exists a prime $p$ satisfying $\frac{n+2}{3} < p < n - 2$, then $G_n(r)$ contains $A_n$.

(ii) If $n \geq \max \{48 - r, 8 + \frac{5r}{3} \}$, then $G_n(r)$ contains $A_n$.

(iii) If $G_n(r)$ contains $A_n$, then

$$G_n(r) = \begin{cases} A_n & \text{if } \Delta_n^{(r)} \text{ is a square,} \\ S_n & \text{otherwise.} \end{cases}$$

If $L_n^{(r)}(x)$ is reducible, it has one factor with degree $\in [1, \frac{n}{7}]$. Thus from now onwards, whenever we consider a factor of degree $k$ of $L_n^{(r)}(x)$, we mean a factor of degree $k$ with $1 \leq k \leq \frac{n}{7}$.

For fixed integers $r \geq 0$ and $n \geq 1$, we write $n = n_0 n_1$ where

$$n_0 := \prod_{p \mid n, p \mid \ell(n) \#} p^{\nu_p(n)} \quad \text{and} \quad n_1 := \prod_{p \mid \gcd(n, \ell(n) \#)} p^{\nu_p(n)}.$$

The following result is contained in the first line of the proof of Hajir [10, Lemma 4.1]

Lemma 2.4. Every factor of $L_n^{(r)}(x)$ has degree divisible by $n_0$.

Next three results are due to Nair and Shorey [13, Corollary 3.2, Corollary 3.3 and Lemma 2.10].

Lemma 2.5. Assume that $L_n^{(r)}(x)$ has a factor of degree $k \geq 2$. Then $r > 1.63k$.

Lemma 2.6. Assume that $L_n^{(r)}(x)$ has a factor of degree $k \geq 2$. Then

$$r > \min \{104, 3.42k + 1 \}.$$

Lemma 2.7. For $n \leq 127$ and $r \leq 103$, $L_n^{(r)}(x)$ is irreducible.

We also need the following statement used in [13] and we give a proof here.

Lemma 2.8. For \( p | n_1 \), we have \( p^{\nu_p(n)} \leq r \).

Proof. Write \( n = pf^s \), where \( d \) is coprime to \( p \) such that \( p^s > r \). We will show that \( \nu_p\left(\binom{n+r}{r}\right) = 0 \).

Let \( r = r_{e-1}p^{e-1} + \cdots + r_1p + r_0 \) be the \( p \)-adic representation of \( r \). Then \( n + r = dp^e + r_{e-1}p^{e-1} + \cdots + r_1p + r_0 \). So we have \( \sigma_p(n) = \sigma_p(d) \), \( \sigma_p(r) = r_{e-1} + \cdots + r_1 + r_0 \) and \( \sigma_p(n + r) = \sigma_p(d) + r_{e-1} + \cdots + r_1 + r_0 \). Thus \( \nu_p\left(\binom{n+r}{r}\right) = \frac{\sigma_p(n) + \sigma_p(r) - \sigma_p(n + r)}{p - 1} = 0 \). \( \square \)

The following result is due to Harborth and Kemnitz [11].

Lemma 2.9. There exists a prime \( p \) satisfying:

(a) \( x < p < \frac{6}{5}x \) for \( x \geq 25 \),
(b) \( x < p < \frac{11}{10}x \) for \( x \geq 116 \).

For real number \( x > 1 \), we denote

\[ \pi(x) = \sum_{p \leq x} 1. \]

We need the following result due to Dusart [3] for the proof of Theorem 1.2.

Lemma 2.10. We have

\[ \pi(x) \leq \frac{x}{\log x} \left(1 + \frac{1.2762}{\log x}\right) \quad \text{for } x > 1. \]

3. Lemmas for the proof of Theorem 1.1

For the proof of Theorem 1.1, we use a number of results which we record here as lemmas and corollaries. These results are general in nature and valid for any positive integers \( n \) and \( r \).

Lemma 3.1. Let \( p | n_1 \) and \( r < p^2 \). Then

\[ \frac{n}{p} \equiv -j \pmod{p} \text{ for some } j \text{ with } 1 \leq j \leq \left\lfloor \frac{r}{p} \right\rfloor. \]

Proof. Since \( p | n_1 \) and \( r < p^2 \), \( \nu_p(n_1) = 1 \). We can write \( n = pd \), where \( d \) is coprime to \( p \) and \( r = r_1p + r_0 \), where \( 0 \leq r_1, r_0 < p \). Then \( n + r = p(d + r_1) + r_0 \). So we have \( \sigma(n) = \sigma(d) \), \( \sigma(r) = r_1 + r_0 \) and \( \sigma(n + r) = \sigma(d + r_1) + r_0 \). Therefore

\[ 1 \leq \nu_p\left(\binom{n + r}{r}\right) = \frac{\sigma_p(n) + \sigma_p(r) - \sigma_p(n + r)}{p - 1} = \frac{\sigma_p(d) + r_1 - \sigma_p(d + r_1)}{p - 1} = \nu_p\left(\binom{d + r_1}{r_1}\right) = \nu_p\left(\frac{(d + 1)(d + 2)\cdots(d + r_1)}{r_1!}\right) = \nu_p(d + 1)(d + 2)\cdots(d + r_1) \text{ (since } r_1 < p) = \nu_p(d + j) \text{ for exactly one } j \text{ with } 1 \leq j \leq \left\lfloor \frac{r}{p} \right\rfloor. \]

Since \( r_1 = \left\lfloor \frac{r}{p} \right\rfloor < p \), we have \( \frac{n}{p} \equiv -j \pmod{p} \), for some \( 1 \leq j \leq \left\lfloor \frac{r}{p} \right\rfloor \). \( \square \)
Corollary 3.2. If \( p | n_1 \) and \( r < p^2 \), then \( d + \left\lfloor \frac{r}{p} \right\rfloor \geq p \) where \( d \equiv \frac{n}{p} \) (mod \( p \)) with \( 1 \leq d < p \).

