Stochastic Orderings among Order Statistics and Sample Spacings

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Abstract

In this paper we review some of the results obtained recently in the area of stochastic comparisons of order statistics and sample spacings. We consider the cases when the parent observations are identically as well as non-identically distributed. But most of the time we shall be assuming that the observations are independent. The case of independent exponentials with unequal scale parameters is discussed in detail.

1 Introduction

The simplest and the most common way of comparing two random variables is through their means and variances. It may happen that in some cases the median of X is larger than that of Y, while the mean of X is smaller than the mean of Y. However, this confusion will not arise if the random variables are stochastically ordered. Similarly, the same may happen if one would like to compare the variability of X with that of Y based only on numerical measures like standard deviation etc. Besides, these characteristics of distributions might not exist in some cases. In most cases one can express various forms of knowledge about the underlying distributions in terms of their survival functions, hazard rate functions, mean residual functions, quantile functions and other suitable functions of probability distributions. These methods are much more informative than those based only on few numerical characteristics of distributions. Comparisons of random variables based on such functions usually establish partial orders among them. We call them as stochastic orders.

Stochastic models are usually sufficiently complex in various fields of statistics, particularly in reliability theory. Obtaining bounds and approximations for their characteristics is of practical importance. That is, the approximation of a stochastic model either by a simpler model or by a model with simple constituent components might lead to convenient bounds and approximations for some particular and desired characteristics of the model. The study of changes in the properties of a model, as the constituent components vary, is also of great interest. Accordingly, since the stochastic components of models involve random variables, the topic of stochastic orders among random variables plays an important role in these areas.

Order statistics and spacings are of great interest in many areas of statistics and they have received a lot of attention from many researchers. Let $X_1, \ldots, X_n$ be n random variables. The ith order statistic, the ith smallest of X_i's, is denoted by $X_{i:n}$. A k-out-of-n system of n components functions if at least k of n components function. The time of a k-out-of-n system of n components with life times $X_1, \ldots, X_n$ corresponds to the $(n-k+1)$th order statistic. Thus, the study of lifetimes of k-out-of-n systems is equivalent to the study of the stochastic properties of order statistics. Spacings, the differences between successive order statistics, and their functions are also important in statistics, in general, and in particular in the context of life testing and reliability models. Lot of work has been done in the literature on different aspects of order statistics and spacings. For a glimpse of this, see the books by David (1981), and Arnold, Balakrishnan and Nagaraja (1992); and two volumes of papers on this topic by Balakrishnan and Rao (1998 a and b). But most of this work has been confined to the case when the observations are i.i.d. In many practical situations, like in reliability theory, the observations are not necessarily i.i.d. Because of the complicated nature of the problem, not much work has been done for the non i.i.d. case. Some references for this case are Sen (1970), David (1981, p.22), Shaked and Tong (1984), Bapat and Beg (1989), Boland et al. (1996), Kochar (1996), and Nappo and Spizzichino (1998), among others.

Some interesting partial ordering results on order statistics and spacings from independent but non-identically random variables have been obtained by Pledger and Proschan (1971), Proschan and Sethuraman (1976), Bapat and Kochar (1994), Boland, El-Neweihi, and Proschan (1994), Kochar and Kirmani (1995), Kochar and Korwar (1996), Kochar and Rojo (1996), Dykstra, Kochar, and Rojo (1997), Kochar and Ma (1999), Bon and Paltanea (1999), Kochar (1999), Khaledi and Kochar (1999), Khaledi and Kochar (2000 a,b,c), and Khaledi and Kochar (2001).

In this chapter, we discuss some newly obtained results on stochastic comparisons of order statistics and spacings. Kochar (1998) and Boland, Shaked and Shanthikumar (1998) have given comprehensive reviews on this topic upto 1998. In Section 2, we introduce the required notation and definitions. Section 3 and 4 are devoted to stochastic comparisons of order statistics in one-sample and two-sample problems, respectively. In Sections 5, we discuss the stochastic ordering among spacings in one-sample problem and two sample problem. Section 6 is devoted to stochastic properties of sample range Throughout this chapter *increasing* means *nondecreasing* and *decreasing* means *nonincreasing*; and we shall be assuming that all distributions under study are absolutely continuous.

2 Definitions

Let X and Y be univariate random variables with distribution functions F and G, survival functions $\overline{F}$ and $\overline{G}$, density functions f and g; and hazard rates r_F $(= f/\overline{F})$ and r_G $(= g/\overline{G})$, respectively. Let $l_X\,(l_Y)$ and $u_X\,(u_Y)$ be the left and the right endpoints of the support of $X\,(Y)$.

Stochastic orderings

Definition 2.1 *X is said to be stochastically smaller than Y (denoted by $X \leq_{st} Y$) if $\overline{F}(x) \leq \overline{G}(x)$ for all x.*

This is equivalent to saying that $Eg(X) \leq Eg(Y)$ for any increasing function g for which expectations exist.

Definition 2.2 *X is said to be smaller than Y in hazard rate ordering (denoted by $X \leq_{hr} Y$) if $\overline{G}(x)/\overline{F}(x)$ is increasing in $x \in (-\infty, max(u_X, u_Y))$.*

It is worth noting that $X \leq_{hr} Y$ is equivalent to the inequalities

$$P[X - t > x | X > t] \leq P[Y - t > x | Y > t], \quad \text{for all } x \geq 0 \text{ and } t.$$

In other words, the conditional distributions, given that the random variables are at least of a certain size, are all stochastically ordered (in the standard sense) in the same direction. Thus, if X and Y represent the survival times of different models of an appliance that satisfy this ordering, one model is better (in the sense of stochastic ordering) when the appliances are new, the same appliance is better when both are one month old, and in fact is better no matter how much time has elapsed. It is clearly useful to know when this strong type of stochastic ordering holds since quantities judgements are then easy to make. In case the hazard rates exist, it is easy to see that $X \leq_{hr} Y$, if and only if, $r_G(x) \leq r_F(x)$ for every x. The hazard rate ordering is also known as uniform stochastic ordering in the literature.

Definition 2.3 *X is said to be smaller than Y in likelihood ratio ordering (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in $x \in (l_X, u_X) \cup (l_Y, u_Y)$.*

When the supports of X and Y have a common left end-point, we have the following chain of implications among the above stochastic orders :

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y. \tag{2.1}$$

Definition 2.4 *The random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is smaller than the random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ in the multivariate stochastic order (denoted by $\mathbf{X} \overset{st}{\preceq} \mathbf{Y}$) if $h(\mathbf{X}) \leq_{st} h(\mathbf{Y})$ for all increasing functions h.*

It is easy to see that multivariate stochastic ordering implies component-wise usual stochastic ordering. For more details on stochastic orderings, see Chapters 1 and 4 of Shaked and Shanthikumar (1994).

One of the basic criteria for comparing variability in probability distributions is that of dispersive ordering. Let F^{-1} and G^{-1} be the right continuous inverses (quantile functions) of F and G, respectively. We say that X is less *dispersed* than Y (denoted by $X \leq_{disp} Y$) if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$, for all $0 \leq \alpha \leq \beta \leq 1$. From this one can easily obtain that

$$X \leq_{disp} Y \Longleftrightarrow g(x) \leq f\left(F^{-1}G(x)\right) \quad \forall\, x, \tag{2.2}$$

when the random variables X and Y admit densities. A consequence of $X \leq_{disp} Y$ is that $|X_1 - X_2| \leq_{st} |Y_1 - Y_2|$ and which in turn implies $var(X) \leq var(Y)$ as well as $E[|X_1 - X_2|] \leq E[|Y_1 - Y_2|]$, where $X_1, X_2\,(Y_1, Y_2)$ are two independent copies of X (Y). For details, see Saunders and Moran (1978), Lewis and Thompson (1981), Deshpande and Kochar (1983), Bagai and Kochar (1986), Bartoszewicz (1986, 1987); and Section 2.B of Shaked and Shanthikumar (1994).

Notions of Majorization and related orderings

One of the basic tools in establishing various inequalities in statistics and probability is the notion of majorization.

Let $\{x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Definition 2.5 *The vector* $\mathbf{x}$ *is said to majorize the vector* $\mathbf{y}$ *(written* $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$*) if* $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ *for* $j = 1, \ldots, n-1$ *and* $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

Functions that preserve the majorization ordering are called Schur-convex functions. The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ weakly (written $\mathbf{x} \overset{w}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n$. Marshall and Olkin (1979) provides extensive and comprehensive details on the theory of majorization and its applications in statistics.

Recently Bon and Paltanea (1999) have considered a pre-order on $\mathbb{R}^{+n}$, which they call as a *p-larger order*.

Definition 2.6 *A vector* $\mathbf{x}$ *in* $\mathbb{R}^{+n}$ *is said to be p-larger than another vector* $\mathbf{y}$ *also in* $\mathbb{R}^{+n}$ *(written* $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$*) if* $\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)}, j = 1, \ldots, n$.

