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On testing dependence between time to failure and cause of failure via conditional probabilities

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Running title: On testing dependence

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ABSTRACT. Dependence structures between the failure time and the cause of failure are expressed in terms of the monotonicity properties of the conditional probabilities involving the cause of failure and the failure time. Further, these properties of the conditional probabilities are used for testing various dependence structures and several U-statistics are proposed. In the process, a concept of concordance and discordance between a continuous and a binary variable is introduced to propose an efficient test. The proposed tests are applied to two illustrative applications.

Key words: Competing risks, Conditional probability, Dependence structures, Subsurvival functions, U-statistics

1 Introduction

The common model for the competing risks situation is the latent lifetimes model. Under this model, the latent lifetimes are never observed together and data are available only on the minimum, T, of these and a variable, δ, identifying the minimum. The problem of identifiability due to such incomplete data is well known. Besides, there is a strong case made out against the latent lifetimes model by many biostatisticians such as Prentice *et al.* (1978) and others. Over the years, the latent lifetimes model has lost much of its lustre. Deshpande (1990), Aras and Deshpande (1992) and others have emphasized an alternative in terms of the observable random pair (T, δ) itself which seems more appropriate.

In this paper we consider the case of two competing risks and study the relations between the various kinds of dependence between $T \geq 0$ and $\delta \in \{0, 1\}$ and the shape of the conditional probability functions $\Phi_1(t) = pr(\delta = 1 \mid T \geq t)$ and $\Phi_0^*(t) = pr(\delta = 0 \mid T < t)$. Examples arise in many fields where such conditional probabilities are of primary importance. It is obvious that the independence of T and δ is equivalent to constancy of $\Phi_1(t)$ and is also equivalent to constancy of $\Phi_0^*(t)$. Many popular bivariate parametric distributions used in survival analysis have constant $\Phi_1(t)$ and $\Phi_0^*(t)$, for example Block and Basu (1974), Farlie-Gumbel-Morgenstern bivariate exponential distribution, Gumbel Type A distribution. However, in many practical situations, this is not the case. In clinical trials carried out to study the performance of an intra-uterine device where termination of the device could be due to several reasons such as pregnancy, expulsion, bleeding and pain, it is often of interest to know the chances of termination due to a specific reason given that the device was intact for some specified period. In such a situation, conditional probabilities are of interest and are expected to vary with time. In the report by Cooke *et al.*(1993) and references therein, it has been shown that different kinds of censoring mechanisms lead to distinct shapes of these functions. Random sign censoring, also known as age-dependent censoring, is a model in which the lifetime of a unit, X, is censored by $Z = X - W\eta$, where $0 < W < X$ is the time at which a warning is emitted by the unit before its failure, and η is a random variable taking values $\{-1, 1\}$ and is independent of X. When $W = aX$ for some $0 < a < 1$ and X is assumed to be exponential, it is easy to see that $\Phi_1(t)$ is increasing. Another model considered in Cooke *et al.* (1993) is a constant warning-constant inspection model in which a warning is emitted at time $X - d$ before it fails, where $d < 1$ is a constant and $\Phi_1(t)$ is a constant. A model where $\Phi_1(t)$ is decreasing is a proportional warning-constant inspection model which is similar to the constant warning-constant inspection model except that the warning is emitted at time X/η if the component fails at X and where $\eta > 1$ is a constant. The important question is that of choosing a model from these three models and it is obvious that the monotonicity of $\Phi_1(t)$ can be used to distinguish between these models.

Section 2 brings out the relationships between the shapes of the conditional probabilities and dependence structures between T and δ. In section 3, we consider the problem of testing

$$H_0 : T \; and \; \delta \text{ are independent}$$

against various alternative hypotheses, characterising the dependence structure of T and δ, which are:

$$\begin{aligned} H_1 &: T \text{ and } \delta \text{ are not independent} \\ H_2 &: T \text{ and } \delta \text{ are positive quadrant dependent} \\ H_3 &: \delta \text{ is right tail increasing in } T \\ H_4 &: \delta \text{ is left tail decreasing in } T. \end{aligned}$$

A test based on the concept of concordance and discordance is proposed for testing H_0 against H_1. Actually a one-sided version of the test is seen to be consistent against H_2 which is a special case of H_1. Two tests are proposed for testing H_0 against H_3 using the properties of $\Phi_1(t)$, and on the same lines two tests are proposed for testing H_0 against H_4 using the properties of $\Phi_0^*(t)$. Note that there is no relationship between H_3 and H_4 but both imply H_2. Two tests are proposed for this weaker hypothesis also. Some of the tests derived here are already in the literature but in other contexts. In section 4, relative efficiencies of these tests are studied and in section 5 the tests are applied to two real data sets. To the best of our knowledge, there are no tests available in the literature to check the dependence structure of T and δ, except $PQD(T, \delta)$.