For the remaining part of this paper, we need the following notation and remark.

Remark 3.3. For \( 1 \leq j \leq n \), we define \( b_j := \binom{n}{j} (r + 1) \cdots (r + j) \). The Newton polygon for \( \mathcal{L}_n^{(r)}(x) = \sum_{j=0}^{\infty} b_{n-j} x^j \) with respect to \( p \) is given by the lower edges along the convex hull of the points \((j, \nu_p(b_j))\) for \( 1 \leq j \leq n \). Thus the slope of the right-most edge of \( NP_p(\mathcal{L}_n^{(r)}(x)) \) is at most \( M_p = \max_{1 \leq j \leq n} \{ \mu_j \} \) where

\[
\mu_j := \frac{\nu_p(b_n) - \nu_p(b_{n-j})}{j} = \frac{\nu_p((r + n)!)}{j} - \frac{\nu_p((r + n - j)!)}{j} - \frac{\nu_p(\binom{n}{j})}{j}
\]

\[
= \frac{j - \sigma_{p}(r + n) + \sigma_{p}(r + n - j) - \sigma_{p}(j) + \sigma_{p}(n - j) - \sigma_{p}(n)}{(p - 1)j} = \frac{j - \sigma_{p}(j) + \sigma_{p}(n) - \sigma_{p}(r + n) - \sigma_{p}(n - j) + \sigma_{p}(r) - \sigma_{p}(r + n - j)}{(p - 1)j} \]

\[
= \frac{j - \sigma_{p}(j)}{(p - 1)j} + \frac{1}{j} \nu_p\left( \binom{r + n}{r} \right) - \frac{1}{j} \nu_p\left( \binom{r + n - j}{r} \right) \quad (\text{since } \nu_p\left( \binom{r + n - j}{r} \right) \geq 0).
\]

Lemma 3.4. Let \( p = p_{\pi(n)} = n - k_n \) be the largest prime less than or equal to \( n \) with \( r + k_n < p \). Then \( \mathcal{L}_n^{(r)}(x) \) cannot have a factor with degree \( > k_n \).

Proof. Clearly \( p \nmid b_0 \). Since \( p \mid n(n - 1) \cdots (n - k_n) \), \( p \mid \binom{n}{j} \) for \( k_n + 1 \leq j < p \). Also, \( p \mid (r + 1) \cdots (r + j) \) for \( j \geq p \). Thus \( p \mid b_j \) for \( k_n + 1 \leq j \leq n \).

Note that \( r + k_n < p \) implies \( p \nmid (r + 1) \cdots (r + k_n) \) and \( p \nmid n(n - 1) \cdots (n - k_n + 1) \). Thus \( p \nmid (r + 1) \cdots (r + j) \) and \( p \nmid \binom{n}{j} \) for \( 1 \leq j \leq k_n \). Therefore \( p \nmid b_j \) for \( 1 \leq j \leq k_n \).

Next \( r + n = r + k_n + p < 2p \) implies \( \nu_p(b_n) = \nu_p((r + 1) \cdots (r + n)) = 1 \). Hence the vertices of first edge of the Newton polygon are \((0, 0)\) and \((k_n, 0)\) and the slope of the right-most edge is at most

\[
\frac{\nu_p(b_n) - \nu_p(b_{k_n})}{n - k_n} \leq \frac{1}{n - k_n} = \frac{1}{p}.
\]

For \( k_n < j < n \), we have \( p \mid b_j \) implying \( \nu_p(b_j) \geq 1 \). Hence \( \nu_p(b_n) - \nu_p(b_j) \leq 1 - 1 = 0 \) for \( k_n < j < n \). For \( j = k_n \), we have

\[
\frac{\nu_p(b_n) - \nu_p(b_{k_n})}{n - k_n} = \frac{1}{n - k_n} = \frac{1}{p}.
\]

Thus we have

\[
\max_{k_n \leq j \leq n} \left\{ \frac{\nu_p(b_n) - \nu_p(b_j)}{n - j} \right\} \leq \frac{1}{p} < \frac{2}{n}.
\]

since \( p > \frac{n}{2} \). Therefore, by Lemma 2.1, \( \mathcal{L}_n^{(r)}(x) \) cannot have a factor with degree in the interval \([k_n + 1, \frac{n}{2}]\) and the assertion follows. \( \square \)
Lemma 3.5. Let \( l_n \in [1, k_n] \) be the least positive integer such that there exists \( p \) with \( p \mid (n - l_n) \), \( p > k_n \), and \( \nu_p \left( \binom{n+r}{r} \right) = 0 \). Then \( L_n^{(r)}(x) \) cannot have a factor with degree in the interval \([l_n + 1, k_n] \).

