Optimal Fractional Factorial Plans for Asymmetric Factorials

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OPTIMAL FRACTIONAL FACTORIAL PLANS FOR ASYMMETRIC FACTORIALS

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Fractional factorial plans for asymmetric factorial experiments are obtained. These are shown to be universally optimal within the class of all plans involving the same number of runs under a model that includes the mean, all main effects and a specified set of two-factor interactions. Finite projective geometry is used to obtain such plans for experiments wherein the number of levels of each of the factors as also the number of runs is a power of m, a prime or a prime power. Methods of construction of optimal plans under the same model are also discussed for the case where the number of levels as well as the number of runs are not necessarily powers of a prime number.

1. Introduction. The study of optimal fractional factorial plans has received considerable attention in the recent past; see e.g., Dey and Mukerjee ((1999a); Chapters 2, 6 & 7). Many of these results relate to situations where all factorial effects involving the same number of factors are considered equally important and, as such, the underlying model involves the general mean and all factorial effects involving up to a specified number of factors. In practice however, all factorial effects involving the same number of factors may not always be equally important and often, an experimenter is interested in estimating the general mean, all main effects and only a specified set of two-factor interactions, all other interactions being assumed negligible. The issue of estimability and optimality in situations of this kind has been addressed by Hedayat and Pesotan (1992, 1997), Wu and Chen (1992) and Chiu and John (1998) in the context of two-level factorials and, by Dey and Mukerjee (1999b) for arbitrary factorials including the asymmetric ones. Using finite projective geometry, Dey and Suen (2002) recently obtained several families of optimal plans under the stated model for *symmetric* factorials of the type m^n, where m is a prime or a prime power.

Continuing with this line of research, in this paper we obtain optimal fractional factorial plans for *asymmetric* (mixed level) factorials under a model that includes the mean, all main effects and a specified set of two-factor interactions. All other interactions are assumed to be negligible. Throughout, the optimality criterion considered is the universal optimality of Kiefer (1975); see also Sinha and Mukerjee (1982). In Section 2, concepts and results from a finite projective geometry are used to obtain optimal plans for asymmetric factorials, where the levels of the factors as also the number of runs are

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powers of the same prime. In Section 3, we obtain of optimal plans for asymmetric experiments where the levels of the factors and the number of runs are *not* necessarily powers of a prime number.

2. Optimal plans based on finite projective geometry. For obtaining the optimal plans in this paper, we make use of a result of Dey and Mukerjee (1999b), giving a combinatorial characterization for a fractional factorial plan to be universally optimal. For completeness, we state this result below in a form that is needed for this paper.

THEOREM 2.1. *Let $\mathcal{D}$ be the class of all N-run fractional factorial plans for an arbitrary factorial experiment involving n factors, such that each member of $\mathcal{D}$ allows the estimability of the mean, the main effects $F_1, \ldots, F_n$ and the k two-factor interactions $F_{i_1}F_{j_1}, \ldots, F_{i_k}F_{j_k}$, where $1 \le i_u, j_u \le n$ for all $u = 1, \ldots, k$. A plan $d \in \mathcal{D}$ is universally optimal over $\mathcal{D}$ if all level combinations of the following sets of factors appear equally often in d:*
(a) $\{F_u, F_v\}$, $1 \le u < v \le n$;
(b) $\{F_u, F_{i_v}, F_{j_v}\}$, $1 \le u \le n,\ 1 \le v \le k$;
(c) $\{F_{i_u}, F_{j_u}, F_{i_v}, F_{j_v}\}$, $1 \le u < v \le k$,
where a factor is counted only once if it is repeated in (b) or (c).

Consider now a factorial experiment involving n factors $F_1, \ldots, F_n$, where for $i = 1, \ldots, n$, the factor F_i has m^{t_i} levels, m is a prime or a prime power and t_i is a positive integer. We shall use an $(r-1)$-dimensional finite projective geometry $PG(r-1, m)$ over the finite (or, Galois) field $GF(m)$ to construct m^r-run plans, r being an integer. For an excellent account of finite projective geometry, see Hirschfeld (1979).

We assign the factor F_i to a $(t_i - 1)$-flat in $PG(r-1, m)$, these flats being distinct for $F_i, F_j,\ i \ne j$. The two-factor interaction F_iF_j is assigned to be the $(m^{t_i} - 1)(m^{t_j} - 1)/(m-1)$ points in the $(t_i + t_j - 1)$-flat through the $(t_i - 1)$-flat F_i and the $(t_j - 1)$-flat F_j but not in F_i and F_j. Making an appeal to Theorem 2.1, one can prove the following result (see also Dey and Suen (2002) for a similar result in the context of symmetric prime-powered factorials).

THEOREM 2.2. *Let $F_1, \ldots, F_n$ be n factors of a factorial experiment, where for $u = 1, \ldots, n$, the factor F_u has m^{t_u} levels, m is a prime or a prime power and t_u is a positive integer. Assign the n main effects $F_1, \ldots, F_n$ and the k two-factor interactions $F_{i_1}F_{j_1}, \ldots, F_{i_k}F_{j_k}$ to points in $PG(r-1, m)$ as described in the previous paragraph. If the $\sum_{u=1}^{n} \frac{m^{t_u}-1}{m-1} + \sum_{u=1}^{k} \frac{(m^{t_{i_u}}-1)(m^{t_{j_u}}-1)}{m-1}$ points corresponding to $F_1, \ldots, F_n, F_{i_1}F_{j_1}, \ldots, F_{i_k}F_{j_k}$ are all distinct, then we can obtain a universally optimal plan for estimating the main effects $F_1, \ldots, F_n$ and two-factor interations $F_{i_1}F_{j_1}, \ldots, F_{i_k}F_{j_k}$ involving m^r runs.*

PROOF. Let A_u be an $r \times t_u$ matrix with the t_u column vectors corresponding to t_u independent points in the $(t_u - 1)$-flat F_u. Then the plan can be generated by the row space of the $r \times \sum_{u=1}^{n} t_u$ matrix $A = [A_1 \vdots \cdots \vdots A_n]$, where the t_u columns of A_u represent the levels of the factor F_u and each element of the row space of A represents a run in the plan. To prove that the plan is universally optimal, it suffices to show, as in Dey and Suen (2002), that the following matrices have full column rank :

(i) $[A_u \vdots A_v], 1 \leq u < v \leq n$;

(ii) $[A_u \vdots A_{i_v} \vdots A_{j_v}], 1 \leq u \leq n, 1 \leq v \leq k$;

(iii) $[A_{i_u} \vdots A_{j_u} \vdots A_{i_v} \vdots A_{j_v}], 1 \leq u < v \leq k$,

where a matrix $A_u (1 \leq u \leq n)$ appears only once if it is repeated in (ii) or (iii).

