Long Term Behavior of a Brownian flow with Jumps

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Abstract

We consider a stochastic jump flow in an interval $(-a, b)$, where $a, b > 0$. Each particle of the flow performs a canonical Brownian motion and jumps to zero when it reaches $-a$ or b. We study the long term behavior of a random measure μ_t which is the image of a finite Borel measure μ_0 under the flow. When a/b is irrational, we show that for almost every driving Brownian path the time averages of the variance of μ_t converge to zero, and the Lebesgue measure of the support of μ_t decreases to zero as time goes to infinity. When a/b is rational, we show that the Lebesgue measure of the support of μ_t decreases to its minimum value in finite time almost surely. In addition, if μ_0 is proportional to Lebesgue measure we show that the number of connected components of the support of μ_t is a recurrent process, which assumes every positive integer value with probability 1.

1 Introduction

Let B_t be the one-dimensional Brownian motion on the canonical probability space $(\Omega, \mathcal{F}_t, P)$ such that $P(B_0 = 0) = 1$. For $x \in (-a, b)$, $a, b > 0$, we consider a right continuous process $\{X_t^x,\, t \geqslant 0\}$. Each sample path of this process is the trajectory of a Brownian particle, which starts at x and jumps to zero each time it reaches the boundary of the interval $(-a, b)$. The particle continues to move in a Brownian fashion between every two consecutive hits. More precisely, let

$$\begin{aligned}
\tau_0^x &= \inf\{t \geq 0 : B_t + x \in \{-a, b\}\}, \\
\tau_n^x &= \inf\{t > \tau_{n-1}^x : B_t - B_{\tau_{n-1}^x} \in \{-a, b\}\},\ n \in \mathbb{N},
\end{aligned}$$

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be the stopping times at which the particle hits the boundary of $(-a, b)$. These stopping times are finite with probability 1. Define

$$X_t^x = \begin{cases} B_t + x & \text{for } 0 \leqslant t < \tau_0^x, \\ B_t - B_{\tau_n^x} & \text{for } \tau_n^x \leqslant t < \tau_{n+1}^x,\ n \in \mathbb{N}. \end{cases} \tag{1}$$

This model was suggested by E. Cinlar at the Seminar on Stochastic Processes 2000. It was motivated by an optimal control problem in economics, for which the random variable X_t^x represents the value of an asset held by an agent. If the asset value reaches level b, the agent sells it and buys a cheaper asset for c dollars. If the asset goes below level $-a$, the agent sells it and buys a more expensive one for c dollars, where $-a < c < b$. In our description we set $c = 0$. Further motivation, the formal construction, and main properties of the single particle process $\{X_t^x\}$ can be found in [1].

Without loss of generality we assume throughout the paper that

$$0 < a < b \text{ and } a + b = 1. \tag{2}$$

Let μ_0 be a Borel probability measure on $(-a, b)$. We are interested in the long time behavior of the random measure μ_t defined by the relation

$$\mu_t(A) \equiv \mu_0\{x \in (-a, b) : X_t^x \in A\}, \tag{3}$$

for every Borel subset A of $(-a, b)$. In the case when μ_0 is the Lebesgue measure on $(-a, b)$ we are able to provide detailed information about the structure of the set

$$X(t) = \operatorname{supp} \mu_t. \tag{4}$$

Let us note that, since X_t^x is a càdlàg process, which almost surely has only finitely many jumps on a finite time interval, the measure μ_t is well defined.

Our main results are stated in Section 1.2. We shall begin with some preliminary observations, examples and definitions.

Notation. For any set $A \subset (-a, b)$, we let $\sup A$ denote the supremum of the set A. We write δ_x for the Dirac measure of mass 1 at x and λ for the Lebesgue measure on $(-a, b)$. The set of all non-negative integers is denoted by $\mathbb{Z}_+$.

1.1 Preliminaries

Invariant measure for the process $\{X_t^x\}$. Let the measure μ on $(-a,b)$ be defined by the density (with respect to the Lebesgue measure)

$$g(y) = \begin{cases} \dfrac{2}{a+b}\left(1+\dfrac{y}{a}\right), & \text{if } -a < y \leqslant 0; \\ \dfrac{2}{a+b}\left(1-\dfrac{y}{b}\right), & \text{if } \ 0 < y < b. \end{cases} \tag{5}$$

One can check that μ is an invariant measure for a single particle process X_t^x. It was shown in [1] that the transition probability densities of the process X_t^x converge to $g(y)$ exponentially fast as $t \to \infty$.

Collisions. Let $\mu_0 = m_1\delta_{x_1} + m_2\delta_{x_2} + m_3\delta_{x_3}$, where $m_1 + m_2 + m_3 = 1$, $-a < x_1 < 0$, $x_2 = x_1 + a$, $x_3 = x_1 + b$. Studying μ_t in this case is equivalent to looking at the trajectories of three particles of masses m_1, m_2, m_3 started at points x_1, x_2, x_3 respectively and driven by the same Brownian path. Let τ_0 be the first time when one of the three particles hits the set $\{-a, b\}$. Clearly, $\tau_0 = \tau_0^{x_1}$ or $\tau = \tau_0^{x_3}$. In the former case the first and the second particles will collide at time τ_0 and $X_t^{x_1} = X_t^{x_2}$ for all $t \geqslant \tau_0$, while in the latter case the first and the third particles will collide and $X_t^{x_1} = X_t^{x_3}$ for all $t \geqslant \tau_0$. It can be shown (see also Lemma 1.1 for a more general result) that in this case with probability 1 all three particles collide in finite time. Therefore, for all t greater than the collision time we have that $\mu_t = \delta_{X_t}$, where $X_t = X_t^{x_i}$, $i = 1, 2, 3$. These collisions are the key to understanding the long term behavior of μ_t.

The above example motivates the following definition. For each $x \in (-a, b)$ let

$$C_x = \{z \in (-a,b) \,|\, z = x + ka + lb,\ k, l \in \mathbb{Z}\}. \tag{6}$$

We write $x \sim y$ if $C_x = C_y$. This defines an equivalence relation on $(-a, b)$, which gives us a continuum of equivalence classes. If a/b is rational then each class C_x consists of finitely many points. If a/b is irrational then each class C_x is a dense countable subset of $(-a, b)$.

Lemma 1.1. *Let $x \sim y$ and $\tau_{\mathrm{col}} = \inf\{t \geqslant 0 \,|\, X_t^x = X_t^y\}$. Then*

$$P(\tau_{\mathrm{col}} < \infty) = 1.$$

A proof of this lemma is given at the end of Section 2 (see also [1] and [2]).

Structure of the set X(t). The next lemma presents a simple fact about the structure of $X(t)$ in a special case.

Lemma 1.2. *Suppose that for some $s \geqslant 0$ the set $X(s)$ is a finite collection of disjoint intervals of positive length. Then with probability 1 for each $t > s$ the set $X(t)$ is a finite collection of disjoint intervals of positive length.*

Proof. Since $X(t)$ is the support of a measure, it is closed in $(-a, b)$. Let $M(t) = \sup X(t)$ and $m(t) = \inf X(t)$. The number of intervals in $X(t)$ can increase only when either $M(t)$ or $m(t)$ hits the boundary (these two events can not happen simultaneously except for $s = 0$). Since the probability that a Brownian path will cross a given level exactly once is equal to zero (see [4], p.105), an interval of positive length will detach and jump to zero once $M(t)$ or $m(t)$ reaches the boundary. This implies that $X(t)$ is a collection of disjoint intervals of positive length with probability 1.