Proof. Clearly \( p \nmid b_0 \). Since \( p \mid n(n - 1) \cdots (n - l) \), \( p \mid \binom{n}{j} \), for \( l + 1 \leq j < p \). Also \( p \mid (r + 1) \cdots (r + j) \) for \( j \geq p \). Thus \( p \mid b_j \) for \( l_n + 1 \leq j \leq n \).

From Remark 3.3, the slope of the right-most edge of \( NP_p(L_n^{(r)}(x)) \) is less than equal to \( M_p \leq \max_{1 \leq j \leq n} \left\{ \frac{\nu_p(n(r/z_0) - \nu_p(n))}{z_0 + 1} \right\} \).

Note that \( \frac{j - \sigma_p(j)}{p - 1} j \leq 0 \) if \( j \leq p - 1 \) and \( \frac{j - \sigma_p(j)}{p - 1} \leq \frac{1}{p - 1} \) if \( j \geq p \). Since \( p > k_n \) and \( \nu_p \left( \binom{n+r}{r} \right) = 0 \), we have \( M_p < \frac{1}{k_n} \).

Therefore, by Lemma 2.1, \( L_n^{(r)}(x) \) cannot have a factor with degree in the interval \([l_n + 1, k_n] \). \( \square \)

Lemma 3.6. Let \( i \) be a positive integer such that \( p \mid (n - 1) \cdots (n - i + 1)(r + 1) \cdots (r + i) \) and let \( \nu_p \left( \binom{n+r}{r} \right) = u \). Then \( L_n^{(r)}(x) \) cannot have a factor of degree equal to \( i \) if any one of the following conditions holds:

(a) \( u = 0 \) and \( p > i \),

(b) \( u > 0 \), \( p > 2 \) and \( \max \left\{ \frac{u + 1}{p}, \frac{\nu_p(n(r/z_0 - \nu_p(n))}{z_0 + 1} \right\} < \frac{1}{i} \), where \( z_0 \equiv n + r \text{mod} \ p \) with \( 1 \leq z_0 < p \).

Proof. Clearly \( p \nmid b_0 \). If \( p \mid (r + 1) \cdots (r + i) \), then \( p \mid b_j \) for \( j \geq i \). If \( p \nmid (r + 1) \cdots (r + i) \), then \( p \mid n(n - 1) \cdots (n - i + 1) \) implies \( p \mid \binom{n}{j} \) for \( i \leq j < p \). Also \( p \mid (r + 1) \cdots (r + j) \) for \( j \geq p \). Thus \( p \mid b_j \) for \( i \leq j \leq n \).

From Remark 3.3, the slope of the right-most edge of \( NP_p(L_n^{(r)}(x)) \) is at most \( M_p = \max_{1 \leq j \leq n} \{ \mu_j \} \) where

\[
\mu_j \leq \frac{j - \sigma_p(j)}{p - 1} j + \frac{u}{j}.
\]

(a) \( u = 0 \) and \( p > i \). For \( 1 \leq j \leq n \), we have

\[
\mu_j \leq \frac{j - \sigma_p(j)}{p - 1} j < \frac{1}{p - 1} \leq \frac{1}{i}.
\]

(b) \( u > 0 \) and \( p > 2 \). We have

\[
\mu_j = \frac{\nu_p((r + n)! - \nu_p((r + n - j)!))}{j} - \frac{\nu_p((r + n) \cdots ((r + n - j + 1)) - \nu_p(n)!)}{j}.
\]

For \( 1 \leq j \leq p \), we have

\[
\mu_j = \begin{cases} 
0 & \text{if } j \leq z_0 \\
\frac{\nu_p(n + r - z_0) - \nu_p(n)}{z_0 + 1} & \text{if } j > z_0 
\end{cases}.
\]
For \( p \leq j \leq p^2 \), we have
\[
\mu_j \leq \frac{j - \sigma_p(j)}{(p-1)j} + \frac{u}{j} \leq \frac{1}{p} + \frac{u}{p} = \frac{u + 1}{p}.
\]

For \( j \geq p^2 \), since \( p > 2 \), we have
\[
\mu_j \leq \frac{j - \sigma_p(j)}{(p-1)j} + \frac{u}{j} < \frac{1}{p-1} + \frac{u}{p^2} < \frac{u + 1}{p}.
\]

Thus, by the assumption on (b), for \( 1 \leq j \leq n \),
\[
\mu_j \leq \max \left\{ \frac{u + 1}{p}, \frac{\nu_p(n + r - z_0) - \nu_p(n)}{z_0 + 1} \right\} < \frac{1}{l}.
\]

Hence \( M_p < \frac{1}{i} \) and therefore, by Lemma 2.1, \( L_n^{(r)}(x) \) cannot have a factor of degree \( i \).

The following lemma is more of general nature which will be useful for higher values of \( r \) when \( l_n \), defined in Lemma 3.5, is large. In our proof of Theorem 1.1, \( l_n \leq 3 \) and Lemma 3.6 suffices.