Case (i) : The columns of A_u and A_v are independent since the $(t_u - 1)$-flat F_u and the $(t_v - 1)$-flat F_v are disjoint.

Case (ii) (a) : If $u = i_v$ or j_v, then the matrix reduces to $[A_{i_v} \vdots A_{j_v}]$ which has full column rank as in *Case* (i).

Case (ii) (b) : If u, i_v, j_v are distinct, then the $(t_u - 1)$-flat F_u and the $(t_{i_v} + t_{j_v} - 1)$-flat, consisting of points in F_{i_v}, F_{j_v}, and $F_{i_v} F_{j_v}$, are disjoint. Hence the columns of A_u are independent of columns of $[A_{i_v} \vdots A_{j_v}]$, and the matrix $[A_u \vdots A_{i_v} \vdots A_{j_v}]$ has full column rank.

Case (iii) (a) : If $i_u = i_v$ or j_v, then the matrix reduces to $[A_{j_u} \vdots A_{i_v} \vdots A_{j_v}]$ which has full column rank as in *Case* (ii) (b).

Case (iii) (b) : If i_u, j_u, i_v, j_v are distinct, then the $(t_{i_u} + t_{j_u} - 1)$-flat, consisting of points in F_{i_u}, F_{j_u}, and $F_{i_u} F_{j_u}$, and the $(t_{i_v} + t_{j_v} - 1)$-flat, consisting of points in F_{i_v}, F_{j_v}, and $F_{i_v} F_{j_v}$, are disjoint. Hence the columns of $[A_{i_u} \vdots A_{j_u}]$ are independent of columns of $[A_{i_v} \vdots A_{j_v}]$, and the matrix $[A_{i_u} \vdots A_{j_u} \vdots A_{i_v} \vdots A_{j_v}]$ has full column rank. This completes the proof. □

Based on Theorem 2.2, we now construct specific families of optimal plans, permitting the estimability of the mean, all main effects and a specified set of two-factor interactions. These families of plans are constructed by a suitable choice of points in $PG(r-1, m)$ satisfying the conditions of Theorem 2.2. Most of the plans reported in this section are saturated. As in Dey and Suen (2002), we introduce the following notations to specify the models:

1. A plan allowing the optimal estimation of the mean, $2u$ main effects $F_1, \ldots, F_{2u}$ and u two-factor interactions $F_1F_2, F_3F_4, \ldots, F_{2u-1}F_{2u}$ will be denoted by

$$(F_1, F_2; F_3, F_4; \ldots; F_{2u-1}, F_{2u})_1.$$

2. A plan allowing the optimal estimation of the mean, $u + v$ main effects $F_1, \ldots, F_{u+v}$ and uv two-factor interactions F_iF_j $(1 \leq i \leq u, u+1 \leq j \leq u+v)$ will be denoted by

$$(F_1, \ldots, F_u; F_{u+1}, \ldots, F_{u+v})_2.$$

3. A plan allowing the optimal estimation of the mean, u main effects $F_1, \ldots, F_u$ and u two-factor interactions $F_1F_2, \ldots, F_{u-1}F_u, F_uF_1$ will be denoted by

$$(F_1, \ldots, F_u)_3.$$

Throughout this section, the m^2-level factors are denoted by $F_1, F_2, \ldots$ etc. and the m-level factors by $G_1, G_2, \ldots$ etc. We now have the following results.

Theorem 2.3. *For any prime or prime power m, we can construct a universally optimal plan*
(a) d_1 for an $(m^2)^2 \times m^{m^2}$ experiment involving m^5 runs where

$$d_1 \equiv \{(F_0; F_1, G_1, \ldots, G_{m^2})_2\};$$

(b) d_2 for an $(m^2) \times m^{3m^2}$ experiment involving m^5 runs where

$$d_2 \equiv \{(F_0; G_1, \ldots, G_{m^2})_2, (G_{1,1}, G_{2,1}; G_{1,2}, G_{2,2}; \ldots; G_{1,m^2}, G_{2,m^2})_1\}.$$

Both d_1 and d_2 are saturated.

Proof. *(a)* Let F_0 and K be disjoint line and plane in $PG(4, m)$. Choose F_1 to be a line on the plane K and $G_1, \ldots, G_{m^2}$ to be the m^2 points on the plane K but not on the line F_1.

(b) Let H be the 3-flat containing lines F_0 and F_1, and let $F_0, L_1, \ldots, L_{m^2}$ be $m^2 + 1$ lines which partition H. For $i = 1, \ldots, m^2$, choose $G_{1,i}$ and $G_{2,i}$ to be two distinct points on the line L_i. □

Theorem 2.4. *For any prime or prime power m, we can construct a universally optimal saturated plan d for an $(m^2)^{m^2+1} \times m$ experiment involving m^5 runs where*

$$d \equiv \{(G; F_1, \ldots, F_{m^2+1})_2\}.$$

Proof. Let H be a 3-flat in $PG(4, m)$, and let $F_1, \ldots, F_{m^2+1}$ be m^2+1 lines which partition H. Choose G to be a point of $PG(4, m)$ not in H. □

Theorem 2.5. *Let F be an m^2-level factor and G be an m-level factor of a universally optimal plan d. If the effects F, G, and FG can be estimated via d and F has no interaction with any other factor except G, then instead of estimating F and FG we can estimate one of the following sets of effects in d :*
(a) $(G; G_1, \ldots, G_{m+1})_2$;
(b) $(G_0; G, G_1, \ldots, G_m)_2$;
(c) $(G_1, G_2, G)_3$ and the main effects of $G_3, \ldots, G_{m^2-2m+3}$;
(d) $(G_1, G_2, G_3)_3$ and the main effects of $G_4, \ldots, G_{m^2-2m+3}$.