Denote by $\Sigma(t)$ the number of disjoint intervals in $X(t)$. If $M(s) < b$ then $\Sigma(t)$ can not be larger than $(\Sigma(s) + 1)$ plus the total number of up-crossings and down-crossings of an interval $[M(s), 1]$ by the Brownian path $M_s + B_{h-s}$, $s \leqslant h \leqslant t$. This number is finite with probability 1. Similar argument applies to the case when $M(s) = b$ and $m(s) > -a$. The situation when $M(s) = b$ and $m(s) = -a$ can arise only for $s = 0$. Indeed, the function $M(t) - m(t)$ is non-increasing and is strictly less than 1 for all $t > 0$ with probability 1. Assume that $M(0) = b$ and $m(0) = -a$. Let $X_+(0) = X(0) \cap [0, b)$ and $X_-(0) = X(0) \cap (-a, 0)$. Then $\inf X_+(0) = 0 < b$ and $\sup X_-(0) = 0 > -a$. Since $\Sigma(t)$ is less or equal than the sum of the numbers of disjoint intervals obtained starting from $X_+(0)$ and $X_-(0)$ separately, the proof is now complete. $\square$

Remark. Case when $a = b$ is trivial. Suppose that μ_0 is the Lebesgue measure. For all $t \geqslant 0$ the set $X(t)$ is an interval. It is easy to see that there exists a finite stopping time after which the set $X(t)$ is an interval of length $a = b$. After this time the density of μ_t with respect to the Lebesgue measure is equal to 2 on $X(t)$. Other initial measures μ_0 could also be considered.

1.2 Main results

The structure of C_x for $x \in (-a, b)$ suggests that the long term behavior of μ_t and $X(t)$ might depend on whether a/b is irrational or rational. This is indeed the case.

Denote the expectation and the variance of μ_t by

$$\bar{X}_t(\omega) \equiv \int_{-a}^{b} x\, \mu_t(dx) = \int_{-a}^{b} X_t^x\, \mu_0(dx) \tag{7}$$

$$\sigma^2(t, \omega) \equiv \int_{-a}^{b} (x - \bar{X}_t)^2\, \mu_t(dx) = \int_{-a}^{b} (X_t^x - \bar{X}_t)^2\, \mu_0(dx). \tag{8}$$

Theorem 1.1. *Assume that a/b is irrational.*

(i) For every $x \in (-a, b)$ and $f \in C([-a, b])$

$$\frac{1}{t} \int_0^t (f(X_s^x) - f(\bar{X}_s))^2\, ds \to 0 \quad P\text{-a.s. as } t \to \infty. \tag{9}$$

(ii) The Lebesgue measure of the set $X(t)$ converges to zero P-a.s as $t \to \infty$.

If we set $f(y) = y$ in part (i) and integrate (9) over $(-a, b)$ with respect to μ_0, then we obtain the time averages of the variance $\sigma^2(t)$ of the random measure μ_t. Theorem 1.1 says that these averages converge to zero. This implies that the proportion of time, when the most part of μ_t is concentrated on a very small interval, approaches 1 as $t \to \infty$. Roughly, this means that for very large t most of the time an observer would see one heavy "lump" about $\bar{X}(t)$, which moves like a single particle plus some negligible "dust" on the rest of the interval. In Section 3 we give a simple example, which shows that without the averaging in time the statement of part (i) does not hold in general .

The proof of Theorem 1.1 is based on the consideration of the distance between two particles driven by the same Brownian path at the stopping times when one of them hits the boundary of the interval $(-a, b)$. We obtain an embedded Markov chain on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ and observe that it is generated by two automorphisms of S^1. We describe the invariant measures of this Markov chain and study its recurrence properties. The statements of Theorem 1.1 are immediate consequences of the results obtained for the embedded Markov chain.

Theorem 1.2. *Assume that $a/b = m/n$, where m and n are coprime numbers. Set $\gamma = a/m = b/n = 1/(m + n)$.*

(i) Let $\alpha = \lambda\{x \in [0, \gamma) \,|\, C_x \cap \operatorname{supp} \mu_0 \neq \emptyset\}$, and $\zeta = \inf\{t \geqslant 0 \,|\, \lambda\{\operatorname{supp} \mu_t\} = \alpha\}$. Then $P(\zeta < \infty) = 1$.

(ii) Let $\mu_0 = \lambda$ and $\Sigma(t, \omega)$ be the number of connected components of the set $X(t)$. Then with probability 1 the process $\Sigma(t, \omega)$ assumes every value in $\mathbb{N}$ and, moreover, is a recurrent process.

Figure 1 illustrates a possible behavior of $X(t)$ when $\mu_0 = \lambda$.

If $\mu_0 = \lambda$ then part (i) of Theorem 1.2 says that the Lebesgue measure of $X(t)$ will reach γ in finite time ζ. In other words, all equivalent points will collide by that time. If μ_0 is an arbitrary probability measure then some C_x may not contain points of $X(0)$. Therefore at time ζ the Lebesgue measure of $X(t)$ attains its minimum value α, which is equal to the Lebesgue measure of the set of initially "occupied" distinct equivalence classes. From Lemma 1.2 we know that when $\mu_0 = \lambda$ the set $X(t)$ consists of a finite number of disjoint intervals. The second part of Theorem 1.2 claims that the number of these intervals is a recurrent process, which takes every positive value with probability 1.

The proof of Theorem 1.2 is accomplished by an explicit path construction. At first, we present the main ideas by considering the discrete analog of our process driven by the standard

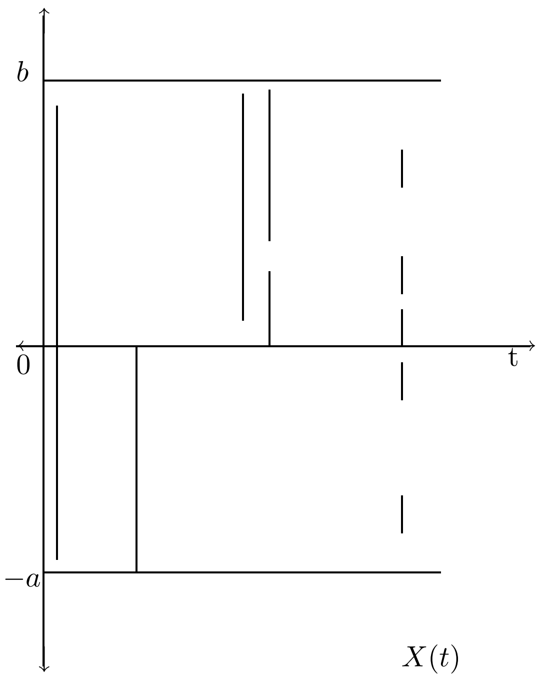

Figure 1: A realization of the set $X(t)$ when $\mu_0 = \lambda$.

random walk. Then we explain how to modify the discrete procedure to make it suitable for the continuous case and obtain the statement of part (i). Part (ii) is proved via a similar explicit path construction. Main tools that we use in the proof are: a number theoretic Lemma A.1, the support theorem (see [5], Theorem 6.6 p. 60) and the second Borel-Cantelli lemma.

1.3 Layout of the paper

The rest of the paper is organized as follows. In Section 2 we describe and study an embedded Markov chain on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. We classify all its invariant measures when b is irrational (Lemma 2.2). Using this key result in Section 3 we prove Theorem 1.1. Section 4 starts with the study of the discrete model driven by the standard random walk. The construction presented there is then adapted to prove part (i) of Theorem 1.2. The proof of part (ii) of Theorem 1.2 is given at the end of Section 4. The technical results that we use in Sections 2 and 4 are proven in the Appendix.