2 Dependence of T and δ

Define $S_i(t) = pr(T > t, \delta = i)$, $i = 0, 1$, and $F_i(t) = pr(T \leq t, \delta = i)$, $i = 0, 1$. The survival function of T is given by $S(t) = pr(T > t) = S_0(t) + S_1(t)$ and the distribution function is given by $F(t) = pr(T \leq t) = F_0(t) + F_1(t)$. Throughout this paper, we assume that the subsurvival functions are continuous. This gives

$$\begin{aligned}\Phi_1(t) &= pr(\delta = 1 \mid T \geq t) = S_1(t-)/S(t-) \text{ and} \\ \Phi_0^*(t) &= pr(\delta = 0 \mid T < t) = F_0(t-)/F(t-),\end{aligned}$$

whenever $S(t-) > 0$ and $F(t-) > 0$. Equivalently, we can define $\Phi_0(t) = pr(\delta = 0 \mid T \geq t) = 1 - pr(\delta = 1 \mid T \geq t)$, and $\Phi_1^*(t) = pr(\delta = 1 \mid T < t) = 1 - pr(\delta = 0 \mid T < t)$. As mentioned earlier, $\Phi_1(t) = \Phi_1^*(t) = \phi$, for all $t > 0$ is equivalent to independence of T and δ. This simplifies the study of competing risks to a greater extent. If T and δ are independent then $S_i(t) = S(t)pr(\delta = i)$. Thus the hypothesis of equality of incidence functions, or that of equality of cause-specific hazard rates reduces to testing whether $pr(\delta = 1) = pr(\delta = 0) = 1/2$. Hence, it allows studying the failure time and the failure types or the risks of failure separately.

Before we study the dependence structure of T and δ, we provide few definitions.

Definition 2.1 *X_2 is Right Tail Increasing in X_1, $RTI(X_2 \mid X_1)$, if $pr(X_2 > t_2 \mid X_1 > t_1)$ is increasing in t_1 for all t_2.*

Definition 2.2 *X_2 is Left Tail Decreasing in X_1, $LTD(X_2 \mid X_1)$, if $pr(X_2 \leq t_2 \mid X_1 \leq t_1)$ is decreasing in t_1 for all t_2.*

Definition 2.3 *X_1 and X_2 are Positively Quadrant Dependent, $PQD(X_1, X_2)$, if $pr(X_1 > t_1, X_2 > t_2) \geq pr(X_1 > t_1)pr(X_2 > t_2)$, for all t_1, t_2 or equivalently, $pr(X_1 \leq t_1, X_2 \leq t_2) \geq pr(X_1 \leq t_1)pr(X_2 \leq t_2)$, for all t_1, t_2*

Definition 2.4 *A function $K(s, t)$ is Totally Positive of Order 2, TP_2, if*

$$K(s_1, t_1)K(s_2, t_2) \geq K(s_2, t_1)K(s_1, t_2)$$

for all $s_1 < s_2, t_1 < t_2$.

Note that, $RTI(X_2 \mid X_1)$ and $LTD(X_2 \mid X_1)$ both imply $PQD(X_1, X_2)$ but there is no hierarchy between $RTI(X_2 \mid X_1)$ and $LTD(X_2 \mid X_1)$.

2.1 Monotonicity of $\Phi_1(t)$ and $\Phi_0^*(t)$

The following results are easy to verify:

(1) Independence of T and δ is equivalent to

(a) $\Phi_1(t) = \phi = pr(\delta = 1)$, for all $t > 0$, a constant and

(b) $\Phi_0^*(t) = 1 - \phi = \phi_0 = pr(\delta = 0)$, for all $t > 0$, a constant.

(2) $PQD(\delta, T)$ is equivalent to

(a) $\Phi_1(t) \geq \Phi_1(0) = \phi,\ for\ all\ t > 0$, and

(b) $\Phi_0^*(t) \geq \Phi_0^*(\infty) = 1 - \phi,\ for\ all\ t > 0$.