Proof. Let K be the plane containing the point G and the line F.
(a) Let L be a line on the plane K which does not pass through the point G. Choose $G_1, \ldots, G_{m+1}$ to be the $m + 1$ points on the line L.
(b) Let L be a line through the point G on the plane K, and let $G, G_1, \ldots, G_m$ be the $m + 1$ points on the line L. Choose G_0 to be a point on the plane K but not on the line L.
(c) Let G_1 and G_2 be points on the plane K such that G, G_1, G_2 are not collinear. Choose $G_3, \ldots, G_{m^2-2m+3}$ to be the $(m - 1)^2$ points on the plane K which are not on the three lines joining the three pairs of points (G, G_1), (G, G_2), (G_1, G_2) separately.

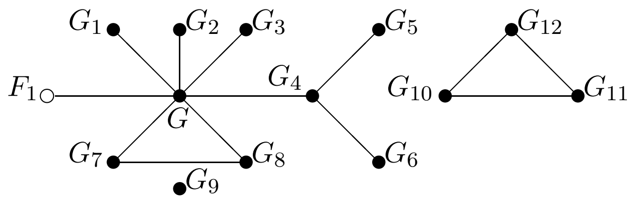
(d) Choose points G_1, G_2, G_3 such that no three of the four points G, G_1, G_2, G_3 are collinear. Now choose $G, G_4, \ldots, G_{m^2-2m+3}$ to be the $(m-1)^2$ points on the plane K which are not on the three lines joining the three pairs of points (G_1, G_2), (G_2, G_3), (G_3, G_1) separately. □

We now consider an example. To save space, only examples for $m = 2$ are given in this section. In the following as well as in subsequent examples in this section, we shall use the numbers $1, \ldots, 2^r - 1$ to represent the $2^r - 1$ points in $PG(r-1, 2)$. A number α represents a point in $PG(r-1, 2)$ with coordinates $(x_0, \ldots, x_{r-1})$ such that $\sum_{i=0}^{r-1} x_i 2^i = \alpha$. For example, the number 19 represent the point $(1, 1, 0, 0, 1)$ in $PG(4, 2)$ and it represents the point $(1, 1, 0, 0, 1, 0)$ in $PG(5, 2)$. A line in $PG(r-1, 2)$ is denoted by two numbers which represent two points on this line. Linear graphs are used to demonstrate the plans, where vertices represent the main effects and an edge joining two vertices represents the interaction of the two factors representing the two vertices. A 2-level factor is denoted by a closed circle $\bullet$ in the graph, and a 4-level factor which is represented by a line in the finite projective geometry, is denoted by an open circle $\circ$.

Example 2.1. With $m = 2$ in Theorem 2.4, we can construct an universally optimal plan d for a $4^5 \times 2$ experiment involving 32 runs where

$$d \equiv \{(G; F_1, F_2, F_3, F_4, F_5)_2\}$$

and $G(16), F_1(1, 2), F_2(4, 8), F_3(5, 10), F_4(6, 11), F_5(7, 9)$. Many universally optimal plans can be obtained by applying Theorem 2.5. For example, by replacing the effects (F_2, GF_2), (F_3, GF_3), (F_4, GF_4), (F_5, GF_5) by *(a), (b), (c), (d)* of Theorem 2.5, we obtain a universally optimal plan for a 4×2^{13} experiment involving 32 runs, whose linear graph is shown below :



where $G_1(4)$, $G_2(8)$, $G_3(12)$, $G_4(31)$, $G_5(5)$, $G_6(10)$, $G_7(6)$, $G_8(11)$, $G_9(29)$, $G_{10}(7)$, $G_{11}(9)$, $G_{12}(30)$. □

Theorem 2.6. *For any prime or prime power m, we can construct a universally optimal saturated plan*
(i) d_1 for an $(m^2) \times m^{m^3+m^2+m}$ experiment involving m^5 runs where

$$d_1 \equiv \{(F_1; G_1, \ldots, G_m)_2, (G_{0,1}; G_{1,1}, \ldots, G_{m,1})_2, \ldots, (G_{0,m^2}; G_{1,m^2}, \ldots, G_{m,m^2})_2\}.$$

(ii) d_2 for an $(m^2) \times m^{m^3+2m^2-m+1}$ experiment involving m^5 runs where

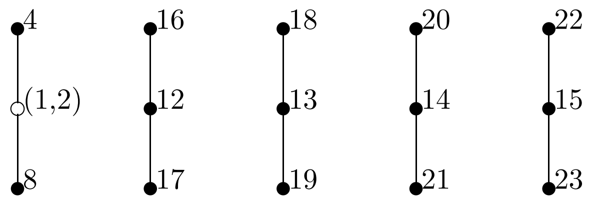
$$d_2 \equiv \{(G_{0,0}; F_2, G_{1,0}, \ldots, G_{m^2-m,0})_2, (G_{0,1}; G_{1,1}, \ldots, G_{m,1})_2, \ldots,$$

$$(G_{0,m^2}; G_{1,m^2}, \ldots, G_{m,m^2})_2\}.$$

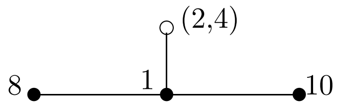
PROOF. Let $G_{0,0}$ be a point on a line F_1 which is on a plane K in $PG(4,m)$. Let $L_1,\ldots,L_m,F_1$ be the $m+1$ lines through the point $G_{0,0}$ on the plane K. For $i=1,\ldots,m$, let $G_{0,0},G_{0,(i-1)m+1},\ldots,G_{0,im}$ be the $m+1$ points on the line L_i. There are $m+1$ 3-flats through the plane K, say $H_0,H_1,\ldots,H_m$. For $i=1,\ldots,m$, let $K,K_{1,i},\ldots,K_{m,i}$ be the $m+1$ planes through the line L_i in the 3-flat H_i. For each $i=1,\ldots,m$ and $j=1,\ldots,m$, choose a line $L_{j,i}$ on the plane $K_{j,i}$ which does not pass through the point $G_{0,(i-1)m+j}$. Choose $G_{1,(i-1)m+j},\ldots,G_{m,(i-1)m+j}$ to be the m points on the line $L_{j,i}$ but not on L_i. For plan *(i)*, let L_0 be a line in the 3-flat H_0 but not on the plane K. Choose $G_1,\ldots,G_m$ to be the m points on the line L_0 but not on the plane K.