2 Embedded Markov chain

In this section we construct a Markov chain $\{Z_i\}$, $i = 0, 1, \ldots$ on the circle $S^1 = \mathbb{R}/\mathbb{Z}$, whose properties will be used in the proof of Theorem 1.1. The idea is to consider the trajectories X_t^x and X_t^y, $x, y \in (-a, b)$, of two particles driven by the same Brownian path and record the changes in the distance between these particles for all $t \geqslant 0$. The distance between the two particles can change only when one of the them hits the boundary $\{-a, b\}$. Since the waiting time of the first hit to the boundary is finite with probability 1, we shall assume without loss of generality that $y = 0$. Let $T_0 = 0$,

$$T_i = \inf\{t > T_{i-1} \,:\, X^x_{t-} \text{ or } X^0_{t-} \in \{-a, b\}\},\ i \in \mathbb{N}, \tag{10}$$

be the hitting times, and

$$\eta_i = T_{i+1} - T_i, \quad i = 0, 1, \dots, \tag{11}$$

be the waiting times between two consecutive hits. Denote by $N(t)$ the number of hits of the boundary up to time t. That is

$$N(t) = \sum_{i=1}^{\infty} 1_{\{T_i \leqslant t\}}. \tag{12}$$

The dependence of T_i, η_i, and $N(t)$ on x will not be reflected in the notation. The following lemma is an easy consequence of standard results about renewal processes, Brownian motion (see [3], Theorem 4.1 p. 204 and Theorem 5.9 p. 401), and convergence of random series, and we omit the proof.

Lemma 2.1. *Let T_i, η_i, $i = 0, 1, \dots$, and $N(t)$ be defined by (10), (11), and (12) respectively. Then*

(i) $P(N(t) < \infty$ for all $t \geqslant 0$ and $N(t) \to \infty$ as $t \to \infty) = 1$;

(ii) With probability 1 the ratio $N(t)/t$ is bounded uniformly in t on the interval $t > 0$;

(iii) Random variables η_i, $i = 0, 1, \dots$, are independent and there is a constant $K > 0$ such that $\sum_{i=0}^{n} \eta_i^2/n \leqslant K$ uniformly in n with probability 1.

We will use the above lemma in the next subsection. Notice that either $X^x_{T_i} = 0$ or $X^0_{T_i} = 0$. Let $Z^x_i = X^x_{T_i} + X^0_{T_i}$, $i = 0, 1, \dots$. In other words, Z^x_i is the position at time T_i of the particle, which is not at zero, unless they both are at zero, in which case $Z^x_i = 0$. The sequence $\{Z^x_i\}$, $i = 0, 1, \dots$, forms a Markov chain, which starts at x and takes values in the set

$$S_x = \{w \in S^1 \,|\, w = \pm x + nb \,(\text{mod } 1) \text{ for some integer } n\}. \tag{13}$$

Instead of looking just at one process $\{Z^x_i\}$, $i = 0, 1, \dots$, we shall look at the collection of such Markov chains $\{Z_i\}$ for all $x \in (-a, b)$. Moreover, we shall identify the ends of the interval $(-a, b)$ and consider a Markov chain on the circle $S^1 = \mathbb{R}/\mathbb{Z}$.

Before we write down the transition probabilities for the above Markov chain we would like to take a closer look at each step. Consider the stopping time T_i. Assume that $Z^x_i \neq 0$, which means that there was no collision before or at T_i. Then there are two possibilities for the next step: either the particle, which currently occupies 0, hits the boundary prior to the other particle or vice versa. The first case corresponds to the transition $x \to x + b \,(\mathrm{mod}\ 1)$, and the second case corresponds to the transition $x \to b - x \,(\mathrm{mod}\ 1)$. Recall that $a = 1 - b$. If the particles collided before or at time T_i then the process reached its absorbing state 0, and its further dynamics are the same as the dynamics for the single particle process.

Taking into account the preceding considerations, define two automorphisms R and Q of the circle S^1 by

$$Rx = x + b \,(\mathrm{mod}\ 1), \;\; Qx = b - x \,(\mathrm{mod}\ 1). \tag{14}$$

These maps have the following properties:

P1. Both R and Q preserve the Lebesgue measure on S^1.

P2. $RQRQ = QRQR = Q^2 = \mathrm{Id}$, where Id is the identity map on S^1.

P3. If b is irrational, then R is uniquely ergodic.

Properties P1 and P2 are straightforward, and P3 is a standard fact from ergodic theory (see [6], p. 153).

The transition probabilities for our Markov chain $\{Z_i\}$, $i = 0, 1, \ldots$, are given by

$$\begin{aligned} &P(Z_1 = 0 \,|\, Z_0 = 0) = 1; \\ &P(Z_1 = Rx \,|\, Z_0 = x) = p(x) \text{ for } x \neq 0; \\ &P(Z_1 = Qx \,|\, Z_0 = x) = q(x) = 1 - p(x) \text{ for } x \neq 0; \end{aligned} \tag{15}$$

where

$$p(x) = \begin{cases} (b - x)/(1 - x) & \text{if } 0 < x \leqslant b; \\ (x - b)/x & \text{if } b < x < 1. \end{cases} \tag{16}$$

Notice that 0 is the only absorbing state. The state space S^1 splits into a continuum of closed (in the terminology of Markov chains) sets S_x (see (13)). Main properties of this chain are summarized in the next two lemmas.

Lemma 2.2. *Assume that b is irrational. Let $\{Z_i\}$, $i = 0, 1, \ldots$, be a Markov chain with the state space S^1 and the transition probabilities (15). Then*

(i) The distribution δ_0 is the only invariant probability distribution for this Markov chain. Moreover, every invariant measure μ_{inv} of this Markov chain is given by

$$\mu_{\mathrm{inv}} = C\mu_c + D\delta_0,$$

where C and D are non-negative constants, and μ_c is defined by the following non-integrable density with respect to the Lebesgue measure on S^1:

$$\rho_c(x) = \begin{cases} (x(b-x))^{-1} & \text{if } 0 < x < b. \\ ((1-x)(x-b))^{-1} & \text{if } b < x < 1. \end{cases}$$

(ii) Let ν_0 be an arbitrary probability distribution on S^1 and ν_i, $i \in \mathbb{N}$, be defined by $\nu_i(A) = \nu_0(x \in S^1 : Z_i^x \in A)$ for every Borel subset A of S^1. Then

$$\frac{1}{n}\sum_{i=0}^{n-1} \nu_i \to \delta_0 \quad \text{as } n \to \infty.$$

Proof. (i) For convenience we slightly modify our transition probabilities by setting

$$\begin{aligned} &P(Z_1 = Rx \,|\, Z_0 = x) = p(x) \text{ for all } x \in S^1; \\ &P(Z_1 = Qx \,|\, Z_0 = x) = q(x) = 1 - p(x) \text{ for all } x \in S^1. \end{aligned} \tag{17}$$

The only change from the original set up is that instead of the absorbing state at 0 we have a closed periodic set $\{0, b\}$ so that not only b always leads to 0 but also 0 always leads to b.