Let $\log(\mathbf{x})$ denote the vector of logarithms of the coordinates of $\mathbf{x}$. It is easy to verify that

$$\mathbf{x} \overset{p}{\succeq} \mathbf{y} \Leftrightarrow \log(\mathbf{x}) \overset{w}{\succeq} \log(\mathbf{y}). \tag{2.3}$$

It is known that $\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow (g(x_1), \ldots, g(x_n)) \overset{w}{\succeq} (g(y_1), \ldots, g(y_n))$ for all concave functions g (cf. Marshal and Olkin, 1979, p. 115). From this and (2.3), it follows that when $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{+n}$

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{p}{\succeq} \mathbf{y}.$$

The converse is, however, not true. For example, the vectors $(0.2, 1, 5) \overset{p}{\succeq} (1, 2, 3)$ but majorization does not hold between these two vectors.

Notions of Aging

Let X be a random variable with distribution function F and let X_t denote a random variable with the same distribution as that of $X - t|X > t$. We will use the following notions of aging in this article.

(*a*) X is said to have an increasing failure rate (denoted by IFR) distribution if $X_t \leq_{st} X_{t'}$, for $t > t'$. This is equivalent to saying that $\overline{F}(x+t)/\overline{F}(t)$ decreasing in t for $x > 0$. It is easy to see that in case the random variable X admits density, F is IFR if and only if, the hazard rate $r_F(t) = f(t)/\overline{F}(t)$ is increasing in t.

(*b*) X is said to have a decreasing failure rate (denoted by DFR) distribution if $X_t \geq_{st} X_{t'}$, for $t > t'$. This is equivalent to $\overline{F}(x+t)/\overline{F}(t)$ increasing in t for $x > 0$.

Next theorem due to Bagai and Kochar (1986) and Bartoszewicz (1987) establishes a connection between dispersive ordering and hazard rate ordering.

THEOREM **2.1** *Let X and Y be random variables with distribution function F and G, respectively. Then,*

(a) $X \leq_{hr} Y$ and F or G being DFR implies $X \leq_{disp} Y$;

(b) $X \leq_{disp} Y$ and F or G being IFR implies $X \leq_{hr} Y$.

3 Stochastic Comparisons of Order Statistics in one-sample problem

Let $X_1, \ldots, X_n$ be a set of independent random variables. It is easy to see that $X_{i:n} \leq_{st} X_{j:n}$, for all $i < j$. Boland, El-Neweihi and Proschan (1994) extended this result from usual stochastic order to hazard rate order. Using the definition of likelihood ratio ordering, it is not hard to prove that $X_{i:n} \leq_{lr} X_{j:n}$ for $i < j$. Shaked and Shanthikumar (1994) considered the problem of comparing order statistics from samples with possibly unequal sample sizes. They showed that if random variables X_i's are iid, then $X_{n:n} \leq_{lr} X_{n+1:n+1}$ and $X_{1:n} \geq_{lr} X_{1:n+1}$. Raqab and Amin (1996) strengthened this result and proved that $X_{i:n} \leq_{lr} X_{j:m}$, whenever $i \leq j$ and $n - i \geq m - j$. Using implications (2.1), we get, for $i \leq j$ and $n - i \geq m - j$, $X_{i:n} \leq_{hr} X_{j:m}$ which in turn implies that $X_{i:n} \leq_{st} X_{j:m}$. Removing the identically distributed assumption, it is interesting to investigate the above stochastic inequalities among order statistics. Boland, El-Neweihi and Proschan (1994) showed that if random variables are independent and $X_k \leq_{hr} X_{n+1}$, $k = 1, \ldots, n$, then $X_{i-1:n} \leq_{hr} X_{i:n+1}$, $i = 1, \ldots, n+1$. They also proved that if X_i's are independent and $X_{n+1} \leq_{hr} X_k$, $k = 2, \ldots, n$, then $X_{i:n} \geq_{hr} X_{i:n+1}$, $i = 1, \ldots, n$. The reader may be wondering whether likelihood ratio ordering among order statistics holds for the case when X_i's are independent but not necessarily identically distributed. Assuming $X_1 \leq_{lr} X_2 \leq_{lr} \ldots \leq_{lr} X_n$, Bapat an Kochar (1994) proved that $X_{i:n} \leq_{lr} X_{j:n}$, $i < j$.

We end this section by discussing some results on dispersive ordering of order statistics. David and Groenveld (1982) proved that if X_i's are iid random variables with a common DFR distribution, then $var(X_{i:n}) \leq var(X_{j:n})$, for $i < j$. Kochar (1996) strengthened this result to prove that under the same conditions, $X_{i:n} \leq_{disp} X_{j:n}$, $i < j$. In Theorem 3.2 below, due to Khaledi and Kochar (2000 a), this result has been further extended. It is proved that if X_i's are iid with DFR distribution, then $X_{i:n} \leq_{disp} X_{j:m}$, whenever $i \leq j$ and $n - i \geq m - j$. We will find the following result useful in proving it.

THEOREM **3.1** *(Saunders (1984)). The random variable X satisfies $X \leq_{disp} X + Y$. for any random variable Y independent of X if and only if X has a log-concave density.*

Using Theorem 3.1, first the result is proved for exponential distribution.

LEMMA 3.1 *Let $X_{i:n}$ be the ith order statistic of a random sample of size n from an exponential distribution. Then*

$$X_{i:n} \leq_{disp} X_{j:m} \quad \text{for } i \leq j \text{ and } n - i \geq m - j. \tag{3.1}$$

PROOF : Suppose we have two independent random samples, $X_1, \ldots, X_n$ and $X_1', \ldots, X_m'$ of sizes n and m from an exponential distribution with failure rate λ. The ith order statistic $X_{i:n}$ can be written as a convolutions of the sample spacings as

$$\begin{aligned} X_{i:n} &= (X_{i:n} - X_{i-1:n}) + \cdots + (X_{2:n} - X_{1:n}) + X_{1:n} \\ &\stackrel{dist}{=} \sum_{k=1}^{i} E_{n-i+k} \end{aligned} \tag{3.2}$$

where for $k = 1, \ldots, i$, E_{n-i+k} is an exponential random variable with failure rate $(n - i + k)\lambda$. It is a well known fact that E_{n-i+k}'s are independent. Similarly we can express $X_{j:m}'$ as

$$X_{j:m}' \stackrel{dist}{=} \sum_{k=1}^{j} E_{m-j+k}' \tag{3.3}$$

where again for $k = 1, \ldots, j$, E_{m-j+k}' is an exponential random variable with failure rate $(m - j + k)\lambda$ and E_{m-j+k}''s are independent. It is easy to verify that $E_{n-i+1} \leq_{disp} E_{m-j+1}'$ for $n - i \geq m - j$.

Since the class of distributions with log-concave densities is closed under convolutions (cf. Dharmadhiakri and Joag-dev (1988), p. 17), it follows from the repeated applications of Theorem 3.1 that

$$\sum_{k=1}^{i} E_{n-i+k} \leq_{disp} \sum_{k=1}^{i} E_{m-j+k}'. \tag{3.4}$$

Again since $\sum_{k=i+1}^{j} E'_{m-j+k}$, being the sum of independent exponential random variables has a log-concave density and since it is independent of $\sum_{k=1}^{i} E'_{n-i+k}$, it follows from Theorem 3.1 that the R.H.S of (3.4) is less dispersed than $\sum_{k=1}^{j} E'_{m-j+k}$ for $i \leq j$. That is,

$$X_{i:n} \stackrel{dist}{=} \sum_{k=1}^{i} E_{n-i+k} \leq_{disp} \sum_{k=1}^{j} E'_{m-j+k} \stackrel{dist}{=} X'_{j:m}.$$

Since $X_{j:m}$ and $X'_{j:m}$ are stochastically equivalent, (3.1) follows from this. ■

The proof of the next lemma can be found in Bartoszewicz (1987).

LEMMA 3.2 *Let $\phi : R_+ \to R_+$ be a function such that $\phi(0) = 0$ and $\phi(x) - x$ is increasing. Then for every convex and strictly increasing function $\psi : R_+ \to R_+$ the function $\psi\phi\psi^{-1}(x) - x$ is increasing.*

In the next theorem we extend Lemma 3.1 to the case when F is a DFR distribution.

THEOREM **3.2** *Let $X_{i:n}$ be the ith order statistic of a random sample of size n from a DFR distribution F. Then*

$$X_{i:n} \leq_{disp} X_{j:m} \quad \textit{for } i \leq j \textit{ and } n - i \geq m - j.$$

PROOF : The distribution function of $X_{j:m}$ is $F_{j:m}(x) = B_{j:m}F(x)$, where $B_{j:m}$ is the distribution function of the beta distribution with parameters $(j, m - j + 1)$.