(3) $RTI(\delta \mid T)$ is equivalent to $\Phi_1(t) \uparrow t$.

(4) Subsurvival functions $S_i(t)$ being TP_2 is equivalent to $\Phi_1(t) \uparrow t$.

(5) $LTD(\delta \mid T)$ is equivalent to $\Phi_0^*(t) \downarrow t$.

(6) Subdistribution function $F_i(t)$ being TP_2 is equivalent to $\Phi_0^*(t) \downarrow t$.

Note that (3) and (4) are equivalent and both imply (2). Similarly, (5) and (6) are equivalent and both imply (2) but there is no relationship between (3) and (5).

2.2 Hazard rate ordering and ageing

Let $r_i(t)$ and $h_i(t)$ denote crude and cause-specific hazard rates, respectively, $i = 0, 1$. Then

$$\begin{aligned} r_i(t) &= \frac{f_i(t)}{S_i(t-)} \\ h_i(t) &= \frac{f_i(t)}{S(t-)}. \end{aligned}$$

Note that $h_i(t) = \Phi_i(t) r_i(t)$. The overall hazard rate of T is $h(t) = f(t)/S(t-) = h_0(t) + h_1(t)$, where $f_i(.), f(.)$ are densities corresponding to $S_i(.)$ and $S(.)$, respectively.

Theorem 2.1 *$\Phi_1(t) \uparrow t$ is equivalent to $r_1(t) \leq h(t) \leq r_0(t)$.*

The proof follows by using the fact that the derivative of $\Phi_1(t)$ is non-negative and the derivative of $1 - \Phi_1(t)$ is non-positive being decreasing function of t.

Thus, $\Phi_1(t)$ is increasing means that the overall failure rate is larger than the failure rate given that the failure is due to risk 1 and is smaller than the failure rate given that the failure is due to risk 0. Another interesting result stated below connects the monotonicity of $\Phi_1(t)$ with the ordering between two survival functions.

Theorem 2.2 *$\Phi_1(t) \uparrow t$ implies that the survival function of T given $\delta = 1$ is larger than that of T given $\delta = 0$, that is, $S_1(t)/\phi \geq S_0(t)/(1 - \phi)$.*

It is important to note that the hazard rates $r_1(t)$ and $r_0(t)$ correspond to the above two distributions. Under the proportional hazards model, $h_1(t) = \phi h(t)$. This is equivalent to independence of T and δ and hence $\Phi_1(t) = \phi$, for all $t > 0$. It is easy to see that $h_1(t) \geq \phi h(t)$ implies $\Phi_1(t) \geq \Phi(0)$, for all t, that is, $PQD(\delta, T)$. Hence, the tests proposed in the next section can be used to test the proportionality of the two casue-specific hazards also. When $\phi \geq 1/2$, $S_1(t) \geq S_0(t)$ for all t and this means that there is stochastic dominance between the two incidence functions as well as the conditional distributions.

A result similar to Theorem 2.1 for cause-specific hazard rates is given below.

Theorem 2.3 *$\Phi_1(t) \uparrow t$ is equivalent to $h_1(t) \leq \Phi_1(t)h(t)$ and $h_0(t) \geq (1 - \Phi_1(t))h(t)$.*

The above theorem implies that $h_1(t)/h_0(t) \leq \Phi_1(t)/\{1 - \Phi_1(t)\}$. This puts functional bounds on the relative rate of ageing of two risks, see Sengupta and Deshpande (1994) for definitions of relative ageing. It is interesting and also useful to express the cause-specific hazard rate in terms of $\Phi_1(t)$. This enables one to study the ageing through the properties of $\Phi_1(t)$.

Theorem 2.4 *(a) $h_1(t) = -\Phi_1'(t) + \Phi_1(t)h(t)$, where $\Phi_1'(t)$ is the first derivative of $\Phi_1(t)$ with respect to t. (b) If $\Phi_1(t)$ is monotone increasing and concave then $h_1(t)$ is an increasing function of t, provided $r(t)$ is IFR.*

Proof : The proof is straightforward and follows from the definitions of $\Phi_1(t)$ and $h_1(t)$.