Let π be an invariant measure of our modified Markov chain. Then π has to satisfy the equation

$$\pi = R(p\pi) + Q(q\pi), \tag{18}$$

where p is given by (16) and $q = 1 - p$. Applying Q to both sides of (18) and then adding the two equations we obtain the equation $p\pi - QR(p\pi) = R(p\pi - QR(p\pi))$. From the unique ergodicity of R we conclude that

$$p\pi - QR(p\pi) = c_1\lambda, \tag{19}$$

where λ is the Lebesgue measure on S^1 and c_1 is a constant. Similarly, applying RQ to both sides of (18), using P2, and then adding the two equations we obtain $\pi - Q(\pi) = R(\pi - Q(\pi))$, which implies that

$$\pi - Q(\pi) = c_2\lambda \tag{20}$$

for some constant c_2. Since R and Q preserve λ, from (19) and (20) we obtain

$$\begin{aligned} c_1\lambda &= QR(p\pi - QR(p\pi)) = QR(p\pi) - p\pi = -c_1\lambda; \\ c_2\lambda &= Q(\pi - Q(\pi)) = Q(\pi) - \pi = -c_2\lambda. \end{aligned}$$

Thus, $c_1 = c_2 = 0$ and the invariant measure π satisfies the equations

$$\pi = Q\pi; \tag{21}$$

$$R(p\pi) = Q(p\pi). \tag{22}$$

It is easy to check that the function $h(x) = |b-x|(1-|b-x|)$ defined on S^1 solves the equation

$$h(Rx)p(x) = h(x)p(Qx). \tag{23}$$

We can use (22), (23), and (21) to conclude that

$$\begin{aligned} h(x)R(p\pi(dx)) =& h(x)Q(p\pi(dx)) = h(x)p(Q^{-1}x)Q\pi(dx) = \\ & h(x)p(Qx)Q\pi(dx) = h(Rx)p(x)\pi(dx). \end{aligned}$$

Therefore,

$$h(Rx)p(x)\pi(dx) = h(x)R(p\pi(dx)) = R(h(Rx)p(x)\pi(dx)),$$

which by the property P3 implies that

$$h(Rx)p(x)\pi(dx) = C\lambda(dx) \tag{24}$$

for some non-negative constant C. Notice that

$$h(Rx)p(x) = \begin{cases} x(b-x) & \text{if } 0 \leqslant x \leqslant b; \\ (1-x)(x-b) & \text{if } b < x < 1. \end{cases}$$

From (24) we see that $\pi(dx)$ can have at most two components: the absolutely continuous (with respect to the Lebesgue measure) component $C\rho_c(x)dx$ and the discrete component supported on the set $\{0, b\}$. The latter one, obviously, has to be of the form $D(\delta_0 + \delta_b)$ for some non-negative constant D. Thus, we obtained all invariant measures for the modified chain. Notice that we only modified the chain at 0. This implies the statements of part (i).

(ii) The space of probability distributions on $(S^1, \mathfrak{B})$, where $\mathfrak{B}$ is the Borel σ-algebra on S^1, is weakly compact. Therefore the sequence $\sum_{i=0}^{n-1} \nu_i/n$ has a weakly convergent subsequence. But the limit point of such a subsequence has to be an invariant distribution for our Markov chain. By part (i) the distribution δ_0 is the only invariant distribution, therefore the sequence $\sum_{i=0}^{n-1} \nu_i/n$ converges weakly to δ_0. □

Corollary 2.1. *If b is irrational then for every $x \in S^1$ and $f \in C(S^1)$*

$$\frac{1}{n}\sum_{i=0}^{n-1} f(Z_i^x) \to f(0) \quad a.s. \text{ as } n \to \infty.$$

Proof. Set $\nu_0 = \delta_x$ in part (ii) of Lemma 2.2. □

Lemma 2.3. *For every $x \in S_0$ the absorption probability is equal to 1. Every state $x \in S_z$, $z \neq 0$, is positive recurrent if b is rational and null recurrent if b is irrational.*

Proof. Let b be rational. Then the set S_0 is a finite closed set. Moreover, the probability to reach 0 from any state $x \in S_0$ is positive. This implies that every $x \in S_0$ reaches 0 in finite time with probability 1. Every set S_z, $z \neq 0$, is a finite irreducible closed set. Therefore each state in S_z, $z \neq 0$, is positive recurrent.

Assume now that b is irrational and $x \in S_0$, that is $x = nb \pmod 1$ for some integer n. Since 0 is the absorbing state and b leads to 0 with probability 1, we can assume that $x \notin \{0, b\}$. Re-enumerate the states of the chain by setting $(k+1)b \mapsto -k$ and $-kb \mapsto k$ for $k \in \mathbb{N}$. Then we obtain a Markov chain with the state space $\mathbb{Z}$, and 0 will be the absorbing state. In Appendix A we prove a lemma (Lemma A.2) about the absorption probabilities for a class of Markov chains, which includes our chain. According to this lemma, to prove that absorption probabilities are equal to 1 it is enough to show the divergence of the series $\sum_{k=1}^{\infty} \beta_k q_{-k}$, where $q_{-k} = 1 - p((k+1)b)$ and

$$\beta_k = \prod_{j=1}^{k} \frac{p(-jb)}{p((j+1)b)}.$$

In our case q_{-k}, $k \in \mathbb{N}$, are bounded below by $(1-b)$. We shall show that there is a positive ε such that $\beta_k > \varepsilon$ for infinitely many values of k. This will finish the proof of the first statement of the lemma. By (23)

$$\beta_k = \prod_{j=1}^{k} \frac{p(-jb)}{p(R(-jb))} = \prod_{j=1}^{k} \frac{h(-jb)}{h(T(-jb))} = \frac{h(-kb)}{h(0)}$$

Since b is irrational, for any $\varepsilon < 1/(4h(0))$ there are infinitely many values of k, for which β_k exceeds ε.

Each set S_z, $z \neq 0$, is an irreducible closed set. The recurrence of each state $x \in S_z$ for $z \neq 0$ can be shown exactly the same way as we proved the absorption. The null recurrence follows from the uniqueness of the invariant distribution for our Markov chain (see part (i) of Lemma 2.2). □

Proof of Lemma 1.1. Consider the first time when one of the trajectories hits the boundary. This stopping time is finite with probability 1. At that time we are in the setting of Section 2. The relation $x \sim y$ implies that we start our Markov chain Z from some state in S_0. By Lemma 2.3 the absorption probability is equal to 1. Also the waiting times between two consecutive hits are almost surely finite. The last two assertions imply the statement of the lemma. □

3 Irrational Case

Proposition 3.1. *Let a/b be irrational. Then for each $x, y \in (-a, b)$*

$$\frac{1}{t}\int_0^t (X_s^x - X_s^y)^2\,ds \to 0 \quad P\text{-a.s. as } t \to \infty.$$

Proof. If $x \sim y$ then by Lemma 1.1 there is nothing to prove. Without loss of generality we may assume that $x \not\sim y$, $x \in (-a, b)$ and $y = 0$. The proof is based on Lemma 2.1 and Corollary 2.1 from Section 2.

Using the notation of Section 2 we obtain

$$\begin{aligned}\frac{1}{t}\int_0^t (X_s^x - X_s^y)^2\,ds &\leqslant \frac{1}{t}\sum_{i=0}^{N(t)} |Z_i^x|^2\eta_i \\ &\leqslant \frac{N(t)}{t}\left(\frac{1}{N(t)}\sum_{i=0}^{N(t)}\eta_i^2\right)^{1/2}\left(\frac{1}{N(t)}\sum_{i=0}^{N(t)}|Z_i^x|^4\right)^{1/2}. \end{aligned} \tag{25}$$

By Lemma 2.1 we have that on a set of full measure the first two terms in the right-hand side of (25) are bounded uniformly in t for all $t > 0$. The last term in the right-hand side of (25) goes to zero almost surely by Corollary 2.1. □

Suppose that we start our measure-valued process $\mu_t(dx, \omega)$ from some probability distribution $\mu_0(dx)$. Recall that we denoted the expectation and the variance of μ_t by $\bar{X}_t$ and $\sigma^2(t)$ respectively (see (7) and (8)).

