Random Oriented Trees: A Model of Drainage Networks

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Abstract Consider the d-dimensional lattice $\mathbb{Z}^d$ where each vertex is 'open' or 'closed' with probability p or $1-p$ respectively. An open vertex v is connected by an edge to the closest open vertex w such that the dth co-ordinates of v and w satisfy $w(d) = v(d) - 1$. In case of non-uniqueness of such a vertex w, we choose any one of the closest vertices with equal probability and independently of the other random mechanisms. It is shown that this random graph is a tree almost surely for $d = 2$ and it is a infinite collection of distinct trees for $d \geq 4$. In addition, for any dimension, we obtain central limit theorems of (a) the number of vertices of a fixed degree ν and (b) of the number of edges of a fixed length l. These results are obtained by using the martingale convergence theorem and a coupling of the process with independent random walks.

Keywords: Random graph, Martingale, Random walk, Central limit theorem.
AMS Classification: 05C80, 60K35

1 Introduction

Leopold and Langbein [1962] introduced a geometric model of natural drainage network which they described as "*using a sheet of rectangular cross-section graph paper, each square is presumed to represent a unit area. Each square is to be drained, but the drainage channel from each square has equal chance of leading off in any of the four cardinal directions, subject only to the condition that, having made a choice, flow in the reverse direction is not possible. Under these conditions it is possible for one or more streams to flow into a unit area, but only one can flow out*". Subsequently Scheidegger [1967] introduced a direction of flow. In his study of Alpine valleys, he imposed conditions on the Leopold and Langbein model by requiring that the drainage paths be in the "*direction of high gradients between watershed and main valleys*". Thus the drainage forms an oriented network, with a square emptying to one of its two neighbours in a preferred direction. Howard [1971] removed the restriction of drainage to a neighbouring square and modelled a network to include "*headward growth and branching in a random fashion*". Rodriguez-Iturbe and Rinaldo [1997] presents a survey of the development of this field.

The random graph we study here follows the one described by Howard [1971] with the caveat that a stream is not permitted to terminate or become inactive. Thus we consider the d-dimensional lattice $\mathbb{Z}^d$ where each vertex is 'open' or 'closed' with probability p or $1-p$ respectively. The open vertices representing the water sources. An open vertex v is connected by an edge to the closest open vertex w such that the dth co-ordinates of v and w satisfy $w(d) = v(d) - 1$. In case of non-uniqueness of such a vertex w, we choose any one of the closest vertices with equal probability and independently of the other random mechanisms. These edges represent the channels of flow in the drainage network.

Our main result is that for $d = 2$, there is one single main river, with all other rivers being tributaries of this river. While for $d \geq 4$, there are infinitely many rivers, with each having its own distinct set of tributaries. Unfortunately, we do not have any result for 3 dimensions. In addition, for any dimension, we obtain central limit theorems of (a) the number of sites where a fixed number ν of tributaries drain, as well as of (b) the number of channels of a fixed length l.

In 2-dimensions we obtain the main result by showing that the distance between two streams starting at two different sites forms a martingale and thereby invoking the martingale convergence theorem. While in 4 or higher dimensions we couple the streams starting at two different sites with two independent and identically distributed random walks starting at these two sites. The limit theorems are obtained by checking that the random processes satisfy the conditions needed to apply Lyapunov's central limit theorem.

The formal details of the model and the statements of results are given in the next section.

2 The model and statement of results

Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$ and $\mathcal{F}$ the σ algebra generated by finite dimensional cylinder sets. On $(\Omega, \mathcal{F})$ we assign a product probability measure P_p which is defined by its marginals as

$$P_p\{\omega : \omega(u) = 1\} = 1 - P_p\{\omega : \omega(u) = 0\} = p, \text{ for } u \in \mathbb{Z}^d \text{ and } 0 \leq p \leq 1.$$

Let $\{U_{u,v} : u, v \in \mathbb{Z}^d,\ v(d) = u(d) - 1\}$ be i.i.d. uniform $(0,1]$ random variables on some probability space $(\Xi, \mathcal{G}, \mu)$. Here and subsequently we express the co-ordinates of a vector u as $u = (u(1), \dots, u(d))$.

Consider the product space $(\Omega \times \Xi, \mathcal{F} \times \mathcal{G}, \mathbb{P} := P_p \times \mu)$. For $(\omega, \xi) \in \Omega \times \Xi$ let $\mathcal{V}(= \mathcal{V}(\omega, \xi))$ be the random vertex set defined by

$$\mathcal{V}(\omega, \xi) = \{u \in \mathbb{Z}^d : \omega(u) = 1\}.$$

Note that if $u \in \mathcal{V}(\omega, \xi)$ for some $\xi \in \Xi$ then $u \in \mathcal{V}(\omega, \xi')$ for all $\xi' \in \Xi$ and thus we say that a vertex u is open in a configuration ω if $u \in \mathcal{V}(\omega, \xi)$ for some $\xi \in \Xi$.

For $u \in \mathbb{Z}^d$ let

$$N_u = N_u(\omega, \xi) = \Big\{v \in \mathcal{V}(\omega, \xi) : v(d) = u(d) - 1 \text{ and } \sum_{i=1}^{d} |v(i) - u(i)| = \\ \min\{\textstyle\sum_{i=1}^{d} |w(i) - u(i)| : w \in \mathcal{V}(\omega, \xi), w(d) = u(d) - 1\}\Big\}.$$

Note that for $p > 0$, N_u is non-empty almost surely and that N_u is defined for all u, irrespective of it being open or closed. For $u \in \mathbb{Z}^d$ let

$$h(u) \in N_u(\omega, \xi) \text{ be such that } U_{u,h(u)}(\xi) = \min\{U_{u,v}(\xi) : v \in N_u(\omega, \xi)\}. \tag{1}$$

Again note that for $p > 0$ and for each $u \in \mathbb{Z}^d$, $h(u)$ is open, almost surely unique and $h(u)(d) = u(d)-1$. On $\mathcal{V}(\omega, \xi)$ we assign the edge set $\mathcal{E} = \mathcal{E}(\omega, \xi) := \{< u, h(u) >: u \in \mathcal{V}(\omega, \xi)\}$.

Consider that graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of the vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. For $p = 0$, $\mathcal{V} = \emptyset$ almost surely, and, for $p = 1$, $< u, v >\in \mathcal{E}$ if and only if $u(i) = v(i)$ for all $i \neq d$ and $|u(d) - v(d)| = 1$. Also, for a vertex $u \in \mathcal{V}(\omega, \xi)$, there is exactly one edge 'going down' from u, i.e., there is a unique edge $< u, v >$ with $v(d) \leq u(d)$; thus the graph $\mathcal{G}$ contains no loops almost surely. Hence, for $0 < p < 1$, the graph $\mathcal{G}$ consists of only trees. Our first result is

Theorem 2.1 *Let $0 < p < 1$. For $d = 2$, $\mathcal{G}$ consists of one single tree $\mathbb{P}$–almost surely; while for $d \geq 4$, $\mathcal{G}$ is a forest consisting of infinitely many disjoint trees $\mathbb{P}$–almost surely.*

Now for $\nu \geq 0$ let S_n be the number of vertices in $\mathcal{V} \cap ([1, n]^d)$ of the graph $\mathcal{G}$ with degree $\nu + 1$. Also, for $l \geq 1$ let L_n be the number of edges of (L_1)length l in the graph $\mathcal{G}$ with one end vertex in $\mathcal{V} \cap ([1, n]^d)$.

Theorem 2.2 *As $n \to \infty$,*
(a) $\frac{S_n - E(S_n)}{n^{d/2}}$ converges weakly to a normal random variable;
(b) $\frac{L_n - E(L_n)}{n^{d/2}}$ converges weakly to a normal random variable.

Finally, for $d = 2$, given that a vertex v is open, the following proposition gives the exact distribution of the degree of v.

Proposition 2.1 *Given that a vertex v is open, the degree of the vertex in the graph $\mathcal{G}$ has the same distribution as that of $1+Y+X_1+X_2$ where Y, X_1 and X_2 are independent non-negative random variables such that*

$$Y = \begin{cases} 0 & \textit{with probability } 1-p \\ 1 & \textit{with probability } p \end{cases}$$

$$P(X_1 \geq r) = P(X_2 \geq r) = \begin{cases} 1 & \textit{for } r = 0 \\ \frac{(1-p)^{2r-1}(2-p)}{2(3-3p+p^2)^r} & \textit{for } r \geq 1. \end{cases}$$

Thus the expected degree of a vertex, given that it is open, is 2.

Remark: As in Lemma 7 of Aldous and Steele [1992], using the ergodicity of the process it may be shown that in any dimension expected degree of a vertex, given that it is open, is 2.