**Lemma 3.7.** Let \( l > 0 \) and let \( p \mid n(r + 1) \) and \( \nu_p\left(\binom{n+r}{r}\right) = u \). Then \( L_n^{(r)}(x) \) cannot have a factor with degree in the interval \([1, l]\) if any one of the following conditions hold:

(a) \( u = 0 \) and \( p > l \),
(b) \( u = 1, p > 2l + 1 \) and \( \mu_j < \frac{1}{l} \) for \( 1 \leq j \leq l \),
(c) \( u > 1, p = l + 1 \) and \( \mu_j < \frac{1}{l} \) for \( 1 \leq j \leq u - \frac{1}{l} \),
(d) \( u > 1, p \neq l + 1 \) and \( \mu_j < \frac{1}{l} \) for \( 1 \leq j \leq ul + \frac{(ul-1)!}{p^{l-1}} \),

where \( \mu_j = \frac{\nu_p((r+n)!)-\nu_p((r+n-j)!)-\nu_p((r)!)}{j} \) (as defined in Remark 3.3).

**Proof.** Clearly \( p \nmid b_0 \). If \( p \nmid (r + 1) \), then \( p \mid b_j \) for all \( 1 \leq j \leq n \). If \( p \mid (r + 1) \), then \( p \mid n \) implies \( p \mid \binom{n}{j} \) for \( 1 \leq j < p \). Also \( p \mid (r + 1) \cdots (r + j) \) for \( j \geq p \). Thus \( p \mid b_j \) for all \( 1 \leq j \leq n \).

From Remark 3.3, the slope of the right-most edge of \( NP_p(L_n^{(r)}(x)) \) is at most \( M_p = \max_{1 \leq j \leq n} \{ \mu_j \} \), where
\[
\mu_j \leq \frac{j - \sigma_p(j)}{(p-1)j} + \frac{u}{j}.
\]

(a) \( u = 0 \) and \( p > l \). For \( 1 \leq j \leq n \), we have
\[
\mu_j \leq \frac{j - \sigma_p(j)}{(p-1)j} < \frac{1}{p-1} \leq \frac{1}{l}.
\]

(b) \( u = 1 \) and \( p > 2l + 1 \). For \( 1 \leq j \leq l \), we have
\[
\mu_j \leq \frac{1}{l}.
\]

For \( l < j < p \), we have
\[
\mu_j \leq \frac{j - \sigma_p(j)}{(p-1)j} + \frac{1}{j} = \frac{1}{l} < \frac{1}{l}.
\]
For \( j \geq p \), we have
\[
\mu_j \leq \frac{j - \sigma_p(j)}{(p-1)j} + \frac{1}{j} < \frac{1}{p-1} + \frac{1}{j} \\
< \frac{1}{2l} + \frac{1}{2l} \quad \text{(since \( p - 1 \geq 2l \) and \( j \geq p > 2l \))} \\
= \frac{1}{l}.
\]

(c) \( u > 1 \) and \( p \neq l + 1 \). For \( 1 \leq j \leq n \), we have
\[
\mu_j \leq \frac{j - \sigma_p(j)}{(p-1)j} + \frac{u}{j} \leq \frac{1}{p-1} - \frac{1}{(p-1)j} + \frac{u}{j} = \frac{1}{p-1} + \frac{u(p-1) - 1}{(p-1)j}.
\]
Thus \( \mu_j < \frac{1}{l} \), if
\[
\frac{u(p-1) - 1}{(p-1)j} < \frac{p - l - 1}{(p-1)l} \quad \text{or} \quad j > ul + \frac{(ul - 1)l}{p - l - 1}.
\]

(d) \( u > 1 \) and \( p = l + 1 \). For \( 1 \leq j \leq n \), we have
\[
\mu_j \leq \frac{j - \sigma_p(j)}{lj} + \frac{u}{j} \leq \frac{1}{l} - \frac{1}{lj} + \frac{u}{j} = \frac{1}{l} + \frac{ul - 1}{lj}.
\]
Thus \( \mu_j < \frac{1}{l} \), if \( \frac{ul - 1}{lj} < 0 \) or \( j > u - \frac{1}{l} \).

Therefore the slope of the right-most edge is less than \( \frac{1}{l} \) and hence, by Lemma 2.1, \( L_n^{(r)}(x) \) cannot have a factor with degree in the interval \([1, l] \). \( \Box \)

We need the following three lemmas for describing the Galois groups of \( L_n^{(r)}(x) \).
The third lemma is computational.

**Lemma 3.8.** Given that \( L_n^{(r)}(x) \) is irreducible, if there is a prime \( p \) with \( \frac{n}{2} < p < n-2 \) and \( r < p \), then \( G_n(r) \) contains \( A_n \).

**Proof.** Let \( n_0 = n - p \) and \( r_0 = p - r \). For \( 1 \leq j \leq n \), we have
\[
\nu_p \left( \binom{n}{j} \right) = \nu_p \left( \frac{n(n-1) \cdots (n-j+1)}{j!} \right) = \begin{cases} 
1 & \text{if} \ n_0 < j < p, \\
0 & \text{otherwise}.
\end{cases}
\]

First assume that \( r + n < 2p \). Note that \( r_0 > n_0 \) and \( r_0 + p = r_0 + n - n_0 > n \).
Thus \( r + r_0 = p \) is the only multiple of \( p \) in the product \( (r+1)(r+2) \cdots (r+n) \). So for \( 1 \leq j \leq n \), we have
\[
\nu_p((r+1)(r+2) \cdots (r+j)) = \begin{cases} 
0 & \text{if} \ j < r_0, \\
1 & \text{otherwise}.
\end{cases}
\]

Therefore \( NP_p(L_n^{(r)}(x)) \) is given by the lower edges along the convex hull of the points:
\[
(0, 0), \ldots, (n_0, 0), (n_0 + 1, 1), \ldots, (r_0 - 1, 1), (r_0, 2), \ldots, (p - 1, 2), (p, 1), \ldots, (n, 1).
\]
Thus the vertices of \( NP_p(L_n^{(r)}(x)) \) are \( (0, 0), (n_0, 0) \) and \( (n, 1) \). Hence \( \frac{1}{p} \) is a slope of \( NP_p(L_n^{(r)}(x)) \) and it follows from Lemma 2.2 that \( G_n(r) \) contains \( A_n \).