The next result is the key to the long term behavior of μ_t. It follows from Proposition 3.1 by the integration over $(-a, b) \times (-a, b)$ with respect to the product measure $\mu_0(dx) \times \mu_0(dy)$ and by the Cauchy-Schwartz inequality.

Corollary 3.1. *Let a/b be irrational. Then*

$$\frac{1}{t}\int_0^t \sigma^2(s)\,ds \to 0 \quad P\text{-a.s. as } t \to \infty.$$

Remark. Without the time averaging the statements of Proposition 3.1 and Corollary 3.1 are false in general. This can be easily seen from the following example. Let $\mu_0 = (\delta_x + \delta_0)/2$, where $x \not\sim 0$. Consider two trajectories X_t^x and X_t^0 and the embedded Markov chain discussed in Section 2. We start our chain from $x \notin S_0$. By Lemma 2.3 every $x \not\sim 0$ is null recurrent. Therefore, the stopping time T when the process returns to the state, where one of the trajectories is at 0 and the other one is at x, is finite with probability 1. Then $|X_T^x - X_T^0| = x$ and $\sigma^2(T) = x^2/4$. This implies that $P(\sigma^2(t) \to 0 \text{ as } t \to \infty) = 0$.

Proof of Theorem 1.1 (i) Choose an arbitrary $\varepsilon > 0$. Let $M = \max_{[-a,b]} |f|$. Since f is continuous on $[-a,b]$, there exists a $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$ and $x_1, x_2 \in [-a,b]$. We have

$$\begin{aligned}
\frac{1}{t}\int_0^t (f(X_s^x) - f(\bar{X}_s))^2\,ds &= \frac{1}{t}\int_0^t (f(X_s^x) - f(\bar{X}_s))^2 1_{\{s:|X_s^x - \bar{X}_s|<\delta\}}\,ds \\
&+ \frac{1}{t}\int_0^t (f(X_s^x) - f(\bar{X}_s))^2 1_{\{s:|X_s^x - \bar{X}_s|\geqslant\delta\}}\,ds \\
&\leqslant \varepsilon^2 + \frac{4M^2}{t\delta^2}\int_0^t |X_s^x - \bar{X}_s|^2\,ds \\
&\leqslant \varepsilon^2 + \frac{4M^2}{t\delta^2}\int_0^t \int_{-a}^b (X_s^x - X_s^y)^2\,\mu_0(dy)\,ds,
\end{aligned}$$

where we used the uniform continuity of f on $[-a,b]$, Chebyshev's inequality, and the Cauchy-Schwartz inequality. From Proposition 3.1 we immediately obtain the statement of part (i).

(ii) It is enough to prove the result in the case when $\mu_0 = \lambda$, where λ is the Lebesgue measure on $(-a,b)$. At first, notice that the Lebesgue measure of $\operatorname{supp}\mu_t$ is a non-increasing function of t. Therefore, it has a (possibly random) limit as $t \to \infty$, which we denote by ξ. Since by Lemma 1.2 the set $X(t)$ is a finite collection of intervals, we observe that $\mu_t(dx) = g_t(x,\omega)\,dx$, where g_t takes only non-negative integer values. For an arbitrary $\varepsilon > 0$ we obtain

$$\begin{aligned}
\sigma^2(t) = \int_{-a}^b (x - \bar{X}_t)^2 g_t(x)\,dx &\geqslant \int_{(-a,b)\cap\{g_t\geqslant 1\}} (x - \bar{X}_t)^2\,dx \\
&\geqslant \varepsilon^2\lambda\{x \in (-a,b) \,:\, g_t(x) \geqslant 1,\ |x - \bar{X}_t| > \varepsilon\} \geqslant \varepsilon^2(\xi - 2\varepsilon).
\end{aligned}$$

By Corollary 3.1

$$\varepsilon^2(\xi - 2\varepsilon) \leqslant \frac{1}{t}\int_0^t \sigma^2(s)\,ds \to 0, \quad \text{as} \quad t \to \infty \quad \text{P-a.s.},$$

which implies that $\xi = 0$ with probability 1. □

4 Rational Case

Throughout this section we will assume that a/b is rational. Before we turn to the proof of Theorem 1.2, we shall discuss a simple discrete analog of our model driven by the standard random walk. This discussion will illustrate the idea of the proof of Theorem 1.2 in a very simple setting. The main point is that a similar construction can be carried out for the continuous case as well.

4.1 Discrete model

Let m and n be coprime numbers, $0 < m < n$, and $\{W_i,\ i \in \mathbb{Z}_+\}$ be the standard symmetric random walk with $W_0 = 0$. We define a discrete time process $\{Y_i^x,\ i \in \mathbb{Z}_+\}$ on the finite set $K = \{-(m-1), -(m-2), \dots, 0, \dots, n-1\}$. For $x \in K$ let

$$\zeta_0^x = \min\{i \geq 0 : W_i + x \in \{-m, n\}\},$$
$$\zeta_j^x = \min\{i > \zeta_{j-1}^x : W_i - W_{\zeta_{j-1}^x} \in \{-m, n\}\}, \text{ for } j \in \mathbb{N}.$$

and

$$Y_i^x = \begin{cases} W_i + x & \text{for } i < \zeta_0^x; \\ W_i - W_{\zeta_n^x} & \text{for } \zeta_n^x \leqslant i < \zeta_{n+1}^x. \end{cases}$$

Initially we place a particle at each $x \in K$ and let each of the particles be driven by the same path. It is a simple consequence of Lemma 2.3 that with probability 1 the set $Y_i = \{Y_i^x : x \in K\}$ collapses to a single point in finite time. Observe also that for any initial configuration of particles the number of occupied sites decreases to 1 in finite time almost surely.

Below we describe the construction of the path, which ensures that all particles collide before some finite time M, which does not depend on the initial configuration of particles. This construction will be adapted later for the continuous case. The main technical difficulties in the continuous case are that we have to "thicken" the explicit path to allow the Brownian motion to stay within that "thickened" path with positive probability, and that we have to deal not with a finite number of particles but with the supports of probability measures on $(-a, b)$.

Assume that initially there is a particle at each $x \in K$.

Preliminary step. For $i \leq n-1$ let the random walk jump downward, so that at time $n-1$ the set of occupied sites is $\{-(m-1), -(m-2), \dots, 0\}$.

Choosing the "jumping particle". Suppose that two particles are separated by a distance $d < m$. Designate one of these two as the "jumping particle". By driving the random walk upward whenever the "jumping particle" is above its partner and downward whenever it is below its partner we could make the "jumping particle" leap repeatedly from the boundary to zero while the other particle would never make such a leap. Lemma A.1 guarantees that we can always choose one of the particles as the "jumping particle" in such a way that the particles collide before they find themselves in adjacent sites, but in the opposite orientation from how they began (see Figure 2). In the continuous case we have a "jumping interval" instead of a "jumping point", and Lemma A.1 will ensure sufficient space between this interval and the remainder of the set.