3 Proof of Theorem 2.1

We fix $0 < p < 1$ and consider the 2-dimensional case first. Consider the graph $\mathcal{H}$ with vertices $\mathcal{V} \cup \{(k,0) : k \in \mathbb{Z}\}$ and edges $\mathcal{E} \cup \{< (k,0), h(k,0) >: k \in \mathbb{Z}\}$. Note that in the graph $\mathcal{H}$ vertices which are not open (i.e. vertices which are not in $\mathcal{V}$) are of degree 1. Clearly, if T_1 and T_2 are two disjoint maximal connected subgraphs of $\mathcal{G}$ then they are subgraphs of two disjoint maximal connected subgraphs of $\mathcal{H}$. Moreover if T_1 and T_2 are two disjoint maximal connected subgraphs of $\mathcal{H}$ then $T_1 \setminus \{< (k,0), h(k,0) >: k \in \mathbb{Z}, (k,0) \notin \mathcal{V}\}$ and $T_2 \setminus \{< (k,0), h(k,0) >: k \in \mathbb{Z}, (k,0) \notin \mathcal{V}\}$ are two disjoint maximal connected subgraphs of $\mathcal{G}$. Thus it suffices to show that the subgraph of $\mathcal{H}$ containing $(0,0)$ and any other point $(k,0)$ (say) is connected.

Now let $(X_j^n, -n) := h^n(j,0)$ for $j \in \mathbb{Z}$, where h^n denotes the n-fold composition of h defined in (1). Observe that for $j < k$, $X_j^n \leq X_k^n$ for every $n \geq 1$. Fix i, j with $i < j$ and let $Z_n := X_j^n - X_i^n$ with $Z_0 = j - i$. Since Z_n depends only on the configuration on the line $\{(k,-n) : k \in \mathbb{Z}\}$ and the value of Z_{n-1}, $\{Z_n, n \geq 0\}$ is a Markov chain.

For $m \geq 1$ and $k = 2m$, it is easy to see that there are two possible ways to obtain $X_1^1 - X_0^1 = k$, *viz.* (see Figure 1)

(a) all vertices between $(-m+1,-1)$ and $(m-1,-1)$ are closed, both vertices $(m,-1)$ and $(-m,-1)$ are open and $U_{(0,0),(-m,-1)} < U_{(0,0),(m,-1)}$,

(b) all vertices between $(-m+2,-1)$ and $(m,-1)$ are closed, both vertices $(-m+1,-1)$ and $(m+1,-1)$ are open and $U_{(1,0),(m+1,-1)} < U_{(1,0),(-m+1,-1)}$;

while, for $k = 2m+1$, the only possible way to obtain $X_1^1 - X_0^1 = k$ is (see Figure 1):

(c) all vertices between $(-m+1,-1)$ and $(m,-1)$ are closed, both vertices $(m,-1)$ and $(-m,-1)$ are open.

Now an easy calculation yields,

$$\mathbb{P}(X_1^1 - X_0^1 = k) = \begin{cases} p & \text{if } k = 0 \\ p^2(1-p)^{k-1} & \text{if } k \geq 1 \end{cases}$$

and so we have, $\mathbb{E}(X_1^1 - X_0^1) = 1$.

By the translation invariance of the model,

$$\mathbb{E}(X_j^1 - X_i^1) = \sum_{k=0}^{j-i-1} \mathbb{E}(X_{i+k+1}^1 - X_{i+k}^1) = (j-i)\mathbb{E}(X_1^1 - X_0^1) = j - i. \tag{2}$$

Also $\mathbb{E}(Z_n | Z_{n-1} = k) = k$, so $\{Z_n, n \geq 0\}$ is a martingale. Since it is a non-negative martingale, by the martingale convergence theorem (see Theorem 35.4, Billingsley [1979] pg. 416), Z_n converges almost surely as $n \to \infty$. Let Z_∞ denote this almost sure limit. We prove the first part of Theorem 2.1 by showing that $Z_\infty = 0$ with probability 1.

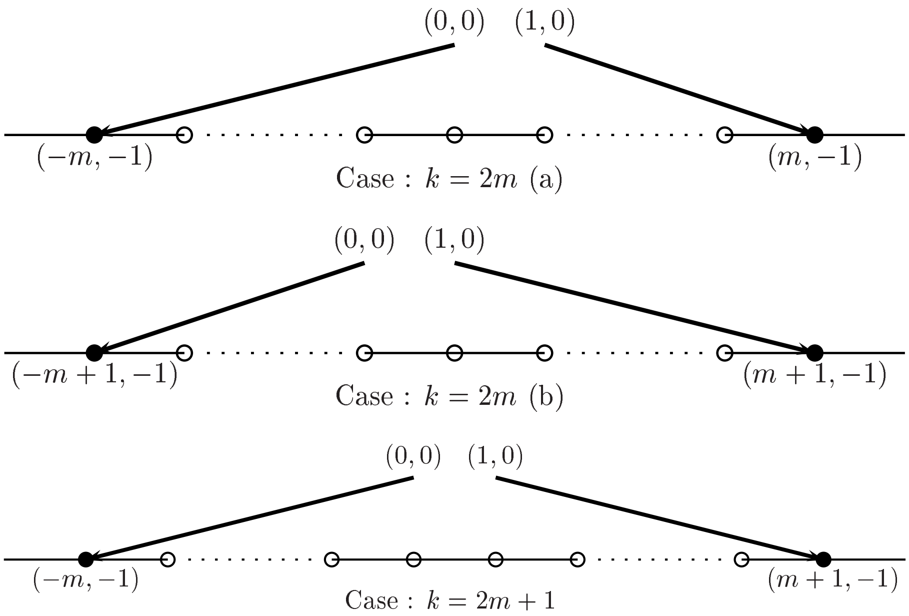

Figure 1: The dark coloured circled points are open and the other circled points are closed.

To this end, for $k, l \geq 0$, let $p_{k,l} := \mathbb{P}(Z_n = l | Z_{n-1} = k)$ denote the 1-step transition probabilities of Z_n. We first calculate $p_{k,0}$. Clearly, $p_{0,0} = 1$. Now for $k \geq 1$, the event $\{h(0,0) = h(k,0) = (m,-1)\}$ occurs if and only if

(a) for $m \leq 0$, all vertices between $(m-1,-1)$ and $(2k-m-1,-1)$ are closed, $(m,-1)$ is open and, either $(2k-m,-1)$ is closed, or if $(2k-m,-1)$ is open then $U_{(k,0),(m,-1)} < U_{(k,0),(2k-m,-1)}$;

(b) for $m \geq k$, all vertices between $(m-1,-1)$ and $(2k-m-1,-1)$ are closed, $(2k-m,-1)$ is open and, either $(m,-1)$ is closed, or if $(m,-1)$ is open then $U_{(k,0),(2k-m,-1)} < U_{(k,0),(m,-1)}$;

(c) for $1 \leq m \leq k-1$, all vertices between $(-m+1,-1)$ and $(2k-m-1,-1)$ except $(m,-1)$ are closed, $(m,-1)$ is open and, either both $(-m,-1)$ and $(2k-m,-1)$ are closed, or if $(-m,-1)$ [respectively, $(2k-m,-1)$] is open then $U_{(0,0),(m,-1)} < U_{(0,0),(-m,-1)}$ [respectively, $U_{(k,0),(m,-1)} < U_{(k,0),(2k-m,-1)}$].

Figure 2 illustrates all these three cases.

Thus, for $k \geq 1$, taking $q = 1-p$ we have

$$
\begin{aligned}
p_{k,0} &= 2\sum_{l=0}^{\infty} pq^{2(l+k)-1}(q+p/2) + \sum_{l=1}^{k-1} pq^{2k-1}(q^2+pq+p^2/4) \\
&= 2pq^{2k-1}\frac{q+p/2}{1-q^2} + (k-1)pq^{2(k-1}(q^2+pq+p^2/4). \qquad (3)
\end{aligned}
$$

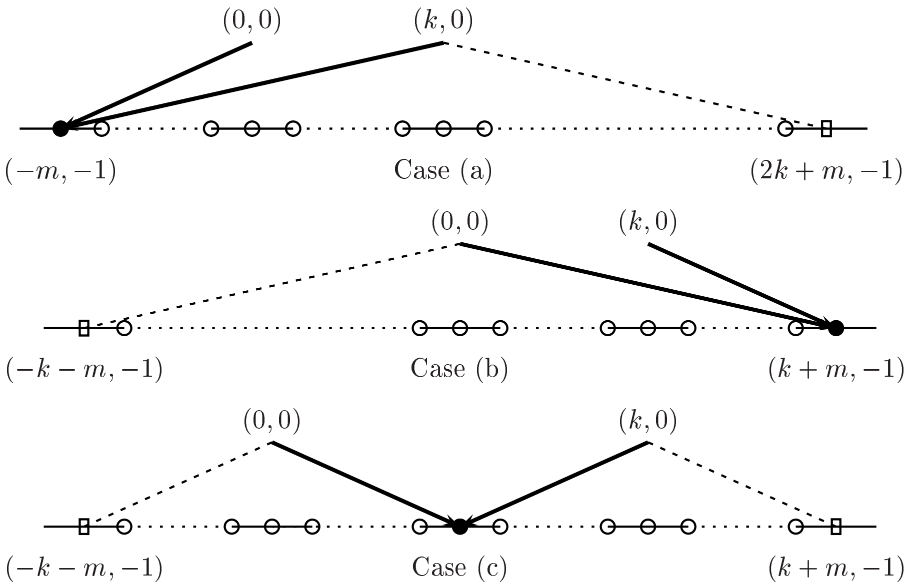

Figure 2: The solid points are open, the other circled points are closed and the points at the edges of each figure depicted by squares may be open or closed.