Next assume that \( r + n \geq 2p \). Since \( r + n < 3p \), \( r + r_0 = p \) and \( r + r_0 + p = 2p \) are the only multiples of \( p \) in the product \( (r+1)(r+2) \cdots (r+n) \). So for \( 1 \leq j \leq n \), we
have
\[ \nu_p((r+1)(r+2)\cdots(r+j)) = \begin{cases} 
0 & \text{if } j < r_0, \\
1 & \text{if } r_0 \leq j < r_0 + p, \\
2 & \text{if } j \geq r_0 + p.
\end{cases} \]

Therefore in this case \( NP_p(L_n^{(r)}(x)) \) is given by the lower edges along the convex hull of the points:
\[(0,0),\ldots,(r_0-1,0),(r_0,1),\ldots,(r_0+p-1,1),(r_0+p,2),\ldots,(n_0,2),(n_0+1,3),\ldots,\]
\[(p-1,3),(p,2),\ldots,(n,2).\]

Thus the vertices of \( NP_p(L_n^{(r)}(x)) \) are \((0,0),(r_0-1,0),(r_0+p-1,1)\) and \((n,2)\). Hence \( \frac{r}{p} \) is one of the slopes of \( NP_p(L_n^{(r)}(x)) \) and it follows from Lemma 2.2 that \( G_n(r) \) contains \( A_n \).

**Lemma 3.9.** Let \( m \geq 197 \) be an odd integer and let \( k \leq 60 \) be an even integer. Then product of any two distinct terms in the set \( \{m+2,m+4,\ldots,m+k\} \) cannot be a square.

**Proof.** Suppose \((m+2i)(m+2j)\) is a square with \( 1 \leq i < j \leq \frac{k}{2} \). We may assume \( m+2i = ax^2 \) and \( m+2j = ay^2 \) where \( y-x \geq 2 \). Then \( k-2 \geq (j-i) = a(y-x)(y+x) \geq 2a(y+x) \geq 4ax \). Therefore \( x \leq \frac{r-2}{4a} \leq \frac{58}{7} = 14 \) which implies \( m \leq 195 \), a contradiction.

**Lemma 3.10.** There is a prime in every set of 20 consecutive positive integers each \( \leq 1129 \).

4. IRREDUCIBILITY OF \( L_n^{(r)}(x) \): PROOF OF THEOREM 1.1(i)

In this section, we give proof of Theorem 1.1(i) by showing that \( L_n^{(r)}(x) \) is irreducible for each \( 23 \leq r \leq 60 \) and \( n \geq 1 \). Recall that for fixed integers \( r \geq 0 \) and \( n \geq 1 \), \( n = n_0n_1 \) where
\[ n_0 := \prod_{p|n, p^{(n+r)}} p^{\nu_p(n)} \text{ and } n_1 := \prod_{p|gcd(n,(n+r))} p^{\nu_p(n)}. \]

Let \( 23 \leq r \leq 60 \) and \( n \geq 1 \) be integers. Suppose \( L_n^{(r)}(x) \) has a factor of degree \( k \). By Lemma 2.4, we have \( n_0k \). So if \( n_0 \geq 2 \), then \( k \geq 2 \) and thus Lemma 2.6 implies \( r > 3.42k + 1 \), i.e., \( n_0 \leq k < \frac{r-1}{3.42} \). Therefore we have \( 1 \leq n_0 \leq \frac{r-1}{3.42} \) for each value of \( r \).

Fix \( r \) with \( 23 \leq r \leq 60 \). For each \( n_0 \), we have
\[ \{n = n_0n_1 : p^{\nu_p(n_1)} \leq r\} \subseteq \{n : p^{\nu_p(n)} \leq r\}. \]

Since \( \frac{r-1}{3.42} \geq \max\{n_0, \sqrt{r}\} \), if \( p|n \) with \( p > \frac{r-1}{3.42} \), then \( p|n_1 \) and \( r < p^2 \). Thus, by Lemma 2.7, Lemma 2.8 and Corollary 3.2, it is enough to check irreducibility of \( L_n^{(r)}(x) \) for \( n \in H_r \) where
\[ H_r = \{n \in \mathbb{N} : n > 127 \text{ and for each } p|n, p^{\nu_p(n)} \leq r \text{ and if } p > \frac{r-1}{3.42} \text{ then } d + \left\lfloor \frac{r}{p} \right\rfloor \geq p\} \]
where \( d \) denotes the remainder of \( \frac{n}{p} \) modulo \( p \).