In case of independent latent lifetimes, the hazard rate of X is expressed in terms of $h_1(t)$. If $h_1(t)$ is IFR then X will also have IFR distribution. Further, let $r_i^*(t)$ and $h_i^*(t)$ denote crude and cause-specific reverse hazard rates, then

$$r_i^*(t) = \frac{f_i(t)}{F_i(t-)}$$

$$h_i^*(t) = \frac{f_i(t)}{F(t-)}.$$

All the above results hold true between these reverse hazards and the $\Phi_0^*(t)$. Since the results are quite similar the details are not given here. The above results bring out the fact that the various kinds of dependence between T and δ can be expressed in terms of various shapes of $\Phi_1(t)$ and $\Phi_0^*(t)$.

3 Test statistics and their distributions

3.1 General dependence between T and δ

Here we consider the problem of testing H_0 against H_1. Note that H_0 and H_1 can equivalently be stated as

$$\begin{aligned} H_0 &: \Phi_1(t) \text{ is a constant} \\ H_1 &: \Phi_1(t) \text{ is not a constant.} \end{aligned}$$

Kendall's τ is expected to work against a very general alternative of dependence. A pair (T_i, δ_i) and (T_j, δ_j) is a concordant pair if $T_i > T_j,\ \delta_i = 1, \delta_j = 0$ or $T_i < T_j,\ \delta_i = 0, \delta_j = 1$ and is a discordant pair if $T_i > T_j,\ \delta_i = 0, \delta_j = 1$ or $T_i < T_j,\ \delta_i = 1, \delta_j = 0$. Define the kernel

$$\psi_k(T_i, \delta_i, T_j, \delta_j) = \begin{cases} 1 & \text{if } T_i > T_j,\ \delta_i = 1, \delta_j = 0 \\ & \text{or } T_i < T_j,\ \delta_i = 0, \delta_j = 1 \\ -1 & \text{if } T_i > T_j,\ \delta_i = 0, \delta_j = 1 \\ & \text{or } T_i < T_j,\ \delta_i = 1, \delta_j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that when both δ_i and δ_j are 1 or $0, \delta_i - \delta_j = 0$. The corresponding U-statistic is given by

$$U_k = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \psi_k(T_i, \delta_i, T_j, \delta_j).$$

Note that

$$E(U_k) = 2\phi + 4 \int_0^\infty S(t) dS_1(t).$$

It is seen that $E(U_k) \geq (\leq) 0$ if T and δ are positive (negative) quadrant dependent. Hence, a one-sided test based on U_k can be used to test $PQD(T, \delta)$ also. It is easy to write the statistic U_k as a function of ranks. Let R_j be the rank of T_j. Let $T_{(1)} < \cdots < T_{(n)}$ be the ordered $T_i's$. Let

$$W_j = \begin{cases} 1 & \text{if } T_{(j)} \text{ corresponds to } \delta = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $V_k = \binom{n}{2} U_k$ can be written as

$$\begin{aligned} V_k &= \sum_{j=1}^n (2R_j - n - 1)\delta_j = \sum_{j=1}^n (2j - n - 1) W_j \\ &= \sum_{j=1}^n a_j W_j \end{aligned} \tag{3.1}$$

where $a_j = 2j - n - 1$.

A test given in equation (2.3), page 214, in Dykstra *et al.* (1996) in a different context, is $-U_k$ and the correct variance of V_n is $(1/3)n(n^2-1)\theta(1-\theta)$ and not the one given on page 215. The null distribution of V_k can be found from its moment generating function. Note that under H_0, $T_1, \ldots, T_n$ and $\delta_1, \ldots, \delta_n$ are independent. Hence, under H_0, $W_1, \ldots, W_n$ are independent and identically distributed with $pr(W_i = 1) = \phi$, $pr(W_i = 0) = 1 - \phi$. From here we obtain that the moment generating function of V_k, under H_0, is given by

$$M(t) = \prod_{j=1}^n [\phi \, exp\{t(2j - n - 1)\} + (1 - \phi)].$$

Hence the null distribution of V_k depends on the unknown ϕ even under H_0. For large n, we can estimate ϕ consistently by $\hat{\phi} = (1/n) \sum_{i=1}^n I(\delta_i = 1)$. Under H_0,

$$\begin{aligned} E(U_k) &= 0, \\ Var(U_k) &= \frac{4(n+1)}{3n(n-1)} \phi(1 - \phi). \end{aligned}$$

Note that $E(U_k) \neq 0$ under H_1. From the results on U-statistics it follows that U_k has asymptotic normal distribution for large n.