Now

$$\begin{aligned}\mathbb{P}(Z_n = 0) &= \sum_{k=0}^{\infty} p_{k,0}\mathbb{P}(Z_{n-1} = k) \\ &= p_{0,0}\mathbb{P}(Z_{n-1} = 0) + \sum_{k=1}^{\infty} p_{k,0}\mathbb{P}(Z_{n-1} = k). \qquad (4)\end{aligned}$$

As $n \to \infty$, Z_n converges almost surely to Z_∞ and thus Z_n also converges in distribution to Z_∞. Moreover, from (3) we see that $\sum_{k=1}^{\infty} p_{k,0}\mathbb{P}(Z_{n-1} = k) \leq \sum_{k=1}^{\infty} p_{k,0} < \infty$. Thus taking limits on both sides of (4), as $n \to \infty$, we have

$$\mathbb{P}(Z_\infty = 0) = \mathbb{P}(Z_\infty = 0) + \sum_{k=1}^{\infty} p_{k,0}\mathbb{P}(Z_\infty = k). \qquad (5)$$

Since $p_{k,0} \neq 0$ for $k \geq 1$, we have our claim, viz., $Z_\infty = 0$ with probability 1.

For the second part of Theorem 2.1 we first show that on $\mathbb{Z}^4$ the graph admits two distinct trees with positive probability, i.e.

$$\mathbb{P}\{\mathcal{G} \text{ is disconnected}\} > 0. \qquad (6)$$

Consider a random vector $X \in \mathbb{Z}^3$ defined as follows: for $k \geq 0$ and $||\cdot||$ being the L_1 norm, let $D_k := \{\mathfrak{v} \in \mathbb{Z}^3 : ||\mathfrak{v}|| \leq k\}$ denote the 'diamond' of radius k and let $\delta D_k := \{\mathfrak{v} \in \mathbb{Z}^3 : ||\mathfrak{v}|| = k\}$ denote its boundary. The distribution of the random vector X is given by

$$\mathbb{P}(X = \mathfrak{v}) = \begin{cases} p & \text{if } \mathfrak{v} = \mathfrak{o} \\ \frac{(1-p)^{\#D_{k-1}}(1-(1-p)^{\#\delta D_k})}{\#\delta D_k} & \text{for } \mathfrak{v} \in \delta D_k,\ k \geq 1 \end{cases} \qquad (7)$$

where $\mathfrak{o} = (0,0,0)$ and $\#A$ denotes the cardinality of the set A. It may easily be checked that $\sum_{\mathfrak{v}\in\mathbb{Z}^3} P(X = \mathfrak{v}) = 1$.

Next, for a fixed vector $\mathbf{u} = (u(1), \dots, u(4)) \in \mathbb{Z}^4$ consider the graph $\mathcal{H} = (\mathcal{V} \cup \{\mathbf{u}\}, \mathcal{E} \cup \{< \mathbf{u}, h(\mathbf{u}) >\})$. For $n \geq 0$, let $h^n(\mathbf{u}) := (g^n(\mathbf{u}), t)$ for $g^n(\mathbf{u}) \in \mathbb{Z}^3$ and $t = u(4) - n \in \mathbb{Z}$. Here we take $h^0(\mathbf{u}) = \mathbf{u}$. Observe that for fixed $\mathbf{u}$, $g^n(\mathbf{u})$ has the same distribution as $(u(1), u(2), u(3)) + \sum_{i=1}^n X_i$, where $X_1, X_2, \dots$ are i.i.d. copies of X. Hence $\{g^n(\mathbf{u}) : n \geq 0\}$ is a symmetric random walk starting at $g^0(\mathbf{u}) = (u(1), u(2), u(3))$, with i.i.d. steps, each step size having distribution X. However, for $\mathbf{v} \in \mathbb{Z}^4$ with $v(4) = u(4)$, in the graph $(\mathcal{V} \cup \{\mathbf{u}, \mathbf{v}\}, \mathcal{E} \cup \{< \mathbf{u}, h(\mathbf{u}) >, < \mathbf{v}, h(\mathbf{v}) >\})$ the processes $\{g^n(\mathbf{u})\}_{n\geq 0}$ and $\{g^n(\mathbf{v})\}_{n\geq 0}$ are not independent and so we cannot use the fact that two independent random walks on $\mathbb{Z}^3$ do not intersect with positive probability to obtain our theorem. Nonetheless, if $\mathbf{u}$ and $\mathbf{v}$ are sufficiently far apart their dependence on each other is sufficiently weak. In the remainder of this section we formalize this notion of weak dependence by coupling two independent random walks and the processes $\{g^n(\mathbf{u}), g^n(\mathbf{v}) : n \geq 0\}$ and obtain the desired result.

For $\mathbf{v} = (v(1), v(2), v(3), 0)$, given $\epsilon > 0$ define the event

$$\begin{aligned} A_{n,\epsilon}(\mathbf{v}) \quad := \quad & \{g^{n^4}(\mathbf{v}) \in g^{n^4}(\mathbf{0}) + (D_{n^{2(1+\epsilon)}} \setminus D_{n^{2(1-\epsilon)}}), \\ & \qquad g^i(\mathbf{v}) \neq g^i(\mathbf{0}) \text{ for all } i = 1, \dots, n^4\}, \end{aligned} \tag{8}$$

where $\mathbf{0} := (0,0,0,0)$.

Lemma 3.1 *For $0 < \epsilon < 1/3$ there exist constants $C, \beta > 0$ and $n_0 \geq 1$ such that, for all $n \geq n_0$,*

$$\inf_{g^0(\mathbf{v}) \in D_{n^{1+\epsilon}} \setminus D_{n^{1-\epsilon}}} \mathbb{P}(A_{n,\epsilon}(\mathbf{v})) \geq 1 - Cn^{-\beta}.$$

Assuming the above lemma we proceed to complete the proof of (6). We shall return to the proof of the lemma later.

For $i \geq 1$ and $n \geq n_0$, let $\tau_i := 1 + n^4 + (n^4)^2 + \cdots + (n^4)^{2^{i-1}}$. For fixed $\mathbf{v}$, we define

$$B_0 = B_0(\mathbf{v}) := \{g(\mathbf{v}) \in g(\mathbf{0}) + (D_{n^{1+\epsilon}} \setminus D_{n^{1-\epsilon}})\},$$

and having defined $B_0, \dots, B_{i-1}$ we define

$$\begin{aligned} B_i = B_i(\mathbf{v}) \quad := \quad & \{g^{\tau_i}(\mathbf{v}) \in g^{\tau_i}(\mathbf{0}) + (D_{n^{2^i(1+\epsilon)}} \setminus D_{n^{2^i(1-\epsilon)}}) \text{ and} \\ & \qquad g^j(\mathbf{v}) \neq g^j(\mathbf{0}) \text{ for all } \tau_{i-1} + 1 \leq j \leq \tau_i\}. \end{aligned}$$

Clearly,

$$\begin{aligned} \mathbb{P}\{g^j(\mathbf{v}) \neq g^j(\mathbf{0}) \text{ for all } j \geq 1\} \quad & \geq \quad \mathbb{P}(\cap_{i=0}^\infty B_i) \\ & = \quad \lim_{i\to\infty} \mathbb{P}(\cap_{j=0}^i B_j) \\ & = \quad \lim_{i\to\infty} \prod_{l=1}^i \mathbb{P}(B_l | \cap_{j=0}^{l-1} B_j) \mathbb{P}(B_0). \end{aligned} \tag{9}$$

Since $\mathbb{P}(B_0) > 0$, from (9) we have that $\mathbb{P}(g^j(\mathbf{v}) \neq g^j(\mathbf{0})$ for all $j \geq 1) > 0$ if $\sum_{l=1}^{\infty} 1 - \mathbb{P}(B_l | \cap_{j=0}^{l-1} B_j) < \infty$.

For fixed $l \geq 1$, let $\mathbf{u}_1 = h^{\tau_l}(\mathbf{0})$ and $\mathbf{v}_1 = h^{\tau_l}(\mathbf{v})$. Now $\{(h^n(\mathbf{0}), h^n(\mathbf{v})) : n \geq 0\}$ being a Markov process and, for $(\omega, \xi) \in B_l(\mathbf{v})$, since $g^0(\mathbf{v}_1)(\omega,\xi) \in g^0(\mathbf{u}_1)(\omega,\xi) + (D_{n^{2^{l-1}}(1+\epsilon)} \setminus D_{n^{2^{l-1}}(1-\epsilon)})$, we have

$$\begin{aligned}
\mathbb{P}\big(B_{l+1} | \cap_{j=0}^{l} B_j\big) &\geq \inf_1 \mathbb{P}\Big\{ g^{(n^4)^{2^l}}(\mathbf{v}_1) \in g^{(n^4)^{2^l}}(\mathbf{u}_1) + (D_{n^{2^l}(1+\epsilon)} \setminus D_{n^{2^l}(1-\epsilon)}), \\
&\qquad g^k(\mathbf{u}_1) \neq g^k(\mathbf{v}_1) \text{ for all } k = 1, 2, \ldots, (n^4)^{2^l} \Big\} \\
&= \inf_2 \mathbb{P}\big(A_{n^{2^l},\epsilon}(\mathbf{u})\big) \\
&\geq 1 - C(n^{2^l})^{-\beta}, \qquad (10)
\end{aligned}$$

where $\inf_1$ is the infimum over all $\mathbf{u}_1, \mathbf{v}_1 \in \mathbb{Z}^4$ with $g^0(\mathbf{v}_1) \in g^0(\mathbf{u}_1) + (D_{n^{2^{l-1}}(1+\epsilon)} \setminus D_{n^{2^{l-1}}(1-\epsilon)})$ and $\inf_2$ is the infimum over all $\mathbf{u}$ with $g^0(\mathbf{u}) \in (D_{n^{2^{l-1}}(1+\epsilon)} \setminus D_{n^{2^{l-1}}(1-\epsilon)})$ and the last inequality follows from Lemma 3.1. Thus $\sum_{l=1}^{\infty} \Big(1 - \mathbb{P}(B_l | \cap_{j=0}^{l-1} B_j)\Big) \leq C \sum_{l=1}^{\infty} (n^{2^l})^{-\beta} < \infty$; thereby completing the proof (6).