For each \( n \in H_r \), we compute \( k_n \) and \( l_n \) (defined respectively in Lemma 3.4 and Lemma 3.5). We find that \( l_n \leq 3 \) for each \( n \in H_r \) and it follows that \( k \leq l_n \leq 3 \). For
1 ≤ i ≤ 3, we define \( H_{i,r} = \{ n \in H_r : l_n ≥ i \} \). To obtain a contradiction, we need to prove non-existence of a factor of degree \( i \) for each \( n \in H_{i,r} \). For this we use Lemma 3.6 and we are left with \((n,r) \in T\) for which \( L_n^{(r)}(x) \) may have a factor of degree 1, where \( T \) is given by

\[
T = \{ (144, 23), (144, 25), (144, 26), (144, 51), (144, 53), (216, 29), (216, 31), (216, 42), (216, 44), (216, 47), (216, 49), (216, 53), (216, 59), (240, 35), (288, 40), (288, 41), (288, 47), (288, 48), (288, 51), (288, 53), (312, 26), (600, 26), (720, 31), (1440, 35), (4320, 55) \}.
\]

Observe that \( p|n \) implies \( p|b_j \) for \( 1 ≤ j ≤ n \) (see the first paragraph in the proof of Lemma 3.7). Since \( 2|n \) and \( 3|n \) for each \( n \) given in \( T \), to remove the existence of a factor of degree 1, by Lemma 2.1 and Remark 3.3, it suffices to show that \( \mu_j < 1 \) for each \( 1 ≤ j ≤ n \), for either \( p = 2 \) or \( p = 3 \), where

\[
\mu_j = \frac{\nu_p((r + n)(r + n - 1) \cdots (r + n - j + 1)) - \nu_p(^{n}_j)}{j} ≤ \frac{j - \sigma_p(j)}{(p-1)j} + \frac{1}{j} \nu_p \left( \binom{n + r}{r} \right).
\]

It can be easily observed that

\[
\frac{j - \sigma_p(j)}{(p-1)j} + \frac{1}{j} \nu_p \left( \binom{n + r}{r} \right) < 1,
\]

if and only if,

\[
(p-1)\nu_p \left( \binom{n + r}{r} \right) < (p-2)j + \sigma_p(j).
\]

For \((n, r) \in T \setminus \{(216, 29), (4320, 55)\} \) and \( p = 3 \), we find the least positive integer \( j_0 \) such that (2) holds for \( j ≥ j_0 \), so that \( \mu_j < 1 \) for \( j ≥ j_0 \). For \( j < j_0 \), we verify that \( \mu_j < 1 \) by using (1). Hence \( L_n^{(r)}(x) \) does not have factor of degree 1.

For \((n, r) \in \{(216, 29), (4320, 55)\} \), we take \( p = 2 \) and proceed as above to verify that \( L_n^{(r)}(x) \) does not have a factor of degree 1. \hfill \Box

5. Galois groups of \( L_n^{(r)}(x) \): Proof of Theorem 1.1(ii)

In this section, we prove Theorem 1.1(ii) by describing the Galois groups of \( L_n^{(r)}(x) \) for \( 23 ≤ r ≤ 60, n ≥ 1 \). From Section 4, we have \( L_n^{(r)}(x) \) is irreducible for each \( 23 ≤ r ≤ 60 \) and \( n ≥ 1 \).
For $23 \leq r \leq 60$, let $B_r$ be given by

$$B_{23} = B_{24} = \cdots = B_{28} = \{1, 2, \ldots, 31\},$$
$$B_{29} = B_{30} = \{1, 2, \ldots, 33\},$$
$$B_{31} = B_{32} = \cdots = B_{36} = \{1, 2, \ldots, 39\},$$
$$B_{37} = B_{38} = \cdots = B_{40} = \{1, 2, \ldots, 43\},$$
$$B_{41} = B_{42} = \{1, 2, \ldots, 45\},$$
$$B_{43} = B_{44} = \cdots = B_{46} = \{1, 2, \ldots, 49\},$$
$$B_{47} = B_{48} = \cdots = B_{52} = \{1, 2, \ldots, 55\},$$
$$B_{53} = B_{54} = \cdots = B_{58} = \{1, 2, \ldots, 61\},$$
$$B_{59} = B_{60} = \{1, 2, \ldots, 63\}.$$

For each $23 \leq r \leq 60$ and $n \in B_r$, we compute $G_n(r)$ using MAGMA online, and in fact, $G_n(r) = A_n$ for $(n, r) \in \{(4, 24), (5, 28), (24, 25), (25, 24), (28, 23), (28, 29), (32, 33), (33, 36), (36, 37), (40, 41), (44, 45), (48, 49), (49, 48), (49, 50), (52, 53), (56, 57)\}$ and $G_n(r) = S_n$ otherwise.

From now onwards, we assume that $n \notin B_r$. We first show that $G_n(r)$ contains $A_n$. Fix $r$ with $23 \leq r \leq 60$. We have $\max\{48 - r, 8 + \frac{5r}{3}\} = 8 + \frac{5r}{3}$. Let $C_r = \{n \in \mathbb{N} : n < 8 + \frac{5r}{3} \text{ and } p \text{ a prime with } \frac{n + r}{2} < p < n - 2\}.$ Observe that $C_r$ is finite and $B_r \subseteq C_r$. By Lemma 2.3 (i) and (ii), we have $G_n(r)$ contains $A_n$ for each $n \notin C_r$. For $n \in C_r$, we now apply Lemma 3.8 to get $G_n(r)$ contains $A_n$ for each $n \in C_r, n \notin B_r$. Hence $G_n(r)$ contains $A_n$ for $n \notin B_r$.