Let G denote the distribution function of a unit mean exponential random variable. Then $H_{j:m}(x) = B_{j:m}G(x)$ is the distribution function of the jth order statistic in a random sample of size m from a unit mean exponential distribution. We can express $F_{j:m}$ as

$$\begin{aligned} F_{j:m}(x) &= B_{j:m}GG^{-1}F(x) \\ &= H_{j:m}G^{-1}F(x). \end{aligned} \tag{3.5}$$

To prove the required result, we have to show that for $i \leq j$ and $n - i \geq m - j$,

$$\begin{aligned} &F_{j:m}^{-1}F_{i:n}(x) - x \quad \text{is increasing in } x \\ \Leftrightarrow \quad &F^{-1}GH_{j:m}^{-1}H_{i:n}G^{-1}F(x) - x \quad \text{is increasing in } x. \end{aligned} \tag{3.6}$$

By Lemma 3.1, $H_{j:m}^{-1}H_{i:n}(x) - x$ is increasing in x for $i \leq j$ and $n - i \geq m - j$. Also the function $\psi(x) = F^{-1}G(x)$ is strictly increasing and it is convex if F is DFR. The required result now follows from Lemma 3.2. ■

REMARK: A consequence of Theorem 3.2 is that if we have random samples from a DFR distribution, then

$$X_{i:n+1} \leq_{disp} X_{i:n} \leq_{disp} X_{i+1:n+1}, \qquad \text{for } i = 1, \ldots, n.$$

4 Stochastic Comparisons of Order Statistics in two-sample problem

Let $X_1, \ldots, X_n$ be a set of independent random variables and $Y_1, \ldots, Y_n$ be another set of independent random variables. Ross (1983) proved that if $X_i \leq_{st} Y_i$, $i = 1, \ldots, n$, then $(X_1, \ldots, X_n) \leq_{st} (Y_1, \ldots, Y_n)$. A consequence of this result is that $X_{i:n} \leq_{st} Y_{i:n}$ for $i = 1, \ldots, n$. Lynch, Mimmack and Proschan (1987) generalized this result from stochastic ordering to hazard rate ordering. They showed that if $X_i \leq_{hr} Y_j$, $i, j \in \{1, \ldots, n\}$, then $X_{i:n} \leq_{hr} Y_{i:n}$, $i = 1, \ldots, n$. A similar result for likelihood ratio ordering has been proved by Chan, Proschan and Sethuraman (1991). They proved that if $X_i \leq_{lr} Y_j$, $i, j \in \{1, \ldots, n\}$, then $X_{i:n} \leq_{lr} Y_{i:n}$, $i = 1, \ldots, n$. Lillo, Nanda and Shaked (2000) strengthened this result to the case when the number of X_i's and Y_i's are possibly different.

THEOREM **4.1** *Let $X_1, \ldots, X_n$ be independent random variables and $Y_1, \ldots, Y_m$ be another set of independent random variables, all having absolutely continuous distributions. Then $X_i \leq_{lr} Y_j$ for all i, j implies $X_{i:n} \leq_{lr} Y_{j:m}$ whenever $i \leq j$ and $n - i \geq m - j$.*

In the next theorem we establish dispersive ordering between order statistics when the random samples are drawn from different distributions.

THEOREM **4.2** *Let $X_1, \ldots, X_n$ be a random sample of size n from a continuous distribution F and let $Y_1 \ldots, Y_m$ be a random sample of size m from another continuous distribution G. If either F or G is DFR, then*

$$X \leq_{disp} Y \Rightarrow X_{i:n} \leq_{disp} Y_{j:m} \quad \text{for } i \leq j \text{ and } n - i \geq m - j. \tag{4.1}$$

PROOF: Let F be a DFR distribution. The proof for the case when G is DFR is similar. By Theorem 3.2, $X_{i:n} \leq_{disp} X_{j:m}$ for $i \leq j$ and $n - i \geq m - j$. Bartoszewicz (1986) proved that if $X \leq_{disp} Y$ then $X_{j:m} \leq_{disp} Y_{j:m}$. Combining these we get the required result. ■

Since the property $X \leq_{hr} Y$ together with the condition that either F or G is DFR implies that $X \leq_{disp} Y$ (Theorem 2.1), we get the following result from the above theorem.

COROLLARY 4.1 *Let $X_1, \ldots, X_n$ be a random sample of size n from a continuous distribution F and $Y_1 \ldots, Y_m$ be a random sample of size m from another continuous distribution G. If either F or G is DFR, then*

$$X \leq_{hr} Y \Rightarrow X_{i:n} \leq_{disp} Y_{j:m}.$$

Stochastic comparisons of order statistics from heterogeneous populations

An assumption often made in reliability models is that the n components have lifetimes with proportional hazards. Let X_i denote the lifetime of the ith component of a reliability system with survival function $\overline{F}_i(t)$, $i = 1, \ldots, n$. Then they have proportional hazard rates (PHR) if there exist constants $\lambda_1, \ldots, \lambda_n$ and a (cumulative hazard) function $R(t) \geq 0$ such that

$\overline{F}_i(t) = e^{-\lambda_i R(t)}$ for $i = 1, \ldots, n$.Clearly then the hazard rate of X_i is $r_i(t) = \lambda_i R'(t)$ (assuming it exists). An example of such a situation is when the components have independent exponential lifetimes with respective hazard rates $\lambda_1, \ldots, \lambda_n$. Many researchers have investigated the effect on the survival function, the hazard rate function and other characteristics of the time to failure of this system when we switch the vector $(\lambda_1, \ldots, \lambda_n)$ to another vector say $(\lambda_1^*, \ldots, \lambda_n^*)$. Pledger and Proschan (1971), for the first time, studied this problem and proved the following interesting result among many other results.

THEOREM **4.3** *Let $(X_1, \ldots, X_n)$ and $(X_1^*, \ldots, X_n^*)$ be two random vectors of independent lifetimes with proportional hazards with $\lambda_1, \ldots, \lambda_n$ and $\lambda_1^*, \ldots \lambda_n^*$ as the constants of proportionality. Suppose that*

$$\boldsymbol{\lambda} \stackrel{m}{\succeq} \boldsymbol{\lambda}^*.$$

Then

$$X_{i:n} \geq_{st} X_{i:n}^*, \qquad i = 1, \ldots, n. \tag{4.2}$$

Proschan and Sethuraman (1976) generalized this result from component wise stochastic ordering to multivariate stochastic ordering. That is, under the same assumptions of Theorem 4.3, they showed that

$$(X_{1:n}, \ldots, X_{n:n}) \geq_{st} (X_{1:n}^*, \ldots, X_{n:n}^*).$$

Boland, El-Neweihi and Proschan (1994) proved that for $n = 2$ the above result can be extended from stochastic ordering to hazard rate ordering. They also showed with the help of a counterexample that for $n > 2$, (4.2) cannot be strengthened from stochastic ordering to hazard rate ordering.

Dykstra, Kochar and Rojo (1997) studied the problem of stochastically comparing the largest order statistic of a set of n independent and non-identically distributed exponential random variables with that corresponding to a set of n independent and identically distributed exponential random variables. Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, for $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\overline{\lambda} = \sum_{i=1}^n \lambda_i / n$, the arithmetic mean of the λ_i's. They proved that $X_{n:n}$ is greater than $Y_{n:n}$ according to dispersive as well as hazard rate orderings. In Theorem 4.4 below we prove that similar results hold if instead, we assume that for $i = 1, \ldots, n$, the random variable Y_i has exponential distribution with hazard rate $\tilde{\lambda} = (\prod_{i=1}^n \lambda_i)^{1/n}$, the geometric mean of the λ_i's. To prove dispersive ordering between $X_{n:n}$ and $Y_{n:n}$ in Theorem 4.4 we shall need the following lemma.

LEMMA 4.1 *For $z > 0$, the functions $g(z) = (1 - e^{-z})/z$ and $h(z) = (z^2 e^{-z})/(1 - e^{-z})^2$ are both decreasing.*

PROOF : The numerator of the derivative of $g(z)$ is $k(z) = (1+z)e^{-z} - 1$, which is a decreasing function of z. This implies that $k(z) < 0$ for $z > 0$, since $k(0) = 0$.

It is easy to see after some simplifications that

$$\frac{d}{dz}\left(\log(h(z))\right) = \frac{2 - 2e^{-z} - z - ze^{-z}}{z(1-e^{-z})} \ . \tag{4.3}$$

Using the fact that $k(z)$ is negative, one can verify that the numerator of (4.3) is decreasing, from which the required result follows. ■

THEOREM **4.4** *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\tilde{\lambda} = (\prod_{i=1}^n \lambda_i)^{1/n}$. Then*

(a) $X_{n:n} \geq_{disp} Y_{n:n}$;

(b) $X_{n:n} \geq_{hr} Y_{n:n}$.