To prove Lemma 3.1, we have to compare the trees $\{h^n(\mathbf{0})\}$ and $\{h^n(\mathbf{v})\}$ and independent "random walks" $\{\mathbf{0} + (\sum_{i=1}^n X_i, -n)\}$ and $\{\mathbf{v} + (\sum_{i=1}^n Y_i, -n)\}$ where $\{X_1, X_2, \ldots\}$ and $\{Y_1, Y_2 \ldots\}$ are independent collections of i.i.d. copies of the random variable X given in (7).

We now describe a method to couple the trees and the independent random walks. Before embarking on the formal details of the coupling procedure we present the main idea. Henceforth, the boldface symbols $\mathbf{u}$, $\mathbf{v}$ denote vectors in $\mathbb{Z}^4$ while the fraktur symbols $\mathfrak{u}$, $\mathfrak{v}$ denote vectors in $\mathbb{Z}^3$.

From a vertex $\mathbf{0}$ we construct the 'path' $\{\mathbf{0} + (\sum_{i=1}^n X_i, -n)\}$. Now consider the vertex $\mathbf{v}$ with $\mathbf{v} = (v_1, v_2, v_3, 0)$. In case the diamond $D = \{\mathfrak{u} \in \mathbb{Z}^3 : ||\mathfrak{u}|| \leq ||X_1||\}$ is disjoint from the diamond $D' = \{\mathfrak{u} \in \mathbb{Z}^3 : ||\mathfrak{u} - (v_1, v_2, v_3)|| \leq ||Y_1||\}$ then we take $h^1(\mathbf{v}) = \{\mathbf{v} + (Y_1, -1)\}$. While if the two diamonds are not disjoint, then we have to define $h^1(\mathbf{v})$ taking into account the configuration inside the diamond D. Similarly, we may obtain $h^2(\mathbf{v})$ by considering the diamonds $\{\mathfrak{u} \in \mathbb{Z}^3 : ||\mathfrak{u} - X_1|| \leq ||X_2||\}$ and $\{\mathfrak{u} \in \mathbb{Z}^3 : ||\mathfrak{u} - g^1(\mathbf{v})|| \leq ||Y_2||\}$. Note that if, for each $i = 1, \ldots, n$ the two diamonds involved in the ith stage are disjoint, then the growth of the tree $\{(h^i(\mathbf{0}), h^i(\mathbf{v})) : 0 \leq i \leq n\}$ is stochastically equivalent to that of the pair of independent 'random walks' $(\mathbf{0} + (\sum_{i=1}^n X_i, -n), \mathbf{v} + (\sum_{i=1}^n Y_i, -n))$.

We start with two vertices $\mathbf{u} = (\mathfrak{u}, 0)$ and $\mathbf{v} = (\mathfrak{v}, 0)$ in $\mathbb{Z}^4$ with $\mathfrak{u}, \mathfrak{v} \in \mathbb{Z}^3$. Let $\{U_1^{\mathfrak{u}}(\mathfrak{z}) : \mathfrak{z} \in \mathbb{Z}^3\}, \{U_2^{\mathfrak{u}}(\mathfrak{z}) : \mathfrak{z} \in \mathbb{Z}^3\}$ and $\{U_1^{\mathfrak{v}}(\mathfrak{z}) : \mathfrak{z} \in \mathbb{Z}^3\}, \{U_2^{\mathfrak{v}}(\mathfrak{z}) : \mathfrak{z} \in \mathbb{Z}^3\}$ be four independent collections of i.i.d. random variables, each of these random variables being uniformly distributed on $[0,1]$.

Let $k_{\mathfrak{u}}$ and $l_{\mathfrak{v}}$ be defined as

$$\begin{aligned}
k_{\mathfrak{u}} &:= \min\{k : U_1^{\mathfrak{u}}(\mathfrak{z}) < p \text{ for some } \mathfrak{z} \in (\mathfrak{u} + D_k)\} \\
l_{\mathfrak{v}} &:= \min\{l : U_1^{\mathfrak{v}}(\mathfrak{z}) < p \text{ for some } \mathfrak{z} \in (\mathfrak{v} + D_l)\}.
\end{aligned}$$

Now define $m_{\mathfrak{v}}$ as

$$\begin{aligned} m_{\mathfrak{v}} &:= \min\{m : \text{ either } U_1^{\mathfrak{v}}(\mathfrak{z}) < p \text{ for some } \mathfrak{z} \in (\mathfrak{v} + D_m) \setminus (\mathfrak{u} + D_{k_{\mathfrak{u}}}) \\ &\qquad \text{or } U_1^{\mathfrak{u}}(\mathfrak{z}) < p \text{ for some } \mathfrak{z} \in (\mathfrak{v} + D_m) \cap (\mathfrak{u} + D_{k_{\mathfrak{u}}})\}. \end{aligned}$$

Also, define the sets

$$\begin{aligned} N_{\mathfrak{u}} &:= \{\mathfrak{z} \in (\mathfrak{u} + D_{k_{\mathfrak{u}}}) : U_1^{\mathfrak{u}}(\mathfrak{z}) < p\} \\ N_{\mathfrak{v}}^1 &:= \{\mathfrak{z} \in (\mathfrak{v} + D_{l_{\mathfrak{v}}}) : U_1^{\mathfrak{v}}(\mathfrak{z}) < p\} \\ N_{\mathfrak{v}}^2 &:= \{\mathfrak{z} \in (\mathfrak{v} + D_{m_{\mathfrak{v}}}) \setminus (\mathfrak{u} + D_{k_{\mathfrak{u}}}) : U_1^{\mathfrak{v}}(\mathfrak{z}) < p\} \\ &\qquad \cup \{\mathfrak{z} \in (\mathfrak{v} + D_{m_{\mathfrak{v}}}) \cap (\mathfrak{u} + D_{k_{\mathfrak{u}}}) : U_1^{\mathfrak{u}}(\mathfrak{z}) < p\}. \end{aligned}$$

We pick

(a) $\phi(\mathfrak{u}) \in N_{\mathfrak{u}}$ such that $U_2^{\mathfrak{u}}(\phi(\mathfrak{u})) = \min\{U_2^{\mathfrak{u}}(\mathfrak{z}) : \mathfrak{z} \in N_{\mathfrak{u}}\}$;

(b) $\zeta(\mathfrak{v}) \in N_{\mathfrak{v}}^1$ such that $U_2^{\mathfrak{v}}(\zeta(\mathfrak{v})) = \min\{U_2^{\mathfrak{v}}(\mathfrak{z}) : \mathfrak{z} \in N_{\mathfrak{v}}^1\}$;

(c) $\psi(\mathfrak{v}) \in N_{\mathfrak{v}}^2$ such that $U_2^{\mathfrak{v}}(\psi(\mathfrak{v})) = \min\{U_2^{\mathfrak{v}}(\mathfrak{z}) : \mathfrak{z} \in N_{\mathfrak{v}}^2\}$.

Taking $\phi^0(\mathfrak{u}) = \mathfrak{u}$, $\phi^n(\mathfrak{u}) = \phi(\phi^{n-1}(\mathfrak{u}))$, and similarly for $\zeta^n(\mathfrak{v})$ and $\psi^n(\mathfrak{v})$, we note that the distribution of $\Big\{\big((\phi^n(\mathfrak{u}), -n), (\zeta^n(\mathfrak{v}), -n)\big) : n \geq 0\Big\}$ is the same as that of $\Big\{\big((\mathfrak{u} + \sum_{i=1}^n X_i, -n), \ (\mathfrak{v} + \sum_{i=1}^n Y_i, -n)\big) : n \geq 0\Big\}$, i.e. two independent "random walks" one starting from $(\mathfrak{u}, 0)$ and the other starting from $(\mathfrak{v}, 0)$. Also the distribution of $\{(h_n(\mathfrak{u}, 0), h_n(\mathfrak{v}, 0)) : n \geq 0\}$ and that of $\{((\phi^n(\mathfrak{u}), -n), (\psi^n(\mathfrak{v}), -n)) : n \geq 0\}$ are identical. Thus, the procedure described above may be used to construct the trees from $(\mathfrak{u}, 0)$ and $(\mathfrak{v}, 0)$.