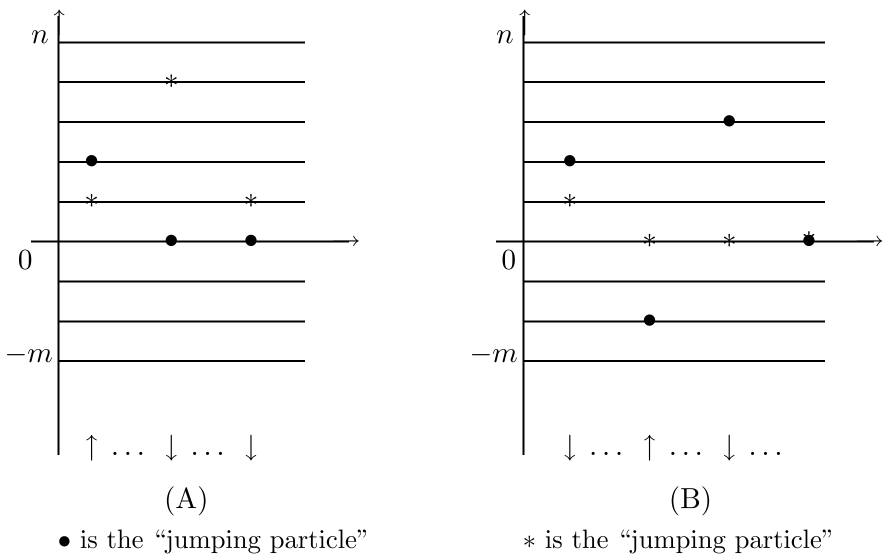

Figure 2: Illustrating the two possibilities with $m = 3$, $n = 5$ and $k = 1$. Down (up) arrow indicates that the random walk is moving down (up) until the "jumping particle" leaps to 0. In case A, the particles land in adjacent positions in opposite orientation. In case B, the particles collapse onto each other. Therefore, we would choose the "$*$" as the "jumping particle" for the next step.

Iterative step (k=m, m-1,. . . , 2). Now suppose that the set of occupied sites is a single block of k consecutive sites. We view the highest and lowest points of that block as the two particles considered in the paragraph above and we choose the jumping particle to be whichever is indicated by Lemma A.1. The remaining portion of the block will be referred to as the "main body". Drive the walk downward or upward depending on whether the jumping particle is below or above the "main body". The "jumping particle" will thus undergo a series of leaps from the boundary to zero, while no particle in the "main body" will undergo such a leap. Lemma A.1 guarantees that, after the particle is detached, it will collide with the "main body" before re-attaching itself to the opposite end of the block. Therefore, at the end of this step the set of occupied sites is a single block of $(k - 1)$ consecutive sites. Repeat this step until only 1 occupied site is left.

Construction of the path. Let M be the time, which is necessary to complete the preliminary step, and $(m - 1)$ repetitions of the iterative step. These steps define a sequence of down and up jumps for the random walk, which ensures the collapse to one point for an arbitrary initial configuration of particles. By the Markov property and the second Borel-Cantelli all particles collide in finite time with probability 1.

4.2 Proof of Theorem 1.2

For given $\varepsilon > 0$, $r > 0$, and a continuous function φ, such that $\varphi(0) = 0$, set

$$\Phi(s, s + r) = \{\omega \in \Omega : |B_{s+t} - B_s - \varphi(t)| \leqslant \varepsilon \text{ for } 0 \leqslant t \leqslant r\}. \tag{26}$$

Let S be a finite stopping time for Brownian motion. We repeatedly use the support theorem (see [5], Theorem 6.6, p. 60), which says that $P(\Phi(S, S + r)) > c > 0$, where c depends only on r, ε and the modulus of continuity of φ.

Part (i) of Theorem 1.2 is an easy consequence of the following proposition.

Proposition 4.1. *Let m, n, and γ be defined as in the statement of Theorem 1.2 and $\mu_0 = \lambda$. Define $\tau = \inf\{t \geqslant 0 : X(t)$ is an interval of length $\gamma\}$. Then $P(\tau < \infty) = 1$.*

Proof. Recall that $a + b = 1$ and $\gamma = 1/(m + n)$. Since we start with $\mu_0 = \lambda$, the Lebesgue measure of $X(t)$ can never be less than γ.

We describe a construction of a set of paths A on a time interval $[0, M_c(\omega)]$, such that $P(A) > 0$, the random time M_c is bounded uniformly on A by some deterministic M, and for every Brownian path in A the set $X(M_c)$ is a single interval of length γ. Then we let $A^{(0)} = A$ and $A^{(i)} = \{\omega \in \Omega : B_{\cdot+Mi}(\omega) \in A\}$, $i \in \mathbb{N}$. It follows from the Markov property of Brownian motion that the events $A^{(i)}$, $i = 0, 1, \ldots$, are independent. Moreover, $P(A^{(i)}) > c$ for some positive c, which does not depend on i. By the second Borel-Cantelli lemma at least one of the events $A^{(i)}$ occurs with probability 1. This will imply the statement of Proposition 4.1.

The set A is essentially built around segments of a piece-wise linear function $\varphi(t)$ with slopes ± 1. Each segment of φ is obtained by the interpolation of the appropriate random walk path from the discrete model. The construction is presented in a sequence of steps. At the end of each step we use a simple stopping time argument, which "aligns" $X(t)$ to allow us to use the discrete framework.

The construction takes only a finite number of steps and at each step every point makes only finitely many leaps to zero. Let ε be small enough so as to satisfy the conditions imposed by each of the steps below.

Preliminary step. Let $\varphi(t) = -t$ for $0 \leqslant t \leqslant \sigma_0 = b + \varepsilon$. Then $P(\Phi(0, \sigma_0)) > 0$ and for any path in $\Phi(0, \sigma_0)$ the set $X(\sigma_0)$ is an interval of length a, $X(\sigma_0) \subset (-a, 2\varepsilon]$, and $\sup X(\sigma_0) \geqslant 0$.

Alignment. Let $\tau_0 = \inf\{t > \sigma_0 : \sup X(t) = \gamma/2\}$. Set

$$A_0 = \Phi(0, \sigma_0) \cap \{\tau_0 - \sigma_0 \leqslant 1\}.$$

Clearly, $P(A_0) > 0$ and $X(\tau_0) = [\gamma/2 - a, \gamma/2]$ (recall that $a = m\gamma$) for each path in A_0.

Iterative step (k=m,m-1,. . . ,2). Let τ_{m-k}, A_{m-k} be defined in the previous step and $X(\tau_{m-k}) = [\gamma/2 - k\gamma, \gamma/2]$ for each path in A_{m-k}. We view $X(\tau_{m-k})$ as the union of k symmetric intervals of length γ around the particles located at $(1-k)\gamma, (2-k)\gamma, \dots, 0$. These particles can be thought of in terms of the embedded discrete process on the lattice $K\gamma = \{-(m-1)\gamma, \dots, (n-1)\gamma\}$.