PROOF : (a) The distribution function of $X_{n:n}$ is

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n}\left(1 - e^{-\lambda_i x}\right),$$

with density function as

$$f_{X_{n:n}}(x) = \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}} \prod_{i=1}^{n}\left(1 - e^{-\lambda_i x}\right). \tag{4.4}$$

Replacing λ_i with $\tilde{\lambda}$ in (4.4), we see that the distribution function and the density function of $Y_{n:n}$ are

$$F_{Y_{n:n}}(x) = \left(1 - e^{-\tilde{\lambda}x}\right)^n \quad \text{and} \quad f_{Y_{n:n}}(x) = n\tilde{\lambda}e^{-\tilde{\lambda}x}\left(1 - e^{-\tilde{\lambda}x}\right)^{n-1},$$

respectively. It is easy to verify that $F^{-1}_{Y_{n:n}}(x) = -\frac{1}{\tilde{\lambda}}\log\left(1 - x^{1/n}\right)$. Using these observations, it follows that

$$f_{Y_{n:n}}\left(F^{-1}_{Y_{n:n}} F_{X_{n:n}}(x)\right) = n\tilde{\lambda}\left(1 - \prod_{i=1}^{n}(1 - e^{-\lambda_i x})^{1/n}\right)\left(\prod_{i=1}^{n}(1 - e^{-\lambda_i x})^{1/n}\right)^{n-1}. \tag{4.5}$$

To prove that $X_{n:n} \geq_{disp} Y_{n:n}$, it follows from relation (2.2) that it is sufficient to show that

$$f_{X_{n:n}}(x) \leq f_{Y_{n:n}}\left(F^{-1}_{Y_{n:n}} F_{X_{n:n}}(x)\right) \quad \forall x > 0. \tag{4.6}$$

Using expressions (4.4) and (4.5) in (4.6), one can see after some simplifications that (4.6) is equivalent to

$$\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} - n\prod_{i=1}^{n}\left(\frac{\lambda_i}{1 - e^{-\lambda_i x}}\right)^{1/n} \leq \sum_{i=1}^{n} \lambda_i - n\prod_{i=1}^{n}(\lambda_i)^{1/n} \ . \tag{4.7}$$

To prove that (4.7) holds for all $\lambda_i > 0, i = 1, \ldots, n$, it is sufficient to show that the L.H.S. of (4.7) (denoted by $h(x)$) is increasing in x since for $x > 0$,

$$h(x) \leq lim_{x \to +\infty} h(x) = \sum_{i=1}^{n} \lambda_i - n\prod_{i=1}^{n}(\lambda_i)^{1/n} \ ,$$

the right hand side of (4.7).

The derivative of $h(x)$ is

$$
\begin{aligned}
h^{'}(x) &= \left(\sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1-e^{-\lambda_i x}}\right)\left(\prod_{i=1}^{n} \frac{\lambda_i}{1-e^{-\lambda_i x}}\right)^{1/n} - \sum_{i=1}^{n} \frac{\lambda_i^2 e^{-\lambda_i x}}{(1-e^{-\lambda_i x})^2} \\
&\geq \left(\sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1-e^{-\lambda_i x}}\right)\left(\frac{n}{\sum_{i=1}^{n} \frac{1-e^{-\lambda_i x}}{\lambda_i}}\right) - \sum_{i=1}^{n} \frac{\lambda_i^2 e^{-\lambda_i x}}{(1-e^{-\lambda_i x})^2} ,
\end{aligned}
$$

since the geometric mean of a set of numbers is always greater than or equal to its harmonic mean. Now $h^{'}(x) \geq 0$ if and only if,

$$
n \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1-e^{-\lambda_i x}} \geq \left(\sum_{i=1}^{n} \frac{\lambda_i^2 e^{-\lambda_i x}}{(1-e^{-\lambda_i x})^2}\right)\left(\sum_{i=1}^{n} \frac{1-e^{-\lambda_i x}}{\lambda_i}\right) . \tag{4.8}
$$

Multiplying both sides of (4.8) by $x(>0)$ and replacing the $\lambda_i x$ with z_i for $i = 1, \ldots, n$, it is enough to prove that

$$
n \sum_{i=1}^{n} \frac{z_i e^{-z_i}}{1-e^{-z_i}} \geq \left(\sum_{i=1}^{n} \frac{z_i^2 e^{-z_i}}{(1-e^{-z_i})^2}\right)\left(\sum_{i=1}^{n} \frac{1-e^{-z_i}}{z_i}\right) . \tag{4.9}
$$

The inequality in (4.9) follows immediately from Čebyšev's inequality (Theorem 1, p. 36 of Mitrinović, 1970), Lemma 4.1 and by writing

$$
\frac{z_i e^{-z_i}}{1-e^{-z_i}} = \left(\frac{z_i^2 e^{-z_i}}{(1-e^{-z_i})^2}\right)\left(\frac{1-e^{-z_i}}{z_i}\right) .
$$

This proves that $h(x)$ is increasing in x and hence the result.

(b) It follows from Theorem 5.8 of Barlow and Proschan (1981) that $Y_{n:n}$ is IFR. Using this and part (a), the required result follows from Theorem 2.1. ■

From the above results, we get the following convenient bounds on the hazard rate and the variance of $X_{n:n}$.

Corollary 4.2 *Under the conditions of Theorem 4.4,*

(a) the hazard rate $r_{X_{n:n}}$ of $X_{n:n}$ satisfies

$$
r_{X_{n:n}}(x; \boldsymbol{\lambda}) \leq \frac{n\tilde{\lambda}\left(1-exp(-\tilde{\lambda}x)\right)^{n-1} exp(-\tilde{\lambda}x)}{1-\left(1-exp(-\tilde{\lambda}x)\right)^{n}} ,
$$

(b)

$$
var(X_{n:n}; \boldsymbol{\lambda}) \geq \frac{1}{\tilde{\lambda}^2} \sum_{i=1}^{n} \frac{1}{(n-i+1)^2} .
$$

Dykstra, Kochar and Rojo (1997) proved a result similar to Theorem 4.4 by assuming that the random variables Y_i's are exponential with common hazard rate $\overline{\lambda} = \sum_{i=1}^{n} \lambda_i/n$ and obtained bounds on the hazard rate and the variance of $X_{n:n}$ in terms of $\overline{\lambda}$. The new bounds given in Corollary 4.2 are better because $r_{Y_{n:n}}$ and $var(Y_{n:n})$ are increasing and decreasing function of $\tilde{\lambda}$, respectively, and the fact that the geometric mean of λ_i's is smaller than their arithmetic mean.

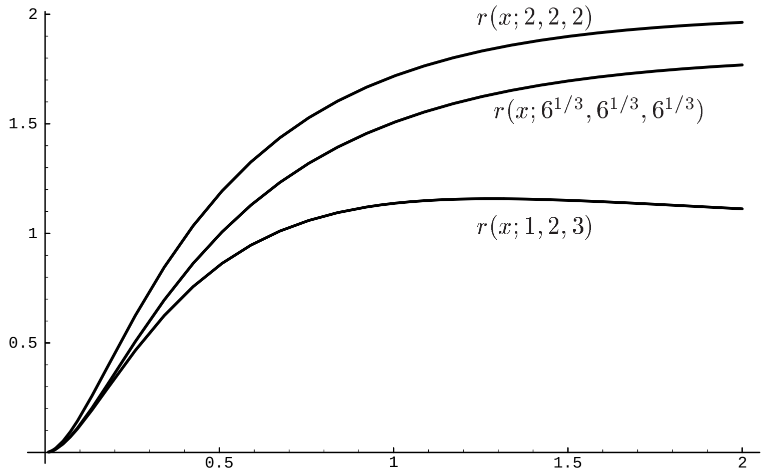

Figure 4.1. Graphs of hazard rates of $X_{3:3}$

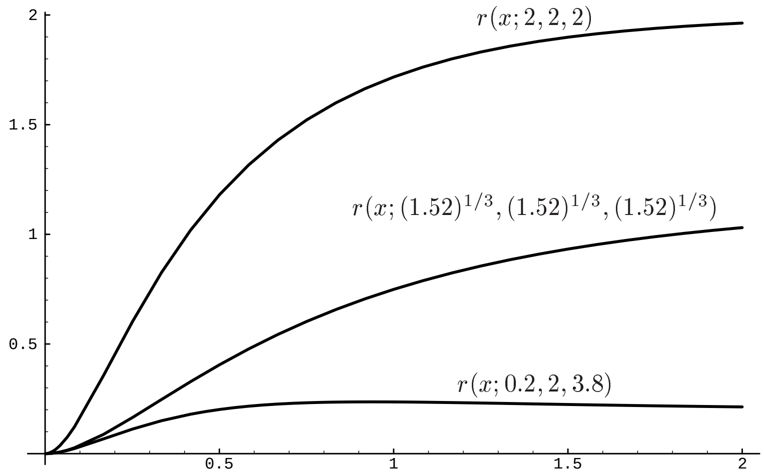

Figure 4.2. Graphs of hazard rates of $X_{3:3}$

In Figures 4.1. and 4.2. above, we plot the hazard rates of parallel systems of three exponential components along with the upper bounds as given by Dykstra, Kochar and Rojo (1997) and the one's given by Corollary 4.2 (a). The vector of parameters in Figure 4.1 is $\boldsymbol{\lambda_1} = (1,2,3)$ and that in Figure 4.2 is $\boldsymbol{\lambda_2} = (0.2,2,3.8)$. Note that $\boldsymbol{\lambda_2} \overset{m}{\succeq} \boldsymbol{\lambda_1}$. It appears from these figures that the improvements in the bounds are relatively more if λ_i's are more dispersed in the sense of majorization. This is true because the geometric mean is Schur-concave and the hazard rate of a parallel system of i.i.d. exponential components with a common parameter $\tilde{\lambda}$ is increasing in $\tilde{\lambda}$.