For plan *(ii)*, let K_0 be a plane in the 3-flat H_0 which does not pass through the $G_{0,0}$. Then the line F_1 intersects K_0 at a point P_0. Choose F_2 to be a line through a point P_0 on the plane K_0 and choose $G_{1,0},\ldots,G_{m^2-m,0}$ to be the m^2-m points on the plane K_0 which are not on the line F_2 or the plane K. □

EXAMPLE 2.2. With $m=2$ in Theorem 2.6, choose the point $G_{0,0}(1)$ and the line $F_1(1,2)$. Let K be the plane through the line F_1 and the point $G_{0,1}(12)$. Let L_0 be the line consisting of points $G_1(4)$, $G_2(8)$, and $G_{0,1}$. Let L_1 be the line consisting of points $G_{0,0}$, $G_{0,1}$, and $G_{0,2}(13)$, and let L_2 be the line consisting of points $G_{0,0}$, $G_{0,3}(14)$, and $G_{0,4}(15)$. Let H_1 be the 3-flat through the plane K and the point $G_{1,1}(16)$, and let H_2 be the 3-flat through the plane K and the point $G_{1,3}(20)$. Following the procedure of Theorem 2.6 *(i)*, we can choose the points $G_{2,1}(17)$, $G_{1,2}(18)$, $G_{2,2}(19)$, $G_{2,3}(21)$, $G_{1,4}(22)$, and $G_{2,4}(23)$ to construct the following universally optimal plan for a 4×2^{14} experiment involving 32 runs :



For plan *(ii)*, we can choose $F_2(2,4)$, $G_{1,0}(8)$, and $G_{2,0}(10)$ to obtain the following universally optimal plan for a 4×2^{15} experiment involving 32 runs. The linear graph is the same as above except that the first component is changed to



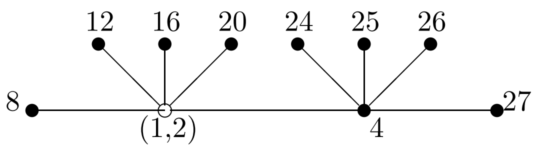
□

THEOREM 2.7. *For any prime or prime power m and integers j, k satisfying $j+k=m+1$, we can construct a universally optimal saturated plan d for an $(m^2)\times m^{km^2+jm+1}$ experiment involving m^5 runs where*

$$d\equiv\{(F_0;G_0,G_{1,1},\ldots,G_{jm,1})_2,(G_0;G_{1,2},\ldots,G_{km^2,\,2})_2\}.$$

PROOF. Let K be a plane in $PG(4,m)$, and let G_0 and F_0 be a point and a line on the plane K such that G_0 is not on F_0. Let $H_1,\ldots,H_{m+1}$ be the $m+1$ 3-flats through the plane K. For $i=1,\ldots,j$, let L_i be a line in the 3-flat H_i which does not intersect the line F_0. Choose $G_{(i-1)m+1,1},\ldots,G_{im,1}$ to be the m points on the line L_i which are not on the plane K. For $i=1,\ldots,k$, let K_i be a plane in the 3-flat H_{j+i} which does not pass through the point G_0. Choose $G_{(i-1)m^2+1,1},\ldots,G_{im^2,1}$ to be the m^2 points on the plane K_i but not on the plane K. □

EXAMPLE 2.3. With $m=2, j=2, k=1$ in Theorem 2.7, choose the point $G_0(4)$ and the line $F_0(1,2)$. Then K is the plane through the line F_0 and the point G_0. Let H_1,H_2,H_3 be the three 3-flats through the plane K and the points $G_{1,1}(8), G_{3,1}(16)$, $G_{1,2}(24)$ respectively. Let L_1 be the line through the points $G_{1,1}$ and $G_{2,1}(12)$, and let L_2 be the line through points $G_{3,1}$ and $G_{4,1}(20)$. Let K_1 be the plane through the line F and the point $G_{1,2}$. Then K_1 has 4 points $G_{2,2}(25)$, $G_{3,2}(26)$, $G_{4,2}(27)$, and $G(1,2)$ which are not on the plane K. We have thus constructed the following universally optimal plan for a 4×2^9 experiment involving 32 runs :



□

THEOREM 2.8. *For any prime or prime power m and integers j, k satisfying $j+k=m$, we can construct a universally optimal plan*
(i) d_1 for an $(m^2)^j\times m^{m^3+km+k+1}$ experiment involving m^5 runs where

$$d_1\equiv\{(G_{0,0};G_{0,1},G_1,\ldots,G_k,G_{1,0},\ldots,G_{j(m^2-m),0},F_1,\ldots,F_j)_2,$$

$$(G_{0,1};G_{1,1},\ldots,G_{(k+1)m^2,1})_2\}.$$

(ii) d_2 for an $(m^2)^j\times m^{m^3+(k+1)m+k}$ experiment involving m^5 runs where

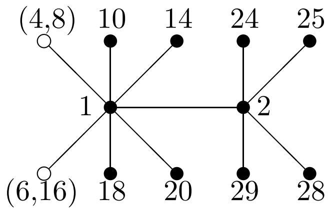
$$d_2\equiv\{(G_{0,0};G_1,\ldots,G_k,G_{1,0},\ldots,G_{j(m^2-m),0},F_1,\ldots,F_j)_2,$$

$$(G_{0,1};G'_{1,1},\ldots,G'_{(k+1)m,1})_2,\ldots,(G_{0,m};G'_{1,m},\ldots,G'_{(k+1)m,m})_2\}.$$

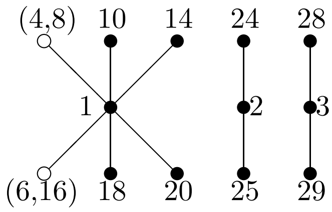
PROOF. Let $G_1,\ldots,G_m$ and $G_{0,1}$ be the $m+1$ points on a line L in $PG(4,m)$, and let $G_{0,0}$ be a point not on the line L. Let K be the plane through the line L and the point $G_{0,0}$. There are $m+1$ 3-flats through the plane K in $PG(4,m)$, say $H_1,\ldots,H_{m+1}$. For $i=1,\ldots,j$, let F_i be a line in the 3-flat H_i which passes through the point G_{k+i} but is not on the plane K. Let K_i be the plane through the lines L and F_i, and choose $G_{(i-1)(m^2-m)+1,0},\ldots,G_{i(m^2-m),0}$ to be the m^2-m points on the plane K_i which are not on the lines L and F_i. To obtain plan *(i)*, for $i=1,\ldots,k+1$, let K_{j+i} be a plane in the 3-flat H_{j+i} which does not pass through the point $G_{0,1}$. Choose $G_{(i-1)m^2+1,1}, \ldots, G_{im^2,1}$ to be the m^2 points on the plane K_i but not on the plane K.