Detachment of "jumping interval". Consider the particles at 0 and at $(1-k)\gamma$ (the midpoints of the highest and the lowest intervals). Either 0 or $(1-k)$ satisfies the "jumping particle" condition stated in the discrete construction. Suppose that it is 0. (The other case is similar and will be omitted.) We drive Brownian motion upward until the interval of length at least γ detaches, and then drive downward to get away from the boundary. Define

$$\varphi_k^{(0)}(t) = \begin{cases} t & \text{for } 0 \leqslant t \leqslant b + \varepsilon + \gamma/2; \\ 2(b + \varepsilon + \gamma/2) - t & \text{for } b + \varepsilon + \gamma/2 < t \leqslant t_0 = b + 2\varepsilon + \gamma. \end{cases}$$

Then for each path in $A_{m-k} \cap \Phi(\tau_{m-k}, \tau_{m-k} + t_0)$ the set $X(\tau_{m-k} + t_0)$ consists of two disjoint intervals: the interval of length between γ and $\gamma + 2\varepsilon$ (the "jumping interval") located between $-(\gamma/2 + 3\varepsilon)$ and $\gamma/2 + \varepsilon$ and the rest of $X(\tau_{m-k} + t_0)$ (the "main body"). The distance between the "jumping interval" and the "main body" is at least γ.

Collapse onto the "main body". We view the "jumping interval" as an interval around 0 and the "main body" as the union of disjoint intervals around the points $(n-k)\gamma, (n-k+1)\gamma, \dots, (n-1)\gamma$ (recall that $b = n\gamma$). At τ_{m-k} the set $\{0, (n-k)\gamma, (n-k+1)\gamma, \dots, (n-1)\gamma\}$ will serve us as a new set of reference particles. The particle at 0 will be referred to as the "jumping particle". We know that there is a random walk path, which ensures the collapse of the "jumping particle" onto the "main body" (which would be the set $\{(n-k)\gamma, (n-k+1)\gamma, \dots, (n-1)\gamma\}$ for the embedded discrete process). Let L be the number of up and down jumps of such a random walk. For $j = 1, 2, \dots, L$ define $\varphi_k^{(j)}$ by the following formulas. If the random walk is to go down in the j-th step then

$$\varphi_k^{(j)}(t) = \begin{cases} t & \text{for } 0 \leq t \leq \gamma/2 + 3\varepsilon \\ 2(\gamma/2 + 3\varepsilon) - t & \text{for } \gamma/2 + 3\varepsilon \leq t \leq 2\gamma + 6\varepsilon \end{cases}$$

If the random walk is to go up in the j-th step then

$$\varphi_k^{(j)}(t) = \begin{cases} -t & \text{for } 0 \leq t \leq \gamma/2 + \varepsilon \\ t - 2(\gamma/2 + \varepsilon) & \text{for } \gamma/2 + \varepsilon \leq t \leq 2\gamma + 2\varepsilon \end{cases}$$

The definition of $\varphi_k^{(j)}$ needs an explanation. There are moments when the random walk switches the direction. For example, it goes down after going up in the previous step (which occurs only if the "jumping particle" makes a leap to 0). If this is the case, then we need to continue driving the Brownian path up until all of the "jumping interval" makes the designated leap. If there is no change in the direction then the above formulas for $\varphi_k^{(j)}$ still work.

Define φ_k to be a continuous, piece-wise linear function, which is the concatenation of all $\varphi_k^{(j)}$, $j = 0, 1, \ldots, L$. Let $\Phi_k(s, s + r)$ be given by (26), where φ is replaced by φ_k. The domain of φ_k is a finite interval $[0, M_k]$, and $P(\Phi_k(\tau_{m-k}, \sigma_{m-k+1})) > c > 0$, where $\sigma_{m-k+1} = \tau_{m-k} + M_k$. Notice that $X(\sigma_{m-k+1})$ is a single interval of length at most $(k-1)\gamma$ (except, possibly, for $k = 2$; see the end of the proof).

Alignment. Define $\tau_{m-k+1} = \inf\{t \geqslant \sigma_{m-k+1} : \sup X(t) = \gamma/2\}$ if $\inf\{t \geqslant \sigma_{m-k+1} : \sup X(t) = b \text{ or } \inf X(t) = a\} \geqslant \inf\{t \geqslant \sigma_{m-k+1} : \sup X(t) = \gamma/2\}$ and $\tau_{m-k+1} = +\infty$ otherwise. Set

$$A_{m-k+1} = A_{m-k} \cap \Phi_k(\tau_{m-k}, \sigma_{m-k+1}) \cap \{\tau_{m-k+1} - \sigma_{m-k+1} \leqslant 1\}.$$

Then $P(A_{m-k+1}) > 0$. The length of $X(\tau_{m-k+1})$ can be less than $(k-1)\gamma$ (except for $k = 2$), but without loss of generality we may assume that $X(\tau_{m-k+1}) = [(1-k)\gamma + \gamma/2, \gamma/2]$ (this is the "worst" case).

We execute the preliminary step and iterative steps for $k = m, m-1, \ldots, 2$.

Case k=2. At time τ_{m-1}, we have finished the case $k = 2$, the interval $X(\tau_{m-1})$ has a length between γ and $\gamma + 2\varepsilon$. Without loss of generality we may assume that the length of $X(\tau_{m-1})$ is $\gamma + 2\varepsilon$. We take a pair of interior points belonging to the same equivalence class, apply a procedure similar to iterative step one final time and obtain the stopping time τ_m and the set A_m, which ensure that $X(\tau_m)$ is a single interval of length γ. We define $A = A_m$ and $M_c = \tau_m$. The set A and the interval $[0, M_c(\omega)]$ have the properties stated at the beginning of the proof. □

Proof of Theorem 1.2 (i). Let μ_0 be an arbitrary probability measure on $(-a, b)$. From Proposition 4.1 it follows that $\operatorname{supp}\mu_\tau$ is contained in an interval of length γ. Let Λ be any interval of the form $[y, y+\gamma) \in (-a, b)$. Then

$$\begin{aligned}\lambda\{x \in \Lambda : C_x \cap \operatorname{supp}\mu_\tau \neq \emptyset\} &= \lambda\{x \in [0,\gamma) : C_x \cap \operatorname{supp}\mu_\tau \neq \emptyset\} \\ &= \lambda\{x \in [0,\gamma) : C_x \cap \operatorname{supp}\mu_0 \neq \emptyset\} = \alpha.\end{aligned}$$

Therefore $\zeta \leqslant \tau$ and $P(\zeta < \infty) = 1$. □

Proof of Theorem 1.2 (ii). Let $l \geqslant 1$ be an integer. Proposition 4.1 gives the result for $l = 1$. Without loss of generality we may assume that $X(0) = [-\gamma, 0]$. We will exhibit an explicit path $\varphi(t)$, $0 \leqslant t \leqslant T$, that breaks the interval into $2l$ disjoint intervals (and therefore necessarily achieves each positive number of components less than $2l$ in the process). We shall describe a path, which detaches intervals of length γ/l and then divides each of them into two pieces. It will be obvious from the construction that all these intervals will be disjoint, and that an application of the support theorem and the Borel-Cantelli lemma will complete the proof.

Set $\delta = \gamma/2l$ and $t_1 = b$. Let $X_\varphi(t)$ denote the process driven by $\varphi(t)$ (as opposed to $X(t)$ which is driven by B_t).

Preliminary step. Let $\varphi(t) = t$ for $0 \leqslant t \leqslant t_1$. Then $X_\varphi(t_1) = [b-\gamma, b) \cup \{0\}$.

Iterative step (k=1,2,...,l). At first, break off an interval of length 2δ from the top by setting

$$\varphi(t_k + t) = \varphi(t_k) + t \quad \text{for } 0 \leqslant t \leqslant 2\delta.$$

Then force the path downward and break off an interval of length δ from the bottom:

$$\varphi(t_k + 2\delta + t) = \varphi(t_k + 2\delta) - t \quad \text{for } 0 \leqslant t \leqslant a + \delta.$$

Notice that at time $t_k + a + 3\delta$ we have $2k+1$ disjoint intervals (one "large" interval and $2k$ intervals of length δ), except for $k = l$, when we have $2l$ disjoint intervals (the "large" interval has disappeared), as required.