Let $\overline{F}$ denote the survival function of a nonnegative random variable X with hazard rate h. According to the PHR model, the random variables $X_1, \ldots, X_n$ are independent with X_i having survival function $\overline{F}^{\lambda_i}(.)$, so that its hazard rate is $\lambda_i h(.)$, $\quad i = 1, \ldots, n$.

Next, we extend Theorem 4.4 from exponential to PHR models. To prove this we need the following theorem due to Rojo and He (1991).

THEOREM **4.5** *Let X and Y be two random variables such that $X \leq_{st} Y$. Then $X \leq_{disp} Y$ implies that $\gamma(X) \leq_{disp} \gamma(Y)$ where γ is a nondecreasing convex function.*

THEOREM **4.6** *Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\overline{F}^{\lambda_i}(x)$, $\quad i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from a distribution with survival function $\overline{F}^{\tilde{\lambda}}(x)$, where $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$. Then*

(a) $X_{n:n} \geq_{hr} Y_{n:n}$; and

(b) if F is DFR, then $X_{n:n} \geq_{disp} Y_{n:n}$.

PROOF : (a)

Let $H(x) = -\log \overline{F}(x)$ denote the cumulative hazard of F. Let $Z_i = H(X_i)$, $i = 1, \ldots, n$ and $W_i = H(Y_i)$, $i = 1, \ldots, n$. Since X_i's follow the PHR model, then it is easy to show that Z_i is exponential with hazard rate λ_i, $i = 1, \ldots, n$. Similarly, W_i is exponential with hazard rate $\tilde{\lambda}$, $i = 1, \ldots, n$. Theorem 4.4 (b) implies that $Z_{n:n} \geq_{hr} W_{n:n}$. Using this fact, (since H^{-1}, the right inverse of H, is nondecreasing) it is easy to show that $H^{-1}(Z_{n:n}) \geq_{hr} H^{-1}(W_{n:n})$ from which the part (a) follows.

(b) Theorem 4.4 (a) and (b), respectively, imply that $Z_{n:n} \geq_{disp} W_{n:n}$ and $Z_{n:n} \geq_{st} W_{n:n}$. The function $H^{-1}(x)$ is convex, since F is DFR, and is nondecreasing. Using these observations, it follows from Theorem 4.5 that $H^{-1}(Z_{n:n}) \geq_{disp} H^{-1}(W_{n:n})$ which is equivalent to $X_{n:n} \geq_{disp} Y_{n:n}$. ■

In Theorem 4.9 below we prove that for the largest order statistic, the conclusion of Theorem 4.3 holds under the weaker p-larger ordering. The proof of this theorem hinges on the following results.

THEOREM **4.7** *(Marshall and Olkin, 1979, p. 57) Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are ϕ is symmetric on I^n and for all $i \neq j$,*

$$(z_i - z_j)[\phi_{(i)}(z_i) - \phi_{(j)}(z_j)] \geq 0 \quad \textit{for all } z \in I^n,$$

where $\phi_{(i)}(z)$ denotes the partial derivative of ϕ with respect to its ith argument.

THEOREM **4.8** *(Marshall and Olkin, 1979, p. 59) A real-valued function ϕ on the set $A \subset I\!R^n$ satisfies*

$$\mathbf{x} \overset{w}{\succeq} \mathbf{y} \text{ on } A \Longrightarrow \phi(\mathbf{x}) \geq \phi(\mathbf{y})$$

if and only if ϕ is decreasing and Schur-convex on A.

LEMMA 4.2 *The function $\psi : I\!R^{+n} \to I\!R$ satisfies*

$$\mathbf{x} \overset{p}{\succeq} \mathbf{y} \Longrightarrow \psi(\mathbf{x}) \geq \psi(\mathbf{y}) \tag{4.10}$$

if and only if,

(i) $\psi(e^{a_1}, \ldots, e^{a_n})$ *is Schur-convex in* $(a_1, \ldots, a_n)$

(ii) $\psi(e^{a_1}, \ldots, e^{a_n})$ *is decreasing in a_i, for $i = 1, \ldots, n$,*

where $a_i = \log x_i$, for $i = 1, \ldots, n$.

PROOF : Using relation (2.3), we see that (4.10) is equivalent to

$$\mathbf{a} \overset{w}{\succeq} \mathbf{b} \Longrightarrow \psi(e^{a_1}, \ldots, e^{a_n}) \geq \psi(e^{b_1}, \ldots, e^{b_n}), \tag{4.11}$$

where $a_i = \log x_i$ and $b_i = \log y_i$, for $i = 1, \ldots, n$.

Taking $\phi(a_1, \ldots, a_n) = \psi(e^{a_1}, \ldots, e^{a_n})$ in Theorem 4.8, we get the required result. ■

Now we are ready to prove the next theorem.

THEOREM **4.9** *Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\overline{F}^{\lambda_i}(x)$, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be another set of random variables with Y_i having survival function $\overline{F}^{\lambda_i^*}(x)$, $i = 1, \ldots, n$. Then*

$$\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow X_{n:n} \geq_{st} Y_{n:n}.$$

PROOF : The survival function of $X_{n:n}$ can be written as

$$\overline{F}_{X_{n:n}}(x) = 1 - \prod_{i=1}^{n} (1 - e^{-e^{a_i} H(x)}), \tag{4.12}$$

where $a_i = \log \lambda_i$, $i = 1, \ldots, n$ and $H(x) = -\log \overline{F}(x)$.

Using Lemma 4.2, we find that it is enough to show that the function $\overline{F}_{X_{n:n}}$ given by (4.12) is Schur-convex and decreasing in a_i's. To prove its Schur-convexity, it follows from Theorem 4.7 that, we have to show that for $i \neq j$, $(a_i - a_j)(\frac{\partial \overline{F}_{X_{n:n}}}{\partial a_i} - \frac{\partial \overline{F}_{X_{n:n}}}{\partial a_j}) \geq 0$. That is,

$$H(x)(a_i - a_j)\left(\prod_{i=1}^{n}(1 - e^{-e^{a_i} H(x)})\right)\left(\frac{e^{a_j} e^{-e^{a_j} H(x)}}{1 - e^{-e^{a_j} H(x)}} - \frac{e^{a_i} e^{-e^{a_i} H(x)}}{1 - e^{-e^{a_i} H(x)}}\right) \geq 0, \text{for } i \neq j \tag{4.13}$$

since

$$\frac{\partial \overline{F}_{X_{n:n}}}{\partial a_i} = -\prod_{i=1}^{n}(1 - e^{-e^{a_i} H(x)})\left(\frac{H(x) e^{a_i} e^{-e^{a_i} H(x)}}{1 - e^{-e^{a_i} H(x)}}\right).$$

It is easy to see that the function $be^{-bH(x)}/(1 - e^{-bH(x)})$ is decreasing in b, for each fixed $x > 0$. Replacing b with e^{a_i}, it follows that the function $e^{a_i}e^{-e^{a_i}H(x)}/(1 - e^{-e^{a_i}H(x)})$ is also decreasing in a_i for $i = 1, \ldots, n$. This proves that (4.13) holds. The partial derivative of $\overline{F}_{X_{n:n}}$ with respect to a_i is negative and which in turn implies that the survival function of $X_{n:n}$ is decreasing in a_i for $i = 1, \ldots, n$. This completes the proof. ■

The following result due to Khaledi and Kochar (2000 b) is a special case of Theorem 4.9.

COROLLARY 4.3 *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be another set of independent exponential random variables with Y_i having hazard rate λ_i^*, $i = 1, \ldots, n$. Then*

$$\boldsymbol{\lambda} \stackrel{p}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow X_{n:n} \geq_{st} Y_{n:n}.$$

Boland, El-Neweihi and Proschan (1994) pointed out that for $n > 2$, (4.2) cannot be strengthened from stochastic ordering to hazard rate ordering. Since majorization implies p-larger ordering, it follows that, in general, Theorem 4.9 cannot be strengthened to hazard rate ordering.

As shown in the next example, a result similar to Theorem 4.9 may not hold for other order statistics.

EXAMPLE 4.1 : Let X_1, X_2, X_3 be independent exponential random variables with $\boldsymbol{\lambda} = (0.1, 1, 7.9)$ and Y_1, Y_2, Y_3 be independent exponential random variables with $\boldsymbol{\lambda}^* = (1, 2, 5)$. It is easy to see that $\boldsymbol{\lambda} \stackrel{p}{\succeq} \boldsymbol{\lambda}^*$. The $X_{1:3}$ and $Y_{1:3}$ have exponential distributions with respective hazard rates 9 and 8 and which implies that $Y_{1:3} \geq_{st} X_{1:3}$.