Thus, by Lemma 2.3(iii), we have

$$G_n(r) = \begin{cases} A_n & \text{if } \Delta_n^{(r)} \text{ is a square,} \\ S_n & \text{otherwise.} \end{cases}$$

Therefore to complete the proof of Theorem 1.1(ii), it suffices to check if $\Delta_n^{(r)}$ is a square or not. In fact, we show that for each $23 \leq r \leq 60$ and $n \notin B_r$, $\Delta_n^{(r)}$ is never a square.

For integers $a$ and $b$, we write $a \sim b$ if $a = bc^2$ for some integer $c > 0$. We consider the following cases:

**Case 1.** $n$ is odd: We have

$$\Delta_n^{(r)} \sim (-1)^{(n-1)/2}(1 \cdot 3 \cdot 5 \cdots n)(n + r - 1)(n + r - 3) \cdots (r + 2).$$

If $n \equiv 3 \pmod{4}$, then $\Delta_n^{(r)}$ is not a square. Thus assume $n \equiv 1 \pmod{4}$.

**Subcase 1(a).** $r$ is even: By re-arranging the factors, we see that

$$\Delta_n^{(r)} \sim (1 \cdot 3 \cdot 5 \cdots (r - 1))(r + 1)(r + 2) \cdots n)(n + 1)(n + 3) \cdots (n + r - 1).$$

For $n > \frac{3^{(r-1)}}{2}$, we have

$$\frac{n + r - 1}{2} < \frac{5}{6}n.$$

By Lemma 2.9 with $x = \frac{5}{6}n$, there is a prime $p$ satisfying

$$\frac{n + r - 1}{2} < p < n.$$
so that \( \nu_p(\Delta_n^{(r)}) \) is odd, and hence \( \Delta_n^{(r)} \) is not a square.

For \( n \leq \frac{3(r-1)}{2} \) with \( n \notin B_r \), we check directly that \( \Delta_n^{(r)} \) is not a square.

**Subcase 1(b).** \( r \) is odd: By re-arranging the factors, we see that

\[
\Delta_n^{(r)} \sim (1 \cdot 3 \cdot 5 \cdots r)(n+2)(n+4)\cdots(n+r-1).
\]

If \( n \leq 1070 \), then \( n+r-1 \leq 1129 \) and since there are at least 10 consecutive odd integers in \( \{n+2, n+4, \ldots, n+r-1\} \), it follows from Lemma 3.10 that there is a prime \( p \) in this set. For \( \frac{r-3}{2} \leq n \leq 1070 \), we have

\[
\frac{r-3}{2} \leq n \leq p-2 < p \leq n+r-1 < 3p.
\]

Since \( n+2, n+4, \ldots, n+r-1 \) are all odd, \( 2p \) is not in the set \( \{n+2, n+4, \ldots, n+r-1\} \) and hence we get \( \nu_p(\Delta_n^{(r)}) \) is odd. Therefore \( \Delta_n^{(r)} \) is not a square.

For \( n < \frac{r-3}{2} \) with \( n \notin B_r \), we check directly that \( \Delta_n^{(r)} \) is not a square. Now suppose that \( n > 1070 \) and \( \Delta_n^{(r)} \) is a square.

Let \( r = 23 \). Then

\[
\Delta_n^{(r)} \sim (3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23)(n+2)(n+4)\cdots(n+22).
\]

There are at most 5 terms in \( \{n+2, n+4, \ldots, n+22\} \) which are divisible by 11, 13, 17, 19 or 23. After removing these terms, we are left with at least 6 terms each of which is either a square or 3 times a square. Therefore there are two distinct terms in \( \{n+2, n+4, \ldots, n+22\} \) whose product is a square. This contradicts Lemma 3.9 for \( m = n \) and \( k = r-1 \). Therefore \( \Delta_n^{(r)} \) is not a square.

Similarly, for \( r \in \{25, 33, 35, 51, 53, 55\} \), we get a contradiction using Lemma 3.9 as above.

Let \( r = 27 \). Then

\[
\Delta_n^{(r)} \sim (11 \cdot 13 \cdot 17 \cdot 19 \cdot 23)(n+2)(n+4)\cdots(n+26).
\]

There are at most 4 terms in \( \{n+2, n+4, \ldots, n+26\} \) which are divisible by 13, 17, 19 or 23 and further 11 divides at most 2 terms of this set. After removing these terms, we are left with 7 terms in this set which are squares. This contradicts Lemma 3.9 for \( m = n \) and \( k = r-1 \). Thus \( \Delta_n^{(r)} \) is not a square.

For \( r \in \{29, 31, 39, 41, 43, 45, 47, 49, 57, 59\} \), we proceed as in the case of \( r = 27 \) and get a contradiction using Lemma 3.9.

Let \( r = 37 \). Then

\[
\Delta_n^{(r)} \sim (3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37)(n+2)(n+4)\cdots(n+36).
\]

The number of terms in \( \{n+2, n+4, \ldots, n+36\} \) divisible by 7, 13 and 17 are at most 3, 2 and 2 respectively. Also each of 19, 23, 29, 31 and 37 divides at most one term in this set. After removing these terms, we are left with 6 terms in the set \( \{n+2, n+4, \ldots, n+36\} \) each of which is either a square or of the form \( ax^2 \) with \( a \in \{3, 5, 15\} \) and it follows that there are two distinct terms in \( \{n+2, n+4, \ldots, n+36\} \) whose product is a square. We get a contradiction using Lemma 3.9 as above.