Now observe that $\{(\phi^n(\mathfrak{u}), -n)\}$ describes both the random walk and the tree starting from $(\mathfrak{u}, 0)$. Also if $D_{k_{\mathfrak{u}}} \cap D_{m_{\mathfrak{v}}} = \emptyset$, then $m_{\mathfrak{v}} = l_{\mathfrak{v}}$ and, more importantly, $\zeta(\mathfrak{v}) = \psi(\mathfrak{v})$. Hence the 'random walk' and the tree from $(\mathfrak{u}, 0)$ are coupled and so are the 'random walk' and the tree from $(\mathfrak{v}, 0)$. In particular, this happens when both $k_{\mathfrak{u}} < [||\mathfrak{u} - \mathfrak{v}||/2]$ and $m_{\mathfrak{v}} < [||\mathfrak{u} - \mathfrak{v}||/2]$. Let $k_0 = ||\mathfrak{u} - \mathfrak{v}||/2$. From the above discussion we have

$$\begin{aligned} &\mathbb{P}\Big(\{\zeta(\mathfrak{v}) \neq \psi(\mathfrak{v})\}\Big) \\ &\leq \mathbb{P}\Big(\{(U_1^{\mathfrak{u}}(\mathfrak{z})) > p \text{ for all } \mathfrak{z} \in (\mathfrak{u} + D_{k_0})\} \\ &\qquad \cup \{(U_1^{\mathfrak{v}}(\mathfrak{z})) > p \text{ for all } \mathfrak{z} \in (\mathfrak{v} + D_{k_0})\}\Big) \\ &= 2\mathbb{P}\Big(\{(U_1^{\mathfrak{u}}(\mathfrak{z})) > p \text{ for all } \mathfrak{z} \in (\mathfrak{u} + D_{k_0})\}\Big) \\ &= 2(1-p)^{\# D_{k_0}}. \end{aligned}$$

Since $(1/2)k^3 \leq \# D_k \leq 2k^3$, the above inequality gives

$$\mathbb{P}(\{\zeta(\mathfrak{v}) = \psi(\mathfrak{v})\}) \geq 1 - C_1 \exp(-C_2 ||\mathfrak{u} - \mathfrak{v}||^3) \tag{11}$$

for constants $C_1 = 2$ and $C_2 = (1/2)|\log(1-p)|$.

With the above estimate at hand, we look at the process $\{(\phi^n(\mathfrak{u}), \zeta^n(\mathfrak{v})) : n \geq 0\}$. Without loss of generality we take $\mathfrak{u} = \mathfrak{o}$. For $\epsilon > 0$ and constant $K > 0$ (to be specified later) define

$$\begin{aligned} B_{n,\epsilon}(\mathfrak{v}) \;:=\; & \{\zeta^{n^4}(\mathfrak{v}) \in \phi^{n^4}(\mathfrak{o}) + (D_{n^{2(1+\epsilon)}} \setminus D_{n^{2(1-\epsilon)}}), \\ & \quad ||\zeta^i(\mathfrak{v}) - \phi^i(\mathfrak{o})|| \geq K \log n \text{ for all } i = 1, \ldots, n^4\}. \end{aligned} \tag{12}$$

This event is an independent random walk version of the event $A_{n,\epsilon}(\mathfrak{v}, 0)$ defined in (8), except that here we require that the two random walks come no closer than $K \log n$ at any stage.

We will show that there exists $\alpha > 0$ such that

$$\sup_{\mathfrak{v} \in (D_{n^{(1+\epsilon)}} \setminus D_{n^{(1-\epsilon)}})} \mathbb{P}\big((B_{n,\epsilon}(\mathfrak{v}))^c\big) < C_3 n^{-\alpha} \tag{13}$$

for some constant $C_3 > 0$.

Since $(B_{n,\epsilon}(\mathfrak{v}))^c \subseteq E_{n,\epsilon}(\mathfrak{v}) \cup F_{n,\epsilon}(\mathfrak{v}) \cup G_{n,\epsilon}(\mathfrak{v})$ where

$$\begin{aligned} E_{n,\epsilon}(\mathfrak{v}) &:= \big\{||\zeta^i(\mathfrak{v}) - \phi^i(\mathfrak{o})|| \leq K \log n \text{ for some } i = 1, \ldots, n^4\big\}, \\ F_{n,\epsilon}(\mathfrak{v}) &:= \big\{\zeta^{n^4}(\mathfrak{v}) \notin \phi^{n^4}(\mathfrak{o}) + D_{n^{2(1+\epsilon)}}\big\}, \\ G_{n,\epsilon}(\mathfrak{v}) &:= \big\{\zeta^{n^4}(\mathfrak{v}) \in \phi^{n^4}(\mathfrak{o}) + D_{n^{2(1-\epsilon)}}\big\}, \end{aligned}$$

to prove (13) it suffices to show

Lemma 3.2 *There exist $\alpha > 0$ and constants $C_4, C_5, C_6 > 0$ such that for all n sufficiently large we have*

(a) $\sup_{\mathfrak{v} \in (D_{n^{(1+\epsilon)}} \setminus D_{n^{(1-\epsilon)}})} \mathbb{P}(E_{n,\epsilon}(\mathfrak{v})) < C_4 n^{-\alpha}$,

(b) $\sup_{\mathfrak{v} \in (D_{n^{(1+\epsilon)}} \setminus D_{n^{(1-\epsilon)}})} \mathbb{P}(F_{n,\epsilon}(\mathfrak{v})) < C_5 n^{-\alpha}$,

(c) $\sup_{\mathfrak{v} \in (D_{n^{(1+\epsilon)}} \setminus D_{n^{(1-\epsilon)}})} \mathbb{P}(G_{n,\epsilon}(\mathfrak{v})) < C_6 n^{-\alpha}$.

Proof: First we fix $\mathfrak{v} \in (D_{n^{(1+\epsilon)}} \setminus D_{n^{(1-\epsilon)}})$. Since $\{(\phi^n(\mathfrak{o}), \zeta^n(\mathfrak{v})) : n \geq 0\}$ and $\{(\sum_{i=1}^n X_i,\ \mathfrak{v} + \sum_{i=1}^n Y_i) : n \geq 0\}$ have the same distribution, we have

$$\begin{aligned} &\mathbb{P}(E_{n,\epsilon}(\mathfrak{v})) \\ &= \mathbb{P}\Big\{||\sum_{j=1}^i X_j - (\mathfrak{v} + \sum_{j=1}^i Y_j)|| \leq K \log n \text{ for some } i = 1, \ldots, n^4\Big\} \\ &= \mathbb{P}\Big\{\sum_{j=1}^i X_j - \sum_{j=1}^i Y_j \in (\mathfrak{v} + D_{K \log n}) \text{ for some } i = 1, \ldots, n^4\Big\} \\ &\leq \mathbb{P}\Big\{\sum_{j=1}^i X_j - \sum_{j=1}^i Y_j \in (\mathfrak{v} + D_{K \log n}) \text{ for some } i \geq 1\Big\} \\ &= \mathbb{P}\Big(\bigcup_{z \in (\mathfrak{v} + D_{K \log n})} \Big\{\sum_{j=1}^i X_j - \sum_{j=1}^i Y_j = z \text{ for some } i \geq 1\Big\}\Big). \end{aligned}$$

Now $\sum_{j=1}^{i}(X_j - Y_j)$ is an aperiodic, isotropic, symmetric random walk whose steps are i.i.d. with each step having the same distribution as $X - Y$ where Y is an independent copy of X. Since $\mathrm{Var}(X - Y) = 2\mathrm{Var}(X) = 2\sigma^2 I$ (where $\sigma^2 = \mathrm{Var}(X(1))$ and $\mathrm{Var}(X)$ denotes the variance-covariance matrix of X) and $\sum_{\mathfrak{u}\in\mathbb{Z}^3}|\mathfrak{u}|^2\mathbb{P}(X - Y = \mathfrak{u}) < \infty$, by Proposition P26.1 of Spitzer [1964] (pg. 308),

$$\lim_{|\mathfrak{z}|\to\infty} |\mathfrak{z}|\,\mathbb{P}\Big\{\sum_{j=1}^{i} X_j - \sum_{j=1}^{i} Y_j = \mathfrak{z} \text{ for some } i \geq 1\Big\} = (4\pi \mathrm{Var}(X(1)))^{-1}. \tag{14}$$

For $\mathfrak{v} \in (D_{n^{(1+\epsilon)}} \setminus D_{n^{(1-\epsilon)}})$ and $z \in \mathfrak{v} + D_{K\log n}$, we must have that for all n sufficiently large $|z| \geq n^{1-\epsilon}/2$. Thus for all n sufficiently large, for some constants $C_7, C_8, C_9 > 0$ we have, using (14),

$$\begin{aligned}
\mathbb{P}\big(E_{n,\epsilon}(\mathfrak{v})\big) &\leq \sum_{z\in(\mathfrak{v}+D_{K\log n})} \mathbb{P}\Big\{\sum_{j=1}^{i} X_j - \sum_{j=1}^{i} Y_j = z \text{ for some } i \geq 1\Big\} \\
&\leq C_7(K\log n)^3 C_8(n^{-(1-\epsilon)}) \\
&\leq C_9 n^{-(1-\epsilon/2)}
\end{aligned}$$

for all n sufficiently large. This completes the proof of Lemma 3.2 (a).