5 Stochastic Comparisons of Sample Spacings

Let $X_1, \ldots, X_n$ be n random variables. The random variables $D_{i:n} = X_{i:n} - X_{i-1:n}$ and $D^*_{i:n} = (n - i + 1)D_{i:n}$, $i = 1, \ldots, n$, with $X_{0:n} \equiv 0$, are respectively called spacings and normalized spacings. They are of great interest in various areas of statistics, in particular, in characterizations of distributions, goodness-of-fit tests, life testing and reliability models. In the reliability context they correspond to times elapsed between successive failures of components in a system. It is well known that the normalized spacings of a random sample from an exponential distribution are i.i.d. random variables having the same exponential distribution. Such a characterization may not hold for other distributions and much of the reliability theory deals with this aspect of spacings. In this section we review stochastic properties of spacings when original random variables are i.i.d. as well as when they are independent but not identically distributed.

Many authors have studied the stochastic properties of spacings from restricted families of distributions. Barlow and Proschan (1966) proved that if $X_1, \ldots, X_n$ is a random sample from a DFR distribution, then the successive normalized spacings are stochastically increasing.

Kochar and Kirmani (1995) strengthened this result from stochastic ordering to hazard rate ordering, that is, for $i = 1, \ldots, n-1$,

$$D^*_{i:n} \leq_{hr} D^*_{i+1:n}. \tag{5.1}$$

The corresponding problem when the random variables are not identically distributed, has also been studied by many researchers, including Pledger and Proschan (1971), Shaked and Tong (1984), Kochar and Korwar (1996), Kochar and Rojo (1996), Nappo and Spizzichino (1998), among others. For a review of this topic see Kochar (1998). Here we give some new results obtained recently by the authors.

Kochar and Korwar (1996) conjectured that a result similar to (5.1) holds in the case when $X_1, \ldots, X_n$ are independent exponential random variables with X_i having hazard rate λ_i, for $i = 1, \ldots, n$. Khaledi and Kochar (2001) proved this conjecture when random variables X_i's follow a single outlier model with parameters λ and λ^*, that is when $\lambda_1 = \ldots = \lambda_{n-1} = \lambda$ and $\lambda_n = \lambda^*$. To prove this we shall be using the following results.

The joint density function of the spacings when λ_i's are possibly different is given by (cf. Kochar and Korwar, 1996),

$$f_{D_{1:n},\ldots,D_{n:n}}(x_1, \ldots, x_n) = \sum_{(\mathbf{r})} \frac{\prod_{i=1}^n \lambda_i}{\prod_{i=1}^n \sum_{j=i}^n \lambda(r_j)} \prod_{i=1}^n (\sum_{j=i}^n \lambda(r_j)) exp\{-x_i \sum_{j=i}^n \lambda(r_j)\}, \tag{5.2}$$

for $x_i \geq 0$, $i = 1, \ldots, n$, where $(\mathbf{r}) = (r_1, \ldots, r_n)$ is a permutation of $(1, \ldots, n)$ and $\lambda(i) = \lambda_i$. It is a mixture of products of exponential random variables. From (5.2) it is easy to find that the joint pdf of $(D_{i:n}, D_{j:n})$ for $1 \leq i < j \leq n$, is

$$\begin{aligned} f_{D_{i:n},D_{j:n}}(x, y) &= \sum_{(\mathbf{r})} \frac{\prod_{i=1}^n \lambda_i}{\prod_{i=1}^n \sum_{j=i}^n \lambda(r_j)} \\ &\times (\sum_{m=i}^n \lambda(r_m)) exp\{-x \sum_{m=i}^n \lambda(r_m)\} (\sum_{m=j}^n \lambda(r_m)) exp\{-y \sum_{m=j}^n \lambda(r_m)\}, \end{aligned} \tag{5.3}$$

for x, $y \geq 0$. Now (5.2) can be written as

$$\begin{aligned} f_{D_{1:n},\ldots,D_{n:n}}(x_1, \ldots, x_n) &= \sum_{\theta=1}^n \frac{(n-1)!\lambda^*(\lambda)^{n-1}}{\prod_{i=1}^\theta ((n-i)\lambda + \lambda^*) \prod_{i=\theta+1}^n (n-i+1)\lambda} \\ &\times \prod_{i=1}^\theta ((n-i)\lambda + \lambda^*) e^{-((n-i)\lambda+\lambda^*)x_i} \prod_{i=\theta+1}^n (n-i+1)\lambda e^{-(n-i+1)\lambda x_i}, \end{aligned} \tag{5.4}$$

which can be further expressed as

$$f_{D_{1:n},\ldots,D_{n:n}}(x_1, \ldots, x_n) = \sum_{\theta=1}^n h(\theta) \prod_{i=1}^\theta \alpha_i^* e^{-\alpha_i^* x_i} \prod_{i=\theta+1}^n \alpha_i e^{-\alpha_i x_i},$$

where $\alpha_i = (n-i+1)\lambda$, $\alpha_i^* = (n-i)\lambda + \lambda^*$, $i = 1, \ldots, n$ and using α_i and α_i^*, the function h is given by

$$h(\theta) = \frac{(n-1)!\lambda^{n-1}\lambda^*}{\prod_{i=1}^\theta \alpha_i^* \prod_{i=\theta+1}^n \alpha_i}, \quad \theta = 1, \ldots, n. \tag{5.5}$$

The marginal density function of $D_{i:n}$ can be expressed as

$$f_{D_{i:n}}(x) = H_i \alpha_i e^{-\alpha_i x} + \overline{H}_i \alpha_i^* e^{-\alpha_i^* x}, \tag{5.6}$$

where

$$H_i = \sum_{\theta=1}^{i-1} h(\theta), \quad i = 2, \ldots, n \text{ and } H_1 = 0. \tag{5.7}$$

Thus, the density function of $D_{i:n}$ is a mixture of two exponential random variables with parameters α_i and α_i^*. Now we prove the main theorem.

THEOREM **5.1** *Let $X_1, \ldots, X_n$ follow the single-outlier exponential model with parameters λ and λ^*. Then*

$$D^*_{i+1:n} \geq_{hr} D^*_{i:n}, \quad i = 1, \ldots, n-1.$$

PROOF : We prove the result when $\lambda^* > \lambda$. The proof for the case $\lambda^* < \lambda$ follows using the same kind of arguments. From (5.6) we find that the survival function of $D^*_{i:n}$ is $\overline{F}_{D^*_{i:n}}(x) = H_i e^{-\lambda x} + \overline{H}_i e^{-\eta_i x}$, where $\eta_i = \frac{(n-i)\lambda + \lambda^*}{n-i+1}$. To prove the theorem we have to show that for any $i \in \{1, \ldots, n-1\}$,

$$g(x) = \frac{\overline{F}_{D^*_{i+1:n}}(x)}{\overline{F}_{D^*_{i:n}}(x)}$$

is increasing in x. The numerator of $g'(x)$, the derivative of $g(x)$ is

$$\begin{aligned}
A(x) &= [H_i e^{-\lambda x} + \overline{H}_i e^{-\eta_i x}][-\lambda H_{i+1} e^{-\lambda x} - \eta_{i+1} \overline{H}_{i+1} e^{-\eta_{i+1} x}] \\
&\quad + [H_{i+1} e^{-\lambda x} + \overline{H}_{i+1} e^{-\eta_{i+1} x}][\lambda H_i e^{-\lambda x} + \eta_i \overline{H}_i e^{-\eta_i x}] \\
&= (\lambda^* - \lambda) \left\{ \frac{\overline{H}_i H_{i+1}}{n-i+1} e^{-(\eta_i + \lambda)x} \right. \\
&\quad \left. - \frac{\overline{H}_{i+1} H_i}{n-i} e^{-(\eta_{i+1} + \lambda)x} - \frac{\overline{H}_i \overline{H}_{i+1}}{(n-i+1)(n-i)} e^{(\eta_i + \eta_{i+1})x} \right\} \\
&\geq (\lambda^* - \lambda) \left\{ \left(\frac{\overline{H}_i H_{i+1}}{n-i+1} - \frac{\overline{H}_{i+1} H_i}{n-i} \right) e^{-(\eta_{i+1} + \lambda)x} \right. \qquad (5.8) \\
&\quad \left. - \frac{\overline{H}_i \overline{H}_{i+1}}{(n-i+1)(n-i)} e^{(\eta_i + \eta_{i+1})x} \right\} \\
&= \frac{(\lambda^* - \lambda)}{(n-i)(n-i+1)} \left\{ \left\{ (n-i)\overline{H}_i - (n-i+1)\overline{H}_{i+1} + \overline{H}_i \overline{H}_{i+1} \right\} \right. \\
&\times \left. e^{-(\eta_{i+1} + \lambda)x} - \overline{H}_i \overline{H}_{i+1} e^{-(\eta_i + \eta_{i+1})x} \right\}. \qquad (5.9)
\end{aligned}$$

The inequality in (5.8) follows, since $\lambda^* > \lambda$ implies $\eta_{i+1} > \eta_i$.