To obtain plan *(ii)*, let L_0 be the line through the points $G_{0,0}$ and $G_{0,1}$, and let $G_{0,2}, \ldots, G_{0,m}$ be the $m-1$ other points on L_0. For $i = 1, \ldots, k+1$, let $K_{1,j+i}, \ldots, K_{m,j+i}$ and K be the $m+1$ planes through the line L_0 in the 3-flats H_{j+i}. For $u = 1, \ldots, m$, let $L_{u,j+i}$ be a line on the plane $K_{u,j+i}$ which does not pass through the point $G_{0,u}$. Now choose $G'_{(i-1)m+1,u}, \ldots, G'_{im,u}$ to be the m points on the line $L_{u,j+i}$ but not on the line L_0. □

EXAMPLE 2.4. With $m = 2, j = 2, k = 0$ in Theorem 2.8, choose the point $G_{0,0}(1)$ and the line L consisting of points $G_1(4)$, $G_2(6)$, and $G_{0,1}(2)$. Then K is the plane through the line L and the point $G_{0,0}$. Choose lines $F_1(4,8)$ and $F_2(6,16)$. Let K_1 be the plane through the lines F_1 and L. Then K_1 has 2 points $G_{1,1}(10)$ and $G_{2,1}(14)$ which are not on the lines F_1 and L. Let K_2 be the plane through the lines F_2 and L. Then K_2 has 2 points $G_{3,1}(18)$ and $G_{4,1}(20)$ which are not on the lines F_2 and L. For plan *(i)*, let K_3 be the plane through the points G_1, $G_{0,0}$, and $G_{1,2}(24)$. Then K_3 has 4 points $G_{1,2}$, $G_{2,2}(25)$, $G_{3,2}(28)$, and $G_{4,2}(29)$ which are not on the plane K. We have thus constructed the following universally optimal plan for a $4^2 \times 2^{10}$ experiment involving 32 runs, whose linear graph is shown below :



For plan *(ii)*, let L_0 be the line consisting of the points $G_{0,0}$, $G_{0,1}$, and $G_{0,2}(3)$. Choose $L_{1,3}$ to be the line through the points $G'_{1,1}(24)$ and $G'_{2,1}(25)$ and choose $L_{2,3}$ to be the line through the points $G'_{1,2}(28)$ and $G'_{2,2}(29)$. We have thus constructed the following universally optimal plan for a $4^2 \times 2^{11}$ experiment involving 32 runs :



□

THEOREM 2.9. *For any prime or prime power m and an integer j, $0 \le j \le m+1$, we can construct a universally optimal plan*
(i) d_1 for an $(m^2)^j \times m^{m^3+3m^2-2j+2}$ experiment involving m^6 runs where

$$d_1 \equiv \{(F_1; G_{1,1}, \ldots, G_{u_1 m^2,1})_2, \ldots, (F_j; G_{1,j}, \ldots, G_{u_j m^2,j})_2,$$

$$(G_1, G_2; \ldots; G_{2m^2-2j+1}, G_{2m^2-2j+2})_1\}, \textit{and} \ \sum_{i=1}^{j} u_i = m+1.$$

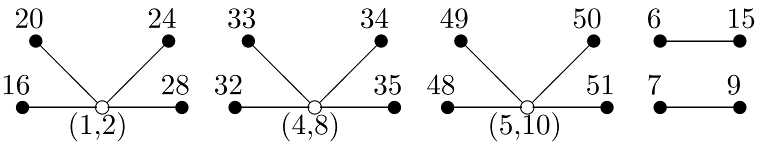
(ii) d_2 *for an* $(m^2)^2 \times m^{m^3+m^2}$ *experiment involving* m^6 *runs where*

$$d_2 \equiv \{(F_1; F_2, G'_{1,1}, \ldots, G'_{jm^2,1})_2, (F_2; G'_{1,2}, \ldots, G'_{(m+1-j)m^2,2})_2\}.$$

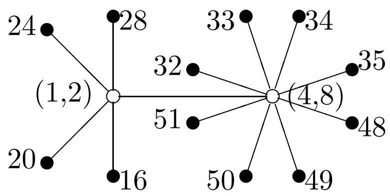
PROOF. Let $F_1, \ldots, F_{m^2+1}$ be m^2+1 lines which partition a 3-flat H in $PG(5,m)$. There are $m+1$ 4-flats through the 3-flat H in $PG(5,m)$, say $M_1, \ldots, M_{m+1}$. To obtain plan *(i)*, for $i = 1, \ldots, j$ and $v = 1, \ldots, u_i$, let $K_{(v-1)m^2-1,i}, \ldots, K_{vm^2,i}$ be the m^2 planes in the 4-flat M_i which pass through the line F_i but are not in the 3-flat H. For $t = 1, \ldots, m^2$, choose $G_{(v-1)m^2+t,i}$ to be a point on the plane $K_{(v-1)m^2+t,i}$ but not on the line F_i. For $i = 1, \ldots, m^2 - j + 1$, choose G_{2i-1} and G_{2i} to be two distinct points on the line F_{j+i}.