For the next part of the Lemma observe that for sufficiently large n and all $\mathfrak{v} \in D_{n^{(1+\epsilon)}} \setminus D_{n^{(1-\epsilon)}}$,

$$\begin{aligned}
\mathbb{P}\big(F_{n,\epsilon}(\mathfrak{v})\big) &= \mathbb{P}\Big\{\mathfrak{v} + \sum_{j=1}^{n^4}(X_j - Y_j) \notin D_{n^{2(1+\epsilon)}}\Big\} \\
&= \mathbb{P}\Big\{||\mathfrak{v} + \sum_{j=1}^{n^4}(X_j - Y_j)|| > n^{2(1+\epsilon)}\Big\} \\
&\leq \mathbb{P}\Big\{||\sum_{j=1}^{n^4}(X_j - Y_j)|| > n^{2(1+\epsilon)} - n^{(1+\epsilon)}\Big\} \\
&\leq \mathbb{P}\Big\{||\sum_{j=1}^{n^4}(X_j - Y_j)|| > n^{2(1+\epsilon)}/2\Big\}.
\end{aligned} \tag{15}$$

To estimate the above probability let $X - Y = Z = (Z(1), Z(2), Z(3))$, where $E(Z(i)) = 0$ and $\mathrm{Var}(Z(i)) = 2\sigma^2$. Then, letting $\sum_{j=1}^{k}(X_j - Y_j)(i)$ denote the ith co-ordinate of the process $\sum_{j=1}^{k}(X_j - Y_j)$ and using Chebychev's inequality, we have

$$\begin{aligned}
&\mathbb{P}\Big\{||\sum_{j=1}^{n^4}(X_j - Y_j)|| > n^{2(1+\epsilon)}/2\Big\} \\
&\leq \mathbb{P}\Big\{\bigcup_{i=1}^{3}\Big\{|\sum_{j=1}^{n^4}(X_j - Y_j)(i)| > n^{2(1+\epsilon)}/6\Big\}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\leq 3\mathbb{P}\Big\{|\sum_{j=1}^{n^4}(X_j - Y_j)(1)| > n^{2(1+\epsilon)}/6\Big\} \\
&\leq \frac{3n^4 \operatorname{Var}(Z(1))}{(n^{2(1+\epsilon)}/6)^2} \\
&\leq \frac{C_{11}}{n^{4\epsilon}},
\end{aligned}
$$

for some constant $C_{11} > 0$. Combining the above inequality with that obtained in (15) we have

$$
\sup_{\mathfrak{v}\in(D_{n^{(1+\epsilon)}}\setminus D_{n^{(1-\epsilon)}})} \mathbb{P}\big(F_{n,\epsilon}(\mathfrak{v})\big) \leq \frac{C_{11}}{n^{4\epsilon}},
$$

which proves Lemma 3.2 (b).

Finally, for the last part of the Lemma, we have that if $0 < \epsilon < 1/3$ and $\mathfrak{v} \in D_{n^{(1+\epsilon)}} \setminus D_{n^{(1-\epsilon)}}$, for all sufficiently large n, $||\mathfrak{v}|| < n^{2(1-\epsilon)}$. Therefore,

$$
\begin{aligned}
\mathbb{P}\big(G_{n,\epsilon}(\mathfrak{v})\big) &\leq \mathbb{P}\Big\{||\mathfrak{v} + \sum_{j=1}^{n^4}(X_j - Y_j)|| < n^{2(1-\epsilon)}\Big\} \\
&\leq \mathbb{P}\Big\{||\sum_{j=1}^{n^4}(X_j - Y_j)|| < ||\mathfrak{v}|| + n^{2(1-\epsilon)}\Big\} \\
&\leq \mathbb{P}\Big\{||\sum_{j=1}^{n^4}(X_j - Y_j)|| < 2n^{2(1-\epsilon)}\Big\} \\
&\leq \mathbb{P}\Big\{\bigcup_{i=1}^{3}\Big\{\Big|\sum_{j=1}^{n^4}(X_j - Y_j)(i)\Big| < 2n^{2(1-\epsilon)}/3\Big\}\Big\} \\
&\leq 3\mathbb{P}\Big\{\Big|\sum_{j=1}^{n^4}(X_j - Y_j)(1)\Big| < 2n^{2(1-\epsilon)}/3\Big\} \\
&= 3\mathbb{P}\Big\{\frac{\big|\sum_{j=1}^{n^4}(X_j - Y_j)(1)\big|}{n^2} < 2n^{-2\epsilon}/3\Big\}. \qquad (16)
\end{aligned}
$$

By the central limit theorem, as $n \to \infty$, $\sum_{j=1}^{n^4}(X_j - Y_j)(1)/(\sqrt{2}\sigma n^2)$ converges in distribution to a random variable N (say) with a standard normal distribution. Thus

$$
\begin{aligned}
&\mathbb{P}\Big\{\frac{\big|\sum_{j=1}^{n^4}(X_j - Y_j)(1)\big|}{n^2} < 2n^{-2\epsilon}/3\Big\} \\
&\leq \Big|\mathbb{P}\Big\{\frac{\big|\sum_{j=1}^{n^4}(X_j - Y_j)(1)\big|}{\sqrt{2}\sigma n^2} < \frac{\sqrt{2}n^{-2\epsilon}}{3\sigma}\Big\} - \mathbb{P}\Big\{|N| < \frac{\sqrt{2}n^{-2\epsilon}}{3\sigma}\Big\}\Big| \\
&\qquad + \mathbb{P}\Big\{|N| < \frac{\sqrt{2}n^{-2\epsilon}}{3\sigma}\Big\}. \qquad (17)
\end{aligned}
$$

Of the terms in the above inequality we have

$$
\mathbb{P}\Big\{|N| \leq \frac{\sqrt{2}n^{-2\epsilon}}{3\sigma}\Big\} = \int_{-\sqrt{2}n^{-2\epsilon}(3\sigma)^{-1}}^{\sqrt{2}n^{-2\epsilon}(3\sigma)^{-1}} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)dx
$$

$$\leq \quad \frac{2\sqrt{2}n^{-2\epsilon}(3\sigma)^{-1}}{\sqrt{2\pi}}, \tag{18}$$

and, we use Berry–Essen bounds (see Chow and Teicher [1978] Corollary 9.4, pg 300), to obtain

$$\begin{aligned}
&\left|\mathbb{P}\left\{\frac{\left|\sum_{j=1}^{n^4}(X_j-Y_j)(1)\right|}{\sqrt{2}\sigma n^2} < \frac{\sqrt{2}n^{-2\epsilon}}{3\sigma}\right\} - \mathbb{P}\left\{|N| < \frac{\sqrt{2}n^{-2\epsilon}}{3\sigma}\right\}\right| \\
&\leq \quad \left|\mathbb{P}\left\{\frac{\sum_{j=1}^{n^4}(X_j-Y_j)(1)}{\sqrt{2}\sigma n^2} < \frac{\sqrt{2}n^{-2\epsilon}}{3\sigma}\right\} - \mathbb{P}\left\{N < \frac{\sqrt{2}n^{-2\epsilon}}{3\sigma}\right\}\right| \\
&\qquad + \left|\mathbb{P}\left\{\frac{\sum_{j=1}^{n^4}(X_j-Y_j)(1)}{\sqrt{2}\sigma n^2} \leq -\frac{\sqrt{2}n^{-2\epsilon}}{3\sigma}\}\right\} - \mathbb{P}\left\{N \leq -\frac{\sqrt{2}n^{-2\epsilon}}{3\sigma}\right\}\right| \\
&\leq \quad 2\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left\{\frac{\sum_{j=1}^{n^4}(X_j-Y_j)(1)}{\sqrt{2}\sigma n^2} \leq x\right\} - P\left\{N \leq x\right\}\right| \\
&\leq \quad \frac{C_{12}E(Z_1^4)}{n^4\sigma^4},
\end{aligned} \tag{19}$$

for some constant $C_{12} > 0$. Combining (16), (17) and (18), we have Lemma 3.2(c). ■

Proof of Lemma 3.1 Let $\mathbf{v} = (\mathfrak{v}, 0) \in \mathbb{Z}^4$. Observe that $A_{n,\epsilon}(\mathbf{v}) \supseteq B_{n,\epsilon}(\mathfrak{v}) \cap \{g^i(\mathbf{0}) = \sum_{j=1}^i X_j,\ g^i(\mathbf{v}) = \mathfrak{v} + \sum_{j=1}^i Y_j$ for all $1 \leq i \leq n^4\}$. Hence