Again $\lambda^* > \lambda$ implies $\lambda < \eta_i$ and which in turn implies $e^{-(\eta_{i+1} + \lambda)x} \geq e^{-(\eta_i + \eta_{i+1})x}$ for every $x \geq 0$. Also for $\lambda^* > \lambda$,

$$\begin{aligned}
\left\{ (n-i)\overline{H}_i - (n-i+1)\overline{H}_{i+1} \right\} &= (n-i)h(i) - \overline{H}_{i+1} \\
&\geq 0, \qquad (5.10)
\end{aligned}$$

since for $\lambda^* > \lambda$, $h(j)$ is a decreasing function of j. Using these results in (5.9) we find that $A(x)$ and hence $g'(x)$ is nonnegative for $x \geq 0$. This proves the required result. ∎

Let $X_1, \ldots, X_n$ be independent exponential random variables with hazard rates $\lambda_1, \ldots, \lambda_n$, respectively. Pledger and Proschan (1971) proved that for $i \in \{1, \ldots, n\}$, $D_{i:n}$ is stochastically larger when the hazard rates are unequal than when they are all equal. Kochar and Rojo (1996) strengthened this result to likelihood ratio ordering. The natural question is to examine whether the survival function of $D_{i:n}$ is Schur-convex in $(\lambda_1, \ldots, \lambda_n)$. Pledger and Proschan (1971) came up with a counterexample to show that this is not true in general. Kochar and Korwar (1996) proved that in the special case of second spacing, whereas the survival function of $D_{2:n}$ is Schur-convex in $(\lambda_1, \ldots, \lambda_n)$, its hazard rate is not Schur-concave. They proved, however, that the hazard rate of $D_{2:2}$ is Schur-concave. We now examine this question when $X_1, \ldots, X_n$ follow the single-outlier exponential model with parameters λ and λ^*. In the rest of this section, we assume that $\lambda^* < \lambda$. We will treat it as a part of the model. It is easy to see that in this case, $(\lambda_1^*, \lambda_1, \ldots, \lambda_1) \overset{m}{\succeq} (\lambda_2^*, \lambda_2, \ldots, \lambda_2)$ if and only if $\lambda_1^* < \lambda_2^* < \lambda_2 < \lambda_1$ and $\lambda_1^* + (n-1)\lambda_1 = \lambda_2^* + (n-1)\lambda_2$. We prove later in this section that for the single-outlier model, for $i \in \{1, \ldots, n\}$, the hazard rate of $D_{i:n}$ is Schur-concave in λ's. To prove it we need the following lemmas.

LEMMA 5.1 *Let $X_1, \ldots, X_n$ follow the single-outlier exponential model with parameters λ and λ^*. Then*

$$\lambda^* < \lambda \Longrightarrow H_i \leq \frac{i-1}{n}, \text{ for } i = 1, \ldots, n, \tag{5.11}$$

where H_i is given by (5.7). *The inequality in* (5.11) *is reversed for $\lambda^* > \lambda$.*

PROOF : $\lambda^* < \lambda$ implies that the function $h(j)$ in (5.5) is increasing in j, $j = 1, \ldots, n$. Note that

$$(h(1), h(2), \ldots, h(n)) \overset{m}{\succeq} (1/n, \ldots, 1/n).$$

The required result follows from the definition of majorization.

The proof for the case $\lambda^* > \lambda$ follows from the same kind of arguments. ∎

LEMMA 5.2 *Let $X_1, \ldots, X_n$ follow the single-outlier exponential model with parameters λ_1 and λ_1^*. Let $Y_1, \ldots, Y_n$ be another set of random variables following the single-outlier exponential model with parameters λ_2 and λ_2^*. If*

(i) $\lambda_1^* < \lambda_2^* < \lambda_2 < \lambda_1$, then $\Theta_1 \geq_{lr} \Theta_2$,

(ii) $\lambda_1 < \lambda_2 < \lambda_2^* < \lambda_1^*$, then $\Theta_1 \leq_{lr} \Theta_2$,

where Θ_1 and Θ_2 correspond to random variable Θ with probability mass function $h(j)$ in (5.5) *for X_i's and Y_i's, respectively.*

PROOF : (i) We prove that

$$\frac{h_2(\theta+1)}{h_1(\theta+1)} \leq \frac{h_2(\theta)}{h_1(\theta)},$$

where h_1 and h_2 are probability mass functions of Θ_1 and Θ_2, respectively. This inequality holds if and only if

$$\frac{(n-\theta-1)\lambda_1+\lambda_1^*}{(n-\theta-1)\lambda_2+\lambda_2^*} \leq \frac{\lambda_1}{\lambda_2}. \tag{5.12}$$

Since $\lambda_1^* < \lambda_2^*$ and $\lambda_2 < \lambda_1$, it is easy to see that (5.12) is true.

(ii) In this case the inequality in (5.12) is reversed which in turn implies that $\Theta_1 \leq_{lr} \Theta_2$. This proves the result. ■

THEOREM **5.2** *Let $X_1, \ldots, X_n$ follow the single-outlier exponential model with parameters λ_1 and λ_1^* with $\lambda_1^* < \lambda_1$. Then for $i \in \{1, \ldots, n\}$, the hazard rate of $D_{i:n}$ is Schur-concave in $\{\lambda_1, \ldots, \lambda_1, \lambda_1^*\}$.*

PROOF : Let $Y_1, \ldots, Y_n$ be another set of random variables following the single-outlier exponential model with parameters λ_2 and λ_2^* $(\lambda_2^* < \lambda_2)$ such that $(\lambda_1^*, \lambda_1, \ldots, \lambda_1) \overset{m}{\succeq} (\lambda_2^*, \lambda_2, \ldots, \lambda_2)$. As discussed above this holds if and only if $\lambda_1^* < \lambda_2^* < \lambda_2 < \lambda_1$ and $\lambda_1^*+(n-1)\lambda_1 = \lambda_2^*+(n-1)\lambda_2$. Without loss of generality, let us assume that $\lambda_1^* + (n-1)\lambda_1 = 1$. We have to prove that under the given conditions for $i = 1, \ldots, n$,

$$D_{i:n}^{(1)} \geq_{hr} D_{i:n}^{(2)},$$

where $D_{i:n}^{(1)}$ $(D_{i:n}^{(2)})$ denotes the ith spacing of X_i's (Y_i's). From (5.6) the survival functions of $D_{i:n}^{(1)}$ and $D_{i:n}^{(2)}$ are

$$\overline{F}_{D_{i:n}^{(1)}}(x) = P_i e^{-\alpha_{i1}x} + \overline{P}_i e^{-\alpha_{i1}^* x},$$

$$\overline{F}_{D_{i:n}^{(2)}}(x) = Q_i e^{-\alpha_{i2}x} + \overline{Q}_i e^{-\alpha_{i2}^* x},$$

where P_i and Q_i correspond to H_i in (5.6) for $D_{i:n}^{(1)}$ and $D_{i:n}^{(2)}$, respectively and $\alpha_{i1} = (n-i+1)\lambda_1$, $\alpha_{i1}^* = (n-i)\lambda_1 + \lambda_1^*$, $\alpha_{i2} = (n-i+1)\lambda_2$ and $\alpha_{i2}^* = (n-i)\lambda_2 + \lambda_2^*$.

We have to show that

$$\phi(x) = \frac{\overline{F}_{D_{i:n}^{(1)}}(x)}{\overline{F}_{D_{i:n}^{(2)}}(x)}$$

is increasing in x. After some simplifications we find that the numerator of $\phi'(x)$, the derivative of $\phi(x)$ is

$$\begin{aligned} g(x) &= -(\alpha_{i1}-\alpha_{i2})P_iQ_ie^{-(\alpha_{i1}+\alpha_{i2})x} + (\alpha_{i2}^*-\alpha_{i1}^*)\overline{P}_i\overline{Q}_ie^{-(\alpha_{i1}^*+\alpha_{i2}^*)x} \\ &- (\alpha_{i1}^*-\alpha_{i2})Q_i\overline{P}_ie^{-(\alpha_{i2}+\alpha_{i1}^*)x} + (\alpha_{i2}^*-\alpha_{i1})\overline{Q}_iP_ie^{-\alpha_{i1}+\alpha_{i2}^*)x}, \end{aligned} \tag{5.13}$$

Using the assumption $\lambda_1^* < \lambda_2^* < \lambda_2 < \lambda_1$ and the fact the $\lambda_i^* + (n-1)\lambda_i = 1$, $i = 1, 2$, it follows, $\alpha_{i1} + \alpha_{i2}^* < \alpha_{i1} + \alpha_{i2}$, $\alpha_{i1} + \alpha_{i2}^* > \alpha_{i1}^* + \alpha_{i2}^*$, $\alpha_{i1} + \alpha_{i2}^* > \alpha_{i1}^* + \alpha_{i2}$ and all $(\alpha_{i1} - \alpha_{i2})$, $(\alpha_{i2}^* - \alpha_{i1}^*)$, $(\alpha_{i2} - \alpha_{i1}^*)$, are nonnegative. Using these observations in (5.13), we see

$$g(x) \geq e^{-(\alpha_{i1}+\alpha_{i2}^*)x}\{-(\alpha_{i1}-\alpha_{i2})P_iQ_i + (\alpha_{i2}^*-\alpha_{i1}^*)\overline{P}_i\overline{Q}_i$$

$$
\begin{aligned}
& -(\alpha_{i1}-\alpha_{i2}^*)\overline{Q}_iP_i+(\alpha_{i2}-\alpha_{i1}^*)Q_i\overline{P}_i\} \\
= \; & \frac{e^{-(\alpha_{i1}+\alpha_{i2}^*)x}}{n-1}\{Q_i-P_i-(nQ_i-(i-1))\lambda_2^*+(nP_i-(i-1))\lambda_1^*\} \\
\geq \; & \frac{e^{-(\alpha_{i1}+\alpha_{i2}^*)x}}{n-1}\{Q_i-P_i-n(Q_i-P_i)\lambda_2^*\} \qquad (5.14)\\
= \; & \frac{e^{-(\alpha_{i1}+\alpha_{i2}^*)x}}{n-1}(Q_i-P_i)(1-n\lambda_2^*) \\
\geq \; & 0. \qquad (5.15)
\end{aligned}
$$