Theorem 3.1 *As n tends to ∞, under H_0, $n^{1/2}\{U_k - E(U_k)\}$ converges in distribution to $N(0, \sigma^2)$ where $\sigma^2 = (4/3)\phi(1 - \phi)$.*

A consistent estimator of variance is $\hat{\sigma^2} = (4/3)\hat{\phi}(1-\hat{\phi})$. A test procedure for testing H_0 against H_1 is then: reject H_0 at $100\alpha\%$ level of significance if $\mid n^{1/2}U_k/\hat{\sigma} \mid$ is larger than $z_{1-\alpha}$, the cut-off point of standard normal distribution.

It is clear that a one-sided test can also be used for testing H_0 against H_2 since it is based on concordance and discordance principle and the number of concordances are expected to be larger than the number of discordances under PQD.

3.2 Testing independence against $PQD(\delta, T)$

Consider testing H_0 against H_2.

A. Test based on $\Phi_1(t)$

H_2 is equivalent to

$$H_2 : \Phi_1(t) \geq \phi \text{ for all } t \text{ with strict inequality for some t.}$$

Consider

$$\Delta_3(S_1, S) = \int_0^\infty [S_1(t) - \phi S(t)]dF(t) = pr(T_2 > T_1, \delta_2 = 1) - \phi/2.$$

Under H_0, $S_1(t)/S(t) = \phi = pr(\delta = 1)$. This implies that $\Delta_3(S_1, S) = 0$. Under H_2, $S_1(t) > \phi S(t)$ and hence $\Delta_3(S_1, S) \geq 0$. Define the symmetric kernel

$$\psi_3(T_i, \delta_i, T_j, \delta_j) = \begin{cases} 1 & \text{if } T_j > T_i,\ \delta_j = 1 \\ & \text{or if } T_i > T_j,\ \delta_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

which is equivalent to

$$\psi_3(T_i, \delta_i, T_j, \delta_j) = \begin{cases} 1 & \text{if } T_j > T_i,\ \delta_j = 1, \delta_i = 0 \\ & \text{if } T_i > T_j,\ \delta_i = 1, \delta_j = 0 \\ & \text{if } \delta_i = \delta_j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the U-statistic corresponding to $\Delta_3(S_1, S)$ is given by

$$U_3 = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \psi_3(T_i, \delta_i, T_j, \delta_j).$$

Note that $E(U_3) = 2\Delta_3(S_1, S) + \phi$. Under H_0, $E(U_3) = \phi$, while under H_2 $E(U_3) \geq \phi$. Note that the statistic U_3 has earlier been proposed by Bagai *et al.* (1989) for testing the equality of failure rates of two independent competing risks. Then, following the arguments for U_k, we see that

$$\binom{n}{2} U_3 = \sum_{i=1}^{n} (R_i - 1)\delta_i = \sum_{i=1}^{n} (i-1)W_i. \tag{3.2}$$

Under H_0, the moment generating function is given by

$$M(t) = \prod_{j=1}^{n} [(1-\phi) + \phi exp\{t(j-1)\}].$$

When $\phi = 1/2$, $M(t)$ is same as that of Wilcoxon signed rank statistics with n replaced by $(n+1)$.

Theorem 3.2 *As n tends to ∞, under H_0, $n^{1/2}\{U_3 - E(U_3)\}$ converges in distribution to $N(0, \sigma_3^2)$, where $\sigma_3^2 = (4/3)\phi(1-\phi)$.*

A consistent estimator of variance is $\hat{\sigma_3^2} = (4/3)\hat{\phi}(1-\hat{\phi})$. We reject the null hypothesis for large values of $Z = n^{1/2}(U_3 - \hat{\phi})/\hat{\sigma_2}$.

B. Test based on $\Phi_0^*(t)$

H_2 is also equivalent to

$$H_2 : \Phi_0^*(t) \geq \phi_0 \text{ for all } t \text{ with strict inequality for some t.}$$

Exactly on the same line as in the earlier section, we have

Theorem 3.3 *As n tends to ∞, $n^{1/2}\{U_3^* - E(U_3^*)\}$ converges in distribution to $N(0, {\sigma_3^*}^2)$, where*

$$\binom{n}{2} U_3^* = n(n-1)/2 - \sum_{i=1}^{n}(n-i)W_i \tag{3.3}$$

and $\sigma_3{*}^2 = (4/3)\phi_0(1-\phi_0)$.

A consistent estimator of variance is $\hat{\sigma_3^2} = (4/3)\hat{\phi}(1-\hat{\phi})$. We reject the null hypothesis for large values of $Z = n^{1/2}(U_3^* - \hat{\phi_0})/\hat{\sigma_3^*}$. ¿From equations (3.1), (3.2) and (3.3), it follows that $U_k = U_3 + U_3^* - 1$.