To obtain plan *(ii)*, for $i = 1, \ldots, j$, let $K'_{(i-1)m^2+1,1}, \ldots, K'_{im^2,1}$ be the m^2 planes in the 4-flat M_i which pass through the line F_1 but are not in the 3-flat H. For $t = 1, \ldots, m^2$, choose $G'_{(i-1)m^2+t,1}$ to be a point on the plane $K'_{(1-1)m^2+t,1}$ but not on the line F_1. For $i = 1, \ldots, m+1-j$, let $K'_{(i-1)m^2+1,2}, \ldots, K'_{im^2,2}$ be the m^2 planes in the 4-flat M_{j+i} which pass through the line F_2 but are not in the 3-flat H. For $t = 1, \ldots, m^2$, choose $G'_{(i-1)m^2+t,2}$ to be a point on the plane $K'_{(1-1)m^2+t,2}$ but not on the line F_2. □

EXAMPLE 2.5. (i) With $m = 2, j = 3, u_1 = u_2 = u_3 = 1$ in Theorem 2.9 *(i)*, we obtain the following universally optimal plan for a $4^3 \times 2^{16}$ experiment involving 64 runs :



(ii) With $m = 2, j = 1$ in Theorem 2.9 *(ii)*, we obtain the following universally optimal plan for a $4^2 \times 2^{12}$ experiment involving 64 runs :



□

THEOREM 2.10. *For any prime or prime power* m*, we can construct a universally optimal plan* d *for an* $(m^2)^{m^2+m} \times m^{m^4-m^2+m+1}$ *experiment involving* m^6 *runs where*

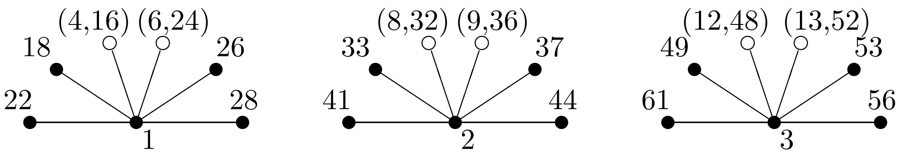
$$d \equiv \{(G_{0,1}; F_{1,1}, \ldots, F_{m,1}, G_{1,1}, \ldots, G_{m^3-m^2,1})_2, \ldots,$$

$$(G_{0,m+1}; F_{1,m+1}, \ldots, F_{m,m+1}, G_{1,m+1}, \ldots, G_{m^3-m^2,m+1})_2\}.$$

PROOF. Let L be a line in a 3-flat H in $PG(5,m)$, and let $G_{0,1}, \ldots, G_{0,m+1}$ be the $m+1$ points on L. There are $m+1$ planes through the line L in the 3-flat H, say $K_1, \ldots, K_{m+1}$. For

$i = 1, \ldots, m+1$, let L_i be a line on the plane K_i which does not pass through the point $G_{0,1}$, and let $P_{1,i}, \ldots, P_{m,i}$ be the m points on L_i but not on L. Let $M_1, \ldots, M_{m+1}$ be the $m+1$ 4-flats through the 3-flat H. For $i = 1, \ldots, m+1$, let $H_{1,i}, \ldots, H_{m,i}$, and H be the $m+1$ 3-flats through the plane K_i in the 4-flat M_i. For $j = 1, \ldots, m$, choose $F_{j,i}$ to be a line through the point $P_{j,i}$ but not on the plane K_i in the 3-flat $H_{j,i}$. Let $K_{j,i}$ be the plane through the lines $F_{j,i}$ and L_i. Choose $G_{(j-1)(m^2-m)+1,i}, \ldots, G_{j(m^2-m),i}$ to be the $m^2 - m$ points on the plane $K_{j,i}$ but not on the lines $F_{j,i}$ and L_i. □

EXAMPLE 2.6. With $m = 2$ in Theorem 2.10, we obtain the following universally optimal plan for a $4^6 \times 2^{15}$ experiment involving 64 runs :



□

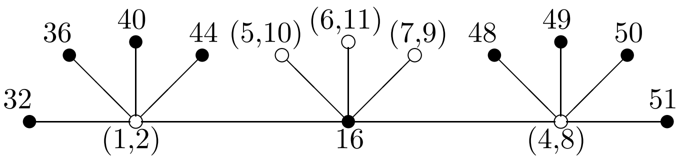
THEOREM 2.11. *For any prime or prime power m and integer j, $1 \leq j \leq m$, we can construct a universally optimal saturated plan d for an $(m^2)^{m^2+1} \times m^{m^3+1}$ experiment involving m^6 runs where*

$$d \equiv \{(G_0; F_1, \ldots, F_{m^2+1})_2, (F_1; G_{1,1}, \ldots, G_{u_1 m^2,1})_2, \ldots,$$

$$(F_j; G_{1,j}, \ldots, G_{u_j m^2,j})_2\}, \textit{and} \quad \sum_{i=1}^{j} u_i = m.$$

PROOF. Let $F_1, \ldots, F_{m^2+1}$ be m^2+1 lines which partition a 3-flat H in $PG(5, m)$. There are $m+1$ 4-flats through the 3-flat H in $PG(5, m)$, say $M_0, \ldots, M_m$. Choose G_0 to be a point in the 4-flat M_0 but not in the 3-flat H. For $i = 1, \ldots, j$ and $v = 1, \ldots, u_i$, let $K_{(v-1)m^2+1,i}, \ldots, K_{vm^2,i}$ be the m^2 planes in the 4-flat $M_{u_1+\cdots+u_{i-1}+v}$ through the line F_i but not in the 3-flat H. For $t = 1, \ldots, m^2$, choose $G_{(v-1)m^2+t,i}$ to be a point on the plane $K_{(v-1)m^2+t,i}$ but not on the line F_i. □

EXAMPLE 2.7. With $m = j = 2$, $u_1 = u_2 = 1$ in Theorem 2.11, we obtain the following universally optimal plan for a $4^5 \times 2^9$ experiment involving 64 runs :



□

REMARK. The plans constructed in this section have some factors at m^2 levels and the others at m levels, where m is a prime or a prime power. In principle, the methods described so far can be extended to obtain optimal plans for experiments of the type $(m^{n_1}) \times \cdots \times (m^{n_u})$ in m^r runs where the $\{n_i\}$ and r are integers. However, such plans generally have too many levels and runs to be attractive to the experimenters. In view of this, we do not report these.