$$\begin{aligned}
&\mathbb{P}\big(A_{n,\epsilon}(\mathbf{v})\big) \\
&\geq \mathbb{P}\Big\{B_{n,\epsilon}(\mathfrak{v}) \cap \Big\{g^i(\mathbf{0}) = \sum_{j=1}^i X_j,\ g^i(\mathbf{v}) = \mathfrak{v} + \sum_{j=1}^i Y_j \text{ for } 1 \leq i \leq n^4\Big\}\Big\} \\
&= \mathbb{P}\Big\{B_{n,\epsilon}(\mathfrak{v}) \cap \Big\{g^i(\mathbf{0}) = \sum_{j=1}^i X_j,\ g^i(\mathbf{v}) = \mathfrak{v} + \sum_{j=1}^i Y_j \text{ for } 1 \leq i \leq n^4-1\Big\}\Big\} \\
&\quad \times \mathbb{P}\Big\{g^{n^4}(\mathbf{0}) = \sum_{j=1}^{n^4} X_j,\ g^{n^4}(\mathbf{v}) = \mathfrak{v} + \sum_{j=1}^{n^4} Y_j \ \Big|\ B_{n,\epsilon}(\mathfrak{v}) \cap \Big\{g^i(\mathbf{0}) = \sum_{j=1}^i X_j, \\
&\qquad\qquad g^i(\mathbf{v}) = \mathfrak{v} + \sum_{j=1}^i Y_j \text{ for } 1 \leq i \leq n^4-1\Big\}\Big\} \\
&\geq \mathbb{P}\Big\{B_{n,\epsilon}(\mathfrak{v}) \cap \Big\{g^i(\mathbf{0}) = \sum_{j=1}^i X_j,\ g^i(\mathbf{v}) = \mathfrak{v} + \sum_{j=1}^i Y_j \text{ for } 1 \leq i \leq n^4-1\Big\}\Big\} \\
&\qquad \times \Big(1 - C_1\exp(-C_2(K\log n)^3\Big),
\end{aligned}$$

where the last inequality follows from (11) after noting that given $B_{n,\epsilon}(\mathfrak{v})$, $g^i(\mathbf{0}) = \sum_{j=1}^i X_j$ and $g^i(\mathbf{v}) = \mathfrak{v} + \sum_{j=1}^i Y_j$ for all $1 \leq i \leq n^4 - 1$, we have $||g^{n^4-1}(\mathbf{0}) - g^{n^4-1}(\mathbf{v})|| \geq K\log n$. Iterating the above argument for $i = 1, \dots, n^4 - 1$ and using (13) we have

$$\begin{aligned}
\mathbb{P}\big(A_{n,\epsilon}(\mathbf{v})\big) \quad &\geq \quad \Big(1 - C_1\exp(-C_2(K\log n)^3)\Big)^{n^4}\mathbb{P}\big(B_{n,\epsilon}(\mathfrak{v})\big) \\
&\geq \quad \Big(1 - C_1 n^4\exp(-C_2K^3\log n)\Big)\Big(1 - C_3n^{-\alpha}\Big)
\end{aligned}$$

$$\begin{aligned} &\geq \left(1 - C_1 n^4 n^{-C_2 K^3}\right)\left(1 - C_3 n^{-\alpha}\right) \\ &= \left(1 - C_1 n^{-C_2 K^3 + 4}\right)\left(1 - C_3 n^{-\alpha}\right). \end{aligned}$$

Taking K such that $C_2 K^3 > 4$ (i.e. $K^3 > 8|\log(1-p)|^{-1}$) we have

$$\begin{aligned} \mathbb{P}\big(A_{n,\epsilon}(\mathbf{v})\big) &\geq 1 - C_1 n^{-C_2 K^3 + 4} - C_3 n^{-\alpha} \\ &\geq 1 - C n^{-\beta}, \end{aligned}$$

for some constant $C > 0$ and $\beta := \min\{\alpha, C_2 K^3 - 4\} > 0$. This completes the proof of Lemma 3.1. ■

Finally to complete the theorem we need to show that $\mathcal{G}$ admits infinitely many trees almost surely. For $k \geq 2$, define $D^k(n,\epsilon) = \{(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k) : \mathbf{u}_i \in \mathbb{Z}^4$ such that $n^{1-\epsilon} \leq ||g^0(\mathbf{u}_i) - g^0(\mathbf{u}_j)|| \leq n^{1+\epsilon}$ for all $i \neq j\}$. Define the event $A(n, \epsilon, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k) = \{n^{2(1-\epsilon)} \leq ||g^{n^4}(\mathbf{u}_i) - g^{n^4}(\mathbf{u}_j)|| \leq n^{2(1+\epsilon)}$ and $g^t(\mathbf{u}_i) \neq g^t(\mathbf{u}_j)$ for all $t = 1, \ldots, n^4$ and for all $i \neq j\}$. Using Lemma 3.1, we can easily show, for $0 < \epsilon < 1/3$ and for all large n

$$\inf\Big\{\mathbb{P}(A(n, \epsilon, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k) : (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k) \in D^k(n,\epsilon)\Big\} \geq 1 - \frac{C_k}{n^\beta} \tag{20}$$

where C_k is a constant independent of n (depending on k) and β is as in Lemma 3.1. We may now imitate the method following the statement of Lemma 3.1 to obtain

$$\mathbb{P}\Big\{g^t(\mathbf{u}_i) \neq g^t(\mathbf{u}_j) \text{ for all } t \geq 1 \text{ and for } 1 \leq i \neq j \leq k\Big\} > 0.$$

Thus, by translation invariance and ergodicity, we have that for all $k \geq 2$

$$\mathbb{P}\Big\{\mathcal{G} \text{ contains at least } k \text{ trees}\Big\} = 1.$$

This shows that $\mathcal{G}$ contains infinitely many trees almost surely.

4 Limit theorem

We first prove Theorem 2.2 (a). The proof of the next part of the theorem is similar and thus omitted. For simplicity in notation we shall prove the result for $d = 2$, however our method is also valid for higher dimensions.

Fix $\nu \geq 0$. Let $B_n = [1,n] \times [1,n]$ be a box of width n and, for $(i,j) \in B_n \cap \mathbb{Z}^2$, define random variables $Y_{i,j}$ as

$$Y_{i,j} := \begin{cases} 1 & \text{if the degree of the vertex } (i,j) \text{ in } B_n \cap \mathcal{V} \text{ is } \nu + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note for a vertex (i,j), $Y_{i,j} = 1$ if and only if there are exactly ν edges 'going up' from (i,j) and one edge going down from it.

Let $Y_j^{(n)} := \sum_{i=1}^n (Y_{i,j} - \mathbb{E}(Y_{i,j}))$ and $S_n := \sum_{j=1}^n Y_j^{(n)}$. To prove Theorem 2.2 we need to show that the distribution of S_n/n is asymptotically normal.

Towards this end, first observe that, for fixed j, $\{Y_{i,j}\}_{i\geq 1}$ is an α-mixing sequence of random variables, i.e., for all $m \geq 1$, $A \in \sigma(Y_{1,j}, Y_{2,j}, \ldots, Y_{m,j})$ and $B \in \sigma(Y_{m+n,j}, Y_{m+n+1,j}, \ldots)$ we have $|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq \alpha_n$ where $\alpha_n \to 0$ as $n \to \infty$. Indeed, given A and B as above, define

$$\begin{aligned} E = \Big\{ & \text{there exists an open vertex in each of the sets} \\ & \big\{(i,j) : m + \frac{n}{4} \leq i \leq m + \frac{3n}{8}\big\}, \big\{(i,j+1) : m + \frac{3n}{8} \leq i \leq m + \frac{n}{2}\big\}, \\ & \big\{(i,j+1) : m + \frac{n}{2} \leq i \leq m + \frac{5n}{8}\big\}, \big\{(i,j) : m + \frac{3n}{8} \leq i \leq m + \frac{3n}{4}\big\}\Big\}. \end{aligned}$$

Now $\mathbb{P}(E) = \Big(1 - (1-p)^{n/8}\Big)^4 \to 1$ as $n \to \infty$. Also, given E, the event A depends only on the configuration of the vertices $\{(i, j-1) : i \leq m + \frac{n}{4}\}$, $\{(i,j) : i \leq m\}$ and $\{(i, j+1) : i < m + \frac{n}{2}\}$, while the event B depends on the vertices $\{(i, j-1) : i \geq m + \frac{3n}{4}\}$, $\{(i,j) : i \geq m+n\}$ and $\{(i, j+1) : i > m + \frac{n}{2}\}$. These sets of vertices being disjoint, given E, A and B are conditionally independent. A simple conditioning argument now yields that, for n large enough,

$$\begin{aligned} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| &\leq 5\mathbb{P}(E^c) \\ &\leq C_1 \exp(-C_2 n) \text{ for constants } C_1, C_2 > 0. \end{aligned} \tag{21}$$

Also observe that for fixed i, $\{Y_{i,j}\}_{j\geq 1}$ is a 1-dependent sequence of random variables, i.e. for fixed i, $Y_{i,j}$ is independent of $Y_{i,j'}$ for $j' \neq j-1, j, j+1$.