**Case 2.** \( n \) is even: We have

\[
\Delta_n^{(r)} \sim (-1)^{n(n-1)/2}(1 \cdot 3 \cdot 5 \cdots (n-1))(n+r-1)(n+r-3)\cdots(r+1).
\]

If \( n \equiv 2 \pmod{4} \), then \( \Delta_n^{(r)} \) is not a square. Thus assume \( n \equiv 0 \pmod{4} \).
Subcase 2(a). $r$ is odd: By re-arranging the factors, we see that

$$
\Delta_n^{(r)} \sim (1 \cdot 3 \cdot 5 \cdots (r - 2))(r(r + 1) \cdots n)(n + 2)(n + 4) \cdots (n + r - 1).
$$

For $n > \frac{3(r-1)}{2}$, we have

$$
\frac{n + r - 1}{2} < \frac{5}{6} n.
$$

By Lemma 2.9 with $x = \frac{5}{6} n$, there is a prime $p$ satisfying

$$
\frac{n + r - 1}{2} < \frac{5}{6} n < p < n
$$

so that $\nu_p(\Delta_n^{(r)})$ is odd, and hence $\Delta_n^{(r)}$ is not a square.

For $n \leq \frac{3(r-1)}{2}$ with $n \notin B_r$, we check directly that $\Delta_n^{(r)}$ is not a square.

Subcase 2(b). $r$ is even: By re-arranging the factors, we see that

$$
\Delta_n^{(r)} \sim (1 \cdot 3 \cdot 5 \cdots (r - 1))(r(r + 1) \cdots n)(n + 2)(n + 4) \cdots (n + r - 1).
$$

If $n \leq 1070$, then $n + r - 1 \leq 1129$ and since there are at least 10 consecutive odd integers in $\{n + 1, n + 3, \ldots, n + r - 1\}$, it follows from Lemma 3.10 that there is a prime $p$ in this set. For $\frac{r - 2}{2} \leq n \leq 1070$, we have

$$
\frac{r - 2}{2} \leq n \leq p - 1 < p \leq n + r - 1 < 3p.
$$

Since $n + 1, n + 3, \ldots, n + r - 1$ are all odd, we get $\nu_p(\Delta_n^{(r)})$ is odd. Hence $\Delta_n^{(r)}$ is not a square.

For $n < \frac{r - 2}{2}$ with $n \notin B_r$, we check directly that $\Delta_n^{(r)}$ is not a square. Now we suppose that $n > 1070$ and $\Delta_n^{(r)}$ is a square.

Let $r = 24$. Then

$$
\Delta_n^{(r)} \sim (3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23)(n + 1)(n + 3) \cdots (n + 23).
$$

There are at most 4 terms in $\{n + 1, n + 3, \ldots, n + 23\}$ which are divisible by 13, 17, 19 or 23 and further 11 divides at most 2 terms of this set. After removing these terms, we are left with 6 terms each of which is either a square or 3 times a square. Thus there are two distinct terms in $\{n + 1, n + 3, \ldots, n + 23\}$ whose product is a square. This contradicts Lemma 3.9 for $m = n - 1$ and $k = r$. Therefore $\Delta_n^{(r)}$ is not a square.

Similarly, for $r \in \{26, 34, 36, 38, 52, 54, 56\}$, we get a contradiction using Lemma 3.9 as above.

Let $r = 28$. Then

$$
\Delta_n^{(r)} \sim (11 \cdot 13 \cdot 17 \cdot 19 \cdot 23)(n + 1)(n + 3) \cdots (n + 27).
$$

There are at most 3 terms in $\{n + 1, n + 3, \ldots, n + 27\}$ which are divisible by 17, 19 or 23 and further each of 11 and 13 divides at most 2 terms of this set. After removing these terms, we are left with 7 terms in this set which are squares. This contradicts Lemma 3.9 for $m = n - 1$ and $k = r$. Thus $\Delta_n^{(r)}$ is not a square.

For $r \in \{30, 32, 40, 42, 44, 46, 48, 50, 58, 60\}$, we proceed as in the case of $r = 36$ and get a contradiction using Lemma 3.9. \qed
IRREDUCIBILITY AND GALOIS GROUPS OF $L_n^{(-1-n-r)}(x)$

6. Proof of Theorem 1.2

Suppose that $L_n^{(r)}(x)$ has a factor of degree $k$. Then by Lemma 2.5, $k < \frac{r}{1.63}$. By Lemma 2.4, we have $n_0 \leq k < \frac{r}{1.63}$. Thus if $p|n_0$, then $p^{\nu_p(n)} < r$ and in fact $p^{\nu_p(n)} = p^{\nu_p(n_0)} < r$. Also by Lemma 2.8, if $p|n_1$, then $p^{\nu_p(n)} \leq r$. Hence

$$n = n_0n_1 = \prod_{p|n} p^{\nu_p(n)} \leq \prod_{p \leq r} r = r^{\pi(r)} = e^{\pi(r) \log r} \leq e^{(1+\frac{2762}{\log r})}$$

by Lemma 2.10. This proves Theorem 1.2. □

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References


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