3. Some more optimal plans for asymmetric experiments. The plans obtained in the previous section are such that the number of levels for each of the factors as also the number of runs is a power of m, which itself is a prime or a prime power. Such plans however are somewhat restrictive in nature in the sense that : (i) except for $m = 2$, the number of levels as also the number of runs generally become too large to be attractive to experimenters and, (ii) the methods cannot be used for obtaining optimal plans for experiments in which the number of levels of the factors and the number of runs are *not* powers of the same prime; for example, the methods described in the previous section cannot produce optimal plans for the practically important experiments of the type $3^{n_1} \times 2^{n_2}$. In this section, we attempt to produce optimal plans for asymmetric experiments where the number of levels of different factors as also the number of runs are *not* necessarily powers of the same prime. We make use of orthogonal arrays in constructing such plans.

Recall that an orthogonal array $OA(N, n, m_1 \times \cdots \times m_n, g)$, having N rows, n columns, $m_1, \ldots, m_n (\geq 2)$ symbols and strength $g (< n)$, is an $N \times n$ matrix with elements in the ith column from a set of m_i distinct symbols $(1 \leq i \leq n)$, in which all possible combinations of symbols appear equally often as rows in every $N \times g$ submatrix. If $m_1 = \cdots = m_n = m$, then we have a *symmetric* orthogonal array, which will be denoted by $OA(N, n, m, g)$.

In what follows, we give a simple, yet powerful method of constructing plans for asymmetric factorials that are universally optimal under a model which includes the mean, all main effects and a specified set of two-factor interactions. Consider an orthogonal array $OA(N, n, m_1 \times \cdots \times m_n, 2)$ of strength two, say A, and suppose for $1 \leq j \leq n$, $m_j = t_{j1} t_{j2} \ldots t_{jk_j}$, where $t_{ji} \geq 2$, $1 \leq i \leq k_j$ are integers. Replace the m_j-symbol column in A by k_j columns, say $F_{j1}, F_{j2}, \ldots, F_{jk_j}$, having $t_{j1}, t_{j2}, \ldots, t_{jk_j}$ symbols respectively and call the derived array B. It is not hard to see that B is an $OA(N, \sum_{j=1}^{n} k_j, \prod_{j=1}^{n} \prod_{u=1}^{k_j} t_{ju}, 2)$. We then have the following result.

Theorem 3.1. *The fractional factorial plan d represented by the orthogonal array B is universally optimal in the class of all N-run plans under a model that includes the mean, all main effects and the two-factor interactions $F_{ji} F_{ji'}$, $1 \leq i < i' \leq k_j, 1 \leq j \leq n$.*

Proof. For the sake of simplicity, we consider the case $n = 2$; the proof for $n > 2$ follows on similar lines. Let F_1 and F_2 represent the columns (factors) having m_1 and m_2 symbols (levels) respectively. For $j = 1, 2$, let $m_j = t_{j1} t_{j2} \ldots t_{jk_j}$ and let F_j be replaced by k_j columns (factors) $F_{j1}, \ldots, F_{jk_j}$ with $t_{j1}, \ldots, t_{jk_j}$ symbols (levels) respectively. From Theorem 2.1, the plan d is universally optimal under the stated model if the combinations of the levels of the following sets of factors occur equally often in d :

$$\begin{aligned}
(F_{ji}, F_{ji'}), &\qquad 1 \leq i < i' \leq k_j,\ j = 1, 2; \\
(F_{1i_1}, F_{2i_2}), &\qquad 1 \leq i_1 \leq k_1, 1 \leq i_2 \leq k_2; \\
(F_{ji_1}, F_{ji_2}, F_{ji_3}), &\qquad 1 \leq i_1 < i_2 < i_3 \leq k_j,\ j = 1, 2; \\
(F_{ji_1}, F_{ji_2}, F_{ji_3}, F_{ji_4}), &\qquad 1 \leq i_1 < i_2 < i_3 < i_4 \leq k_j,\ j = 1, 2.
\end{aligned}$$

$$
\begin{aligned}
&(F_{1i}, F_{2i_1}, F_{2i_2}), && 1 \le i \le k_1,\ 1 \le i_1 < i_2 \le k_2;\\
&(F_{2i}, F_{1i_1}, F_{1i_2}), && 1 \le i \le k_2,\ 1 \le i_1 < i_2 \le k_1;\\
&(F_{1i_1}, F_{1i_2}, F_{2j_1}, F_{2j_2}) && 1 \le i_1 < i_2 \le k_1,\ 1 \le j_1 < j_2 \le k_2.
\end{aligned}
$$

From the method of construction of B, the above conditions are clearly satisfied by d and hence the claimed universal optimality of d is established. □

We now give a few examples to illustrate Theorem 3.1.

EXAMPLE 3.1. Consider the orthogonal array $OA(16, 9, 4^3 \times 2^6, 2)$ displayed below (in transposed form) :

$$
\begin{bmatrix}
0000 & 1111 & 2222 & 3333\\
0321 & 3012 & 0312 & 0132\\
2103 & 0321 & 0312 & 1023\\
0011 & 0011 & 1100 & 1010\\
1010 & 1010 & 0110 & 1001\\
0110 & 0110 & 0101 & 1100\\
1100 & 0011 & 1100 & 0101\\
1001 & 1001 & 0101 & 1100\\
1010 & 0101 & 0110 & 0110
\end{bmatrix}'.
$$

Replacing i of the 4-symbol (level) columns by two 2-symbol (level) columns each, we get an $OA(16, 9+i, 4^{3-i} \times 2^{6+2i}, 2)$, $i = 1, 2, 3$. For example, for $i = 1$, we get the following array :

F_1	0000	0000	1111	1111
F_2	0000	1111	0000	1111
	0321	3012	0312	0132
	2103	0321	0312	1023
	0011	0011	1100	1010
	1010	1010	0110	1001
	0110	0110	0101	1100
	1100	0011	1100	0101
	1001	1001	0101	1100
	1010	0101	0110	0110

This array, with columns as runs, represents a 16-run plan for a $4^2 \times 2^8$ experiment that is universally optimal for estimating the mean, all main effects and the two-factor interaction F_1F_2; it is also saturated. □