The inequality in (5.14) follows, since by Lemma 5.1 $P_i \leq \frac{i-1}{n}$ and $\lambda_1^* < \lambda_2^*$. From Lemma 5.2 it follows that $Q_i \geq P_i$, since it is known the likelihood ratio ordering implies usual stochastic ordering. This observation along with the fact that $\lambda_2^* \leq 1/n$ implies the inequality in (5.15). ■

Remark : The conclusion of Theorem 5.2 holds if instead of $\lambda_1^* < \lambda_1$ and $\lambda_2^* < \lambda_2$ we assume that $\lambda_1^* > \lambda_1$ and $\lambda_2^* > \lambda_2$.

It is known that spacings of independent exponential random variables have DFR distributions (cf. Kochar and Korwar, 1996). Combining this observation with Theorem 2.1, we have proved the following corollary.

Corollary 5.1 *Under the assumptions of Theorem 5.2,*

$$D_{i:n}^{(1)} \geq_{disp} D_{i:n}^{(2)}.$$

A consequence of Corollary 5.1 is that $var(D_{i:n}^{(1)}) \geq var(D_{i:n}^{(2)})$, $i = 1, \ldots, n$.

6 Stochastic ordering for sample range

Sample range is one of the criteria for comparing variabilities among distributions and hence it is important to study its stochastic properties. First we study the stochastic properties of the range of a random sample from a continuous distribution. Let $X_1, \ldots, X_n$ be a random sample from F and let $Y_1, \ldots, Y_n$ be an independent random sample from another distribution G. It follows from Lemma 3(c) of Bartoszewic (1986) that $X \geq_{disp} Y \Rightarrow X_{n:n} - X_{1:n} \geq_{st} Y_{n:n} - Y_{1:n}$. This observation along with Theorem 2.1 (a) leads to the following theorem.

Theorem **6.1** *Let $X \geq_{hr} Y$ and let either F or G be DFR. Then*

$$X_{n:n} - X_{1:n} \geq_{st} Y_{n:n} - Y_{1:n}. \qquad (6.1)$$

Next we consider the case when the parent observations are independent exponentials but with unequal parameters. Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\overline{\lambda}$, the arithmetic mean of the λ_i's. Finally, let $R_X = X_{n:n} - X_{1:n}$ and $R_Y = Y_{n:n} - Y_{1:n}$ denote the sample ranges of X_i's and Y_i's, respectively. Kochar and Rojo (1996) proved that $R_X \geq_{st} R_Y$. Khaledi and Kochar (2000 c) proved the following result which is in terms of $\tilde{\lambda}$, the geometric mean of the λ_i's.

Theorem 6.2 *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, for $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\tilde{\lambda}$. Then,*

$$R_X \geq_{st} R_Y.$$

Proof : The distribution function of R_X (see David, 1981, p. 26) is

$$F_{R_X}(x) = \frac{1}{\sum_{i=1}^n \lambda_i} \sum_{i=1}^n \frac{\lambda_i}{1 - e^{-\lambda_i x}} \prod_{i=1}^n (1 - e^{-\lambda_i x}). \tag{6.2}$$

and that of R_Y is

$$G_{R_Y}(x) = \left(1 - e^{-\tilde{\lambda} x)}\right)^{n-1}. \tag{6.3}$$

Using (6.2) and (6.3), we have to show that

$$\sum_{i=1}^n \frac{\lambda_i}{1 - e^{-\lambda_i x}} \prod_{i=1}^n (1 - e^{-\lambda_i x}) \leq \sum_{i=1}^n \lambda_i \left(1 - e^{-\tilde{\lambda} x)}\right)^{n-1}. \tag{6.4}$$

Multiplying both sides of (6.4) by $x (> 0)$, it is sufficient to prove that

$$\sum_{i=1}^n \frac{\lambda_i x}{1 - e^{-\lambda_i x}} \prod_{i=1}^n (1 - e^{-\lambda_i x}) \leq \left(\sum_{i=1}^n \lambda_i x\right) \left(1 - e^{-\tilde{\lambda} x)}\right)^{n-1}. \tag{6.5}$$

Dykstra, Kochar and Rojo (1997) proved that

$$\sum_{i=1}^n \frac{y_i}{1 - e^{-y_i}} \leq \left(\sum_{i=1}^n y_i\right) \prod_{i=1}^n (1 - e^{-y_i})^{-\frac{1}{n}},$$

where $y_i > 0$ for $i = 1, \ldots, n$. Making use of this inequality on the L.H.S. of (6.5), we get

$$\sum_{i=1}^n \frac{\lambda_i x}{1 - e^{-\lambda_i x}} \prod_{i=1}^n (1 - e^{-\lambda_i x}) \leq \left(\sum_{i=1}^n \lambda_i x\right) \prod_{i=1}^n \left(1 - e^{-\lambda_i x}\right)^{\frac{n-1}{n}} \tag{6.6}$$

A consequence of Theorem 4.4 (b) is that $X_{n:n} \geq_{st} Y_{n:n}$, which is equivalent to $\prod_{i=1}^n (1 - e^{-\lambda_i x})^{1/n} \leq 1 - e^{-\tilde{\lambda} x}$. Using this result, we find that the expression on the R.H.S. of (6.6) is less than or equal to that on the R.H.S. of (6.5) and from which the required result follows. ■

As a consequence of this result we get the following upper bound on the distribution function of R_X in terms $\tilde{\lambda}$.

Corollary 6.1 *Under the conditions of Theorem 6.2, for $x > 0$,*

$$P[X_{n:n} - X_{1:n} \leq x] \leq \left[1 - e^{-\tilde{\lambda} x}\right]^{n-1}. \tag{6.7}$$

This bound is better than the one obtained in Kochar and Rojo (1996) in terms of $\overline{\lambda}$, since the expression on the R.H.S. of (6.7) is increasing in $\tilde{\lambda}$ and $\tilde{\lambda} \leq \overline{\lambda}$.

Now we extend Theorem 6.1 to the PHR model. We assume that F is *new worse than used* (NWU), that is,

$$\overline{F}(x + y) \geq \overline{F}(x)\overline{F}(y), \quad \text{for } x, y \geq 0,$$

or equivalently,

$$H(x+y) \leq H(x) + H(y), \quad \text{for } x, y \geq 0,$$

where $H(x) = -\log \overline{F}(x)$ denotes the cumulative hazard of F.

THEOREM **6.3** *Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\overline{F}^{\lambda_i}(x)$, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from a distribution with survival function $\overline{F}^{\tilde{\lambda}}(x)$, where $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$. If F is NWU, then $X_{n:n} - X_{1:n} \geq_{st} Y_{n:n} - Y_{1:n}$.*

PROOF :

The distribution function of the sample range $X_{n:n} - X_{1:n}$ (see David, 1981, p. 26) is

$$\begin{aligned}
F_{R_n^X}(x) &= \sum_{i=1}^{n} \int_0^{+\infty} \lambda_i h(t) e^{-\lambda_i H(t)} \prod_{j\neq i}^{n} \left(e^{-\lambda_j H(t)} - e^{-\lambda_j H(t+x)}\right) dt \\
&\leq \sum_{i=1}^{n} \int_0^{+\infty} \lambda_i h(t) e^{-\lambda_i H(t)} \prod_{j\neq i}^{n} \left(e^{-\lambda_j H(t)} - e^{-\lambda_j H(t)} e^{-\lambda_j H(x)}\right) dt \\
&\quad \text{(since } F \text{ is NWU)} \\
&= \sum_{i=1}^{n} \lambda_i \prod_{j\neq i} (1 - e^{-\lambda_j H(x)}) \int_0^{+\infty} h(t) \prod_{j=1}^{n} e^{-\lambda_j H(t)} dt \\
&= \sum_{i=1}^{n} \lambda_i \prod_{j\neq i} (1 - e^{-\lambda_j H(x)}) \int_0^{+\infty} h(t) e^{-H(t) \sum_{j=1}^{n} \lambda_j} \\
&= \frac{1}{\sum_{i=1}^{n} \lambda_i} \sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i H(x)}} \prod_{i=1}^{n} (1 - e^{-\lambda_i H(x)}), \quad x > 0,
\end{aligned}$$

Now, replacing x with $H(x)$ in the proof of Theorem 6.2, it is easy to see that

$$F_{R_n^X}(x) \leq F_{R_n^Y}(x).$$

■

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