Next we go up until the topmost point of $X_\varphi(t)$ reaches b:

$$\varphi(t_k + a + 3\delta + t) = \varphi(t_k + a + 3\delta) + t \quad \text{for } 0 \leqslant t \leqslant a + \delta.$$

If $k \leq l$ then define $t_{k+1} = t_k + 2a + 4\delta$. If $k+1 \leqslant l$ then we repeat the procedure.

Define $T = t_{l+1} = b + 2l(a + 2\delta)$. At time T , $X_\varphi(T)$ will have $2l$ disjoint intervals of length δ. More precisely, if $D_1 = [\delta, 2\delta] \cup [3\delta, 4\delta] \cup \cdots \cup [\gamma - \delta, \gamma]$ then $X_\varphi(T) = D_1 \cup \{D_1 + a - \delta\}$.

Explicit Path for X. We apply the preliminary step and then the iterative steps as indicated. Let $\varepsilon < \delta/4$. For each path in $\Phi(0,T)$ we have that $X(T)$ is a union of $2l$ disjoint intervals of some random length strictly between 0 and 2δ. Since $P(\Phi(0,T)) > 0$, we can argue exactly the same way as in the proof of part (i) to say that with probability one $\Sigma(t,\omega)$ assumes every value in $\mathbb{N}$, and $\Sigma(t,\omega)$ is recurrent. $\square$

A Appendix

In this section we state and prove two technical lemmas, that we used in Sections 2 and 4.

Lemma A.1. *Let m and n be coprime numbers, d be an integer, $1 \leqslant d < m < n$. Consider two sequences $F_0 = \{jm,\ j = 1, 2, \ldots, m+n-1\}$ and $F_k = \{d + jm,\ j = 1, 2, \ldots, m+n-1\}$*

in $\mathbb{Z}/(m+n)\mathbb{Z}$. Then either in the sequence F_0 the remainder d occurs prior to $(d+1)$ or in the sequence F_d the remainder 0 occurs prior to -1.

Remark. To apply this lemma to our discrete model it is convenient to identify the boundary points $-m$ and n and consider the random walk on the circle $\mathbb{Z}/(m+n)\mathbb{Z}$. Since we are only interested in the location of the two particles with respect to each other, we may choose to keep the particles, where they are, say, at 0 and d, and let the boundary point to perform the random walk. When the boundary point hits one of the particles that particle jumps and its coordinate increases by m. If the boundary point always hits the particle, which was originally at 0 (d respectively), then this particle consecutively occupies the sites listed in the sequence F_0 (F_d respectively). Therefore, the statement of the lemma implies the claim that for each d we can choose the "jumping" particle and a path for the random walk in such a way that the particles will collide before they land in adjacent sites in the opposite orientation from how they began.

Proof of Lemma A.1. We write $i \equiv j$ whenever $i = j \,(\mathrm{mod}\,(m+n))$. For $l = 0, 1$ define

$$\alpha_l = \min\{j \in \mathbb{N} : jm \equiv d + 1 - l\};$$
$$\beta_l = \min\{j \in \mathbb{N} : d + jm \equiv l - 1\}.$$

We want to show that either $\alpha_1 < \alpha_0$ or $\beta_1 < \beta_0$. Clearly, $\alpha_0 \neq \alpha_1$, $\beta_0 \neq \beta_1$ and $\alpha_l, \beta_l < (m+n)$, $l = 0, 1$. Observe that $(\alpha_l + \beta_l)m \equiv 0$, which implies that $\alpha_l + \beta_l = m + n$ for $l = 0, 1$. Since $(\alpha_1 - \alpha_0) + (\beta_1 - \beta_0) \equiv 0$, we conclude that either $\alpha_1 - \alpha_0 < 0$ or $\beta_1 - \beta_0 < 0$. $\square$

Now we prove a lemma about absorption probabilities that was required in Section 3. Consider a Markov chain on $\mathbb{Z}$ with the transition probabilities

$$P_{ij} = \begin{cases} 1 & \text{if } \; i = j = 0, \\ p_i & \text{if } \; j = i - 1, \; |i| \geqslant 1, \\ q_i & \text{if } \; j = -i, \; |i| \geqslant 1, \\ 0 & \text{otherwise} \end{cases}$$

where $p_i, q_i \in (0, 1)$ and $p_i + q_i = 1$ for all $i \in \mathbb{Z} \setminus \{0\}$. This chain is irreducible, state 0 is the only absorbing state. Let $a(i)$ be the absorption probability starting from i.

Lemma A.2. *For $k \in \mathbb{N}$ the absorption probabilities for the above Markov chain are given by*

$$a(k) = \frac{\beta_k p_{-k} + \sum_{i=k}^{\infty} \beta_i q_{-i}}{1 + \sum_{i=1}^{\infty} \beta_i q_{-i}} \quad \text{and} \quad a(-k) = \frac{\sum_{i=k}^{\infty} \beta_i q_{-i}}{1 + \sum_{i=1}^{\infty} \beta_i q_{-i}}$$

where $\beta_k = \prod_{j=1}^{k} \dfrac{p_i}{p_{-i}}$.
In particular, $a(i) = 1$ for all $i \in \mathbb{Z}$ if and only if the series $\sum_{k=1}^{\infty} \beta_k q_{-k}$ diverges.

Proof. Fix a large $N \in \mathbb{N}$, and make the state $-N$ into the absorbing state. Let $a_N(i)$, be the probability starting from i $(-N \leqslant i \leqslant N-1)$ to reach state 0 before $-N$. Then for $k = 1, 2, \dots, N-1$ we have the system of equations

$$a_N(k) = p_k a_N(k-1) + q_k a_N(-k) \tag{27}$$

$$a_N(-k) = p_{-k} a_N(-k-1) + q_{-k} a_N(k) \tag{28}$$

Adding (27) and (28), simplifying, and incorporating the initial data $a_N(0) = 1$, $a_N(-N) = 0$ yields

$$\begin{aligned} a_N(k) - a_N(-k-1) &= \frac{p_k}{p_{-k}}(a_N(k-1) - a_N(-k)) = \dots \\ &= \beta_k(a_N(0) - a_N(-1)) = \beta_k(1 - a_N(-1)). \end{aligned} \tag{29}$$

Solving (28) for $a_N(k)$ and substituting into (29) we obtain

$$a_N(-k-1) = a_N(-k) - \beta_k q_{-k}(1 - a_N(-1)),$$

which leads to the equation

$$a_N(-k-1) = a_N(-1) - (1 - a_N(-1)) \sum_{i=1}^{k} \beta_i q_{-i}.$$

Using the fact that $a_N(-N) = 0$ we find that

$$a_N(-1) = \frac{\sum_{i=1}^{N-1} \beta_i q_{-i}}{1 + \sum_{i=1}^{N-1} \beta_i q_{-i}},$$

and

$$a_N(-k) = \frac{\sum_{i=k}^{N-1} \beta_i q_{-i}}{1 + \sum_{i=1}^{N-1} \beta_i q_{-i}},$$

$$a_N(k) = \frac{\beta_k p_{-k} + \sum_{i=k}^{N-1} \beta_i q_{-i}}{1 + \sum_{i=1}^{N-1} \beta_i q_{-i}}$$

Clearly, $a(i) = \lim_{N\to\infty} a_N(i)$, which implies the statement of the lemma. □

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