EXAMPLE 3.2. As a second example, consider an $OA(48, 13, 12 \times 4^{12}, 2)$ (*cf.* Suen (1989)). Replacing the 12-symbol column in this orthogonal array by two columns with 3 and 4 symbols respectively, we get a 48-run saturated plan for a 3×4^{13} experiment, permitting the optimal estimation of the mean, all main effects and a two-factor interaction between the 3-level factor and a 4-level factor. Similarly, replacing the 12-symbol column by two columns having 6 and 2 symbols respectively, one gets a 48-run plan for a $6\times2\times4^{12}$ experiment, permitting the optimal estimation of the two-factor interaction between the 6-level factor and the 2-level factor, apart from the mean and all main effects. Again, replacing the 12-symbol column by three columns having 3, 2 and 2 symbols respectively, one gets a 48-run plan for a $2^2\times3\times4^{12}$ experiment that allows the optimal estimation of all two-factor interactions among the 3-level factor and the 2-level factors, apart from the mean and all main effects. If in the 48-run plan for a $6\times2\times4^{12}$ experiment obtained above, $t(1\le t\le 12)$ of the 4-level factors, say $F_1,\dots,F_t$, are replaced by two 2-level factors each, say $F_i, 1\le i\le t$ being replaced by F_{i1},F_{i2}, then one obtains a plan for a $6\times4^{12-t}\times2^{2t+1}$, $1\le t\le 12$ experiment in 48 runs that is universally optimal for the mean, all main effects, the two factor interaction between the 6-level factor and a 2-level factor (other than $F_1,\dots,F_{12}$) and, the two-factor interactions $F_{i1}F_{i2}, 1\le i\le t, 1\le t\le 12$. The plan is clearly saturated. □

Such examples can obviously be multiplied by referring to the vast literature on orthogonal arrays of strength two; see e.g., Hedayat, Solane and Stufken (1999). All these orthogonal arrays can be used in conjunction with Theorem 3.1 to yield a large number of universally optimal plans under the stated model. Obviously, the method of Theorem 3.1 also applies to the situation where all the factors have levels that are powers of the same prime.

Next, suppose there exists a universally optimal plan d^* for an $m_1\times\cdots\times m_n$ factorial in N/t runs, where $N,t\ge 2$ are integers and, in the notation of Section 2,

$$d^*\equiv(G_1;G_2,\dots,G_n)_2,$$

the factor G_1 being at m_1 levels and for $2\le i\le n$, G_i is at m_i levels. Let the treatment combinations of d^* be represented by an $(N/t)\times n$ matrix A. Let B be an orthogonal array $OA(t,m,s_1\times\cdots\times s_u,2)$ of strength two. Form N treatment combinations of an $s_1\times\cdots\times s_u\times m_1\times\cdots\times m_n$ factorial as

$$[B\otimes\mathbf{1}_{N/t}\vdots\mathbf{1}_t\otimes A],$$

where for a pair of matrices E,F, $E\otimes F$ denotes their Kronecker (tensor) product. Let d be the plan represented by these N treatment combinations. Furthermore, for $1\le i\le u$, let F_i denote the factor at s_i levels. Then, one can prove the following result.

Theorem 3.2. *The N treatment combinations forming the fractional factorial plan d is universally optimal for estimating the mean, all main effects and the interactions F_iG_j; $1\le i\le u, 1\le j\le n$ and G_1G_j, $2\le j\le n$.*

Proof. From Theorem 2.1, the plan d is universally optimal within the relevant class of competing plans under the stated model if the combinations of the levels of the following sets of factors occur equally often in d:

$$\begin{aligned}&(F_i,F_{i'}), && 1\le i<i'\le u;\\ &(G_j,G_{j'}), && 1\le j<j'\le n;\\ &(F_i,G_j), && 1\le i\le u, 1\le j\le n;\end{aligned}$$

$$
\begin{array}{rc}
(F_i, F_{i'}, G_j), & 1 \leq i < i' \leq u, 1 \leq j \leq n; \\
(F_i, G_j, G_{j'}), & 1 \leq i \leq u, 1 \leq j < j' \leq n; \\
(F_i, F_{i'}, G_j, G_{j'}), & 1 \leq i < i' \leq u, 1 \leq j < j' \leq n; \\
(G_1, G_j, G_{j'}), & 2 \leq j < j' \leq n; \\
(F_i, G_1, G_j, G_{j'}), & 1 \leq i \leq u, 2 \leq j < j' \leq n.
\end{array}
$$

Clearly, from the method of construction of d, the above conditions hold and the claimed universal optimality of d is established. □

EXAMPLE 3.3. Let $N = 48,\ t = 4$ in Theorem 3.2. Consider the following 12-run plan for a 3×2^3 experiment which is universally optimal for the estimation of the mean, all main effects and the two-factor interactions $G_1G_j,\ 2 \leq j \leq 4$, where G_1 is at 3 levels while G_2, G_3, G_4 are at 2 levels each; columns are runs :

$$
A' = \begin{bmatrix} 0000 & 1111 & 2222 \\ 0011 & 0011 & 0011 \\ 0101 & 0101 & 0101 \\ 0110 & 0110 & 0110 \end{bmatrix}.
$$

Also, let B be a symmetric orthogonal array $OA(4, 3, 2, 2)$. Then, following Theorem 3.2, we get a 48-run plan for a 3×2^6 experiment, shown below :

$$
\begin{array}{ccccc}
F_1 & 00\cdots0 & 00\cdots0 & 11\cdots1 & 11\cdots1 \\
F_2 & 00\cdots0 & 11\cdots1 & 00\cdots0 & 11\cdots1 \\
F_3 & 00\cdots0 & 11\cdots1 & 11\cdots1 & 00\cdots0 \\
 & A' & A' & A' & A'
\end{array}
$$

This plan in universally optimal under a model that includes the mean, all main effects and the two-factor interactions $F_iG_j,\ 1 \leq i \leq 3, 1 \leq j \leq 4$ and $G_1G_k,\ 2 \leq k \leq 4$.

REMARK. If one chooses A to be an orthogonal array of strength three, then obviously the conditions required for d to be universally optimal under the stated model are satisfied. Furthermore, the plan d of Theorem 3.2, apart from being universally optimal for estimating the mean, all main effects and the specified two-factor interactions, is also optimal when the three-factor interactions $F_iG_1G_j,\ 1 \leq i \leq u, 2 \leq j \leq n$ are also in the model, under the assumption that all other factorial effects are negligible in magnitude.

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