Now, for some $0 < \delta < 1$ to be chosen later and for $0 \leq k < r_n$, where $r_n := \lfloor \frac{n}{\lfloor n^\delta \rfloor + 1} \rfloor$, let

$$\begin{aligned} W_{k+1}^{(n)} &:= Y_{k\lfloor n^\delta\rfloor + k+1}^{(n)} + \cdots + Y_{(k+1)\lfloor n^\delta \rfloor + k}^{(n)}, \\ \eta_{k+1}^{(n)} &:= Y_{(k+1)\lfloor n^\delta\rfloor + k+1}^{(n)}, \\ E_n &:= Y_{r_n(\lfloor n^\delta \rfloor + 1)+1}^{(n)} + \cdots + Y_n^{(n)}. \end{aligned}$$

First we show that, for any $r \geq 1$, there exists a constant $C > 0$ such that

$$\mathbb{E}(Y_1^{(n)} + \cdots + Y_r^{(n)})^4 \leq C r^2 n^2. \tag{22}$$

Indeed note that, as in the proof of the first part of Theorem 27.5, Billingsley [1979], we have $\mathbb{E}(Y_i^{(n)})^4 = \mathbb{E}(Y_1^{(n)})^4 \leq K n^2$ for some constant $K > 0$. Now

$$\mathbb{E}(\sum_{k=1}^r Y_k^{(n)})^4 = \sum_{k,l,s,t=1}^r \mathbb{E}(Y_k^{(n)} Y_l^{(n)} Y_s^{(n)} Y_t^{(n)}), \tag{23}$$

and using the fact that $\{Y_k^{(n)}\}_{k\geq 1}$ is 1-dependent sequence of random variables, the Cauchy-Schwarz inequality and that $\mathbb{E}Y_1^{(n)} = 0$, we obtain after some elementary calculations $\mathbb{E}(\sum_{k=1}^r Y_k^{(n)})^4 \leq 2r\mathbb{E}(Y_1^{(n)})^4 + r^2 \mathbb{E}(Y_1^{(n)})^4$. Here the term $2r\mathbb{E}(Y_1^{(n)})^4$ comes from the terms in the sum $\sum_{j,k,s,t=1}^{r} \mathbb{E}(Y_j^{(n)} Y_k^{(n)} Y_s^{(n)} Y_t^{(n)})$ when j,k,s,t are close to each other so as to have dependence among all the 4 random variables making the product, while the term $r^2 \mathbb{E}(Y_1^{(n)})^4$ comes from the terms of the sum when j,k are close to each other, s,t are close to each other, but there is independence between $(Y_j^{(n)}, Y_k^{(n)})$ and $(Y_s^{(n)}, Y_t^{(n)})$. This proves (22).

Now taking $r = \lfloor n^\delta \rfloor$, and using the fact that $W_1^{(n)}, W_2^{(n)}, \ldots$ are i.i.d. random variables, we have from (22) that $\mathbb{E}(W_k^{(n)})^4 \le C n^{2+2\delta}$ for all $k \ge 1$.

Also

$$\begin{aligned} \mathrm{Var}(W_1^{(n)}) &= \mathbb{E}\Big(\sum_{j=1}^{\lfloor n^\delta \rfloor} Y_j^{(n)} \Big)^2 \\ &= \lfloor n^\delta \rfloor \mathbb{E}(Y_1^{(n)})^2 + 2 \sum_{j=1}^{\lfloor n^\delta \rfloor - 1} \mathrm{Cov}(Y_j^{(n)}, Y_{j+1}^{(n)}) \\ &= \lfloor n^\delta \rfloor \mathbb{E}(Y_1^{(n)})^2 + 2(\lfloor n^\delta \rfloor - 1) \mathrm{Cov}(Y_1^{(n)}, Y_2^{(n)}). \end{aligned} \tag{24}$$

In the above expression,

$$\begin{aligned} \mathbb{E}(Y_1^{(n)})^2 &= n\mathrm{Var}(Y_{1,1}) + 2\sum_{s=1}^{n-1}\sum_{t=1}^{n-s} \mathrm{Cov}(Y_{s,1}, Y_{s+t,1}) \\ &= n\mathrm{Var}(Y_{1,1}) + 2\sum_{s=1}^{n-1}(n-s)\ \mathrm{Cov}(Y_{1,1}, Y_{1+s,1}) \\ &= O(n) \text{ as } n \to \infty, \end{aligned}$$

where the last equality follows because from the α-mixing of the sequence $\{Y_{t,1}\}_{t\ge1}$ we have $\sum_{t=2}^{\infty} \mathrm{Cov}(Y_{1,1}, Y_{t,1}) \le C \sum_{t=2}^{\infty} \alpha_t < \infty$ for some constant $C > 0$. Moreover, by Cauchy-Schwarz inequality,

$$\mathrm{Cov}(Y_1^{(n)}, Y_2^{(n)}) \le \mathbb{E}(Y_1^{(n)})^2.$$

Thus, from (24), we have $\mathrm{Var}(W_1^{(n)}) = O(n^{1+\delta})$ as $n \to \infty$ and

$$\mathrm{Var}(\sum_{k=1}^{r_n} W_k^{(n)}) = O(n^{(1-\delta)+(1+\delta)}) = O(n^2) \text{ as } n \to \infty. \tag{25}$$

Finally, for $0 < \delta < 1$,

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=1}^{r_n} \frac{1}{(\mathrm{Var} \sum_{k=1}^{r_n} W_k^{(n)})^2} \mathbb{E}(W_k^{(n)})^4 &\le \lim_{n\to\infty} \sum_{k=1}^{r_n} C \frac{n^{2+2\delta}}{n^4} \\ &= \lim_{n\to\infty} C n^{\delta-1} \\ &= 0. \end{aligned}$$

Thus by Lyapunov's central limit theorem (see Theorem 27.3, pg 312 Billingsley [1979]) we have that for $0 < \delta < 1$, $\frac{1}{\sqrt{\sum_{k=1}^{r_n} Var(W_k^{(n)})}} \sum_{k=1}^{r_n} W_k^{(n)}$ converges in probability to a standard normal random variable.

Now let $\eta_n := \sum_{k=1}^{r_n} \eta_k^{(n)}$. We will show that

$$\eta_n/n \to 0 \text{ in probability as } n \to \infty. \tag{26}$$

Indeed,

$$\begin{aligned}
\mathbb{E}(\eta_k^{(n)})^2 &\leq \sum_{i=1}^{n} \mathrm{Var}(Y_{i,k}) + 2n \sum_{i=2}^{n} \mathrm{Cov}(Y_{1,k}, Y_{i,k}) \\
&\leq n \,\mathrm{Var}(Y_{1,1}) + 2n \sum_{i=2}^{\infty} C_1 \exp(-C_2 i) \\
&\leq Mn \text{ for some constant } M > 0.
\end{aligned}$$

Thus, using the fact that $r_n = O(n^{1-\delta})$ as $n \to \infty$, we have, for $\epsilon > 0$,

$$\begin{aligned}
\mathbb{P}(|\eta_n| > n\epsilon) &\leq \frac{\mathbb{E}(\eta_n^2)}{n^2\epsilon^2} \\
&= \frac{MnO(n^{1-\delta})}{n^2\epsilon^2} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

This proves (26).

To complete the proof, we have to show that $\frac{E_n}{n} \to 0$ in probability as $n \to \infty$. First observe that number of terms in E_n is at most $\lfloor n^\delta \rfloor$. Therefore taking $\delta = 1/2$, from (22) we have $\mathbb{E}(E_n^4) \leq Cn^3$. Hence, for $\epsilon > 0$,

$$\mathbb{P}(|E_n| > n\epsilon) \leq \frac{\mathbb{E}(E_n^4)}{n^4\epsilon^4} \to 0 \text{ as } n \to \infty. \tag{27}$$

Theorem 2.2 (a) now follows by combining the equations (26) and (27) and the fact that $\sum_{k=1}^{r_n} W_k{}^{(n)}/n$ has asymptotically a $N(0, s^2)$ distribution, where $s^2 = \mathrm{Var}(Y_{1,1}) + 2\sum_{i=2}^{\infty} \mathrm{Cov}(Y_{1,1}, Y_{i,1}) + 2\sum_{i=1}^{\infty} \mathrm{Cov}(Y_{1,1}, Y_{i,2}) + 2\sum_{i=2}^{\infty} \mathrm{Cov}(Y_{1,2}, Y_{i,1})$. Note that to compute s^2 we use the fact that $\{(Y_{i,j}, Y_{i,j+1})\}_{i\geq 1}$ is an α-mixing sequence.

5 Degree of a vertex

To prove Proposition 2.1 observe that, given the vertex $(0, -1)$ is open, let

$$Y = \begin{cases} 1 & \text{if the vertex } (0,0) \text{ is open} \\ 0 & \text{otherwise,} \end{cases}$$

$$X_1 = \#\{(i, 0) : i \leq -1 : (i, 0) \text{ is connected by an edge to } (0, -1)\},$$

$$X_2 = \#\{(i, 0) : i \geq 1 : (i, 0) \text{ is connected by an edge to } (0, -1)\}.$$

Clearly the degree of $(0, -1)$ equals $Y + X_1 + X_2$. Now given the vertex $(0, -1)$ is open, the probability that the vertex $(-l, 0)$ is connected to $(0, -1)$ and that there are exactly $r - 1$

vertices in $\{(i,0) : -l+1 \leq i \leq -1\}$ which are connected to $(0,-1)$ equals $\binom{l-1}{r-1} p^r (1-p)^{l-r} (1-p)^{2l-1}((1-p)+\frac{1}{2}p)$. Thus $P(X_1 \geq r) = \sum_{l=r}^{\infty} \binom{l-1}{r-1} p^r (1-p)^{l-r}(1-p)^{2l-1}((1-p)+\frac{1}{2}p)$. An easy calculation now completes the proof of the proposition.

Similarly in 2-dimensions, given that a vertex v is open, the distribution of the number of edges of length l 'going up' from v is binomial with parameters 2 and $(1-\frac{p}{2})(1-p)^{2l-1}$.

Remark From the above distributions we may calculate the quantities $\mathbb{E}(S_n)$, $\mathrm{Var}(S_n)$, s^2 and the related quantities involving L_n required in Theorem 2.2 for 2-dimensions.

6 References

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