3.3 Testing independence against $RTI(\delta \mid T)$

Here, we consider testing H_0 against H_3. Note that H_3 is equivalent to

$$H_3 : \Phi_1(t) \uparrow t, t > 0.$$

A. Test I - U_1

$\Phi_1(t) \uparrow t$ is equivalent to $\Phi_1(t_1) \leq \Phi_1(t_2)$, whenever $t_1 \leq t_2$. This gives $\delta(t_1, t_2) = S_1(t_2)S(t_1) - S_1(t_1)S(t_2) \geq 0, t_1 \leq t_2$ with strict inequality for some (t_1, t_2). Define

$$\begin{aligned} \Delta_1(S_1, S) &= \int\int_{t_1 \leq t_2} \delta(t_1, t_2) dF_1(t_1) dF_1(t_2) \\ &= \int_0^\infty [S_1^2(t) - \phi^2/2] S(t) dF_1(t). \end{aligned} \tag{3.4}$$

Under H_0, $S_1(t)/S(t) = \phi$. This implies that $\Delta_1(S_1, S) = 0$. Under H_3, $\Delta_1(S_1, S) \geq 0$. Define the kernel

$$\psi_1^*(T_i, \delta_i, T_j, \delta_j, T_k, \delta_k, T_l, \delta_l) = \begin{cases} 1 & \text{if } T_k > T_j > T_l > T_i, \\ & \delta_i = \delta_j = \delta_k = 1, \delta_l = 0 \\ -1 & \text{if } T_l > T_j > T_k > T_i, \\ & \delta_i = \delta_j = \delta_k = 1, \delta_l = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then the U-statistic corresponding to $\Delta_1(S_1, S)$ is given by

$$U_1 = \frac{1}{\binom{n}{4}} \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} \psi_1(T_{i_1}, \delta_{i_1}, T_{i_2}, \delta_{i_2}, T_{i_3}, \delta_{i_3}, T_{i_4}, \delta_{i_4}),$$

where ψ_1 is the symmetric version corresponding to ψ_1^*. Note that $E(U_1) = 24\Delta_1(S_1, S)$. Under H_0, $E(U_1) = 0$ and under H_3, $E(U_1) \ge 0$. Now we will express U_1 as a function of ranks. Let $T's$ corresponding to $1's$ be called $X's$ and those corresponding to $0's$ be called $Y's$. Then the number of $X's$ is $n_1 = \sum_{i=1}^n \delta_i$, and there are $n_2 = n - n_1$ $Y's$. Let $R_{(i)}(S_{(j)})$ be the rank of $X_{(i)}(Y_{(j)})$ be the $ith(jth)$ ordered statistic in the $X(Y)$ sample in the combined arrangement of $n_1 X's$ and $n_2 Y's$ (in fact $nT's$). Hence

$$\binom{n}{4} U_1 = \sum_{j=1}^{n_2} (S_{(j)} - j) \binom{n_1 + j - S_{(j)}}{2} - \sum_{j=1}^{n_2} \binom{S_{(j)} - j}{3}.$$

It is interesting to note that in terms of $X's$ and $Y's$ the above statistic is the same as that proposed by Kochar (1979) for testing equality of failure rates, the only difference being that the number of $X's$ and $Y's$ is random.

Theorem 3.4 *As n tends to ∞, under H_0, $n^{1/2}\{U_1 - E(U_1)\}$ converges in distribution to $N(0, \sigma_1^2)$, where $\sigma_1^2 = (96/35)\phi^5(1-\phi)$.*

The null hypothesis is rejected for large values of $n^{1/2}U_1/\hat{\sigma_1}$ where $\hat{\sigma_1^2} = (96/35)\hat{\phi}^5(1-\hat{\phi})$.

B. Test II - U_2

As mentioned earlier, H_3 is equivalent to $S_i(t)$ being TP_2. Under TP_2, $S_1(t_2)S_0(t_1) - S_1(t_1)S_0(t_2) > 0, t_1 < t_2$. Consider

$$\Delta_2(S_1, S) = \int_{t_1 < t_2} [S_1(t_2)S_0(t_1) - S_1(t_1)S_0(t_2)] d[F_1(t_1)F_0(t_2) + F_1(t_2)F_0(t_1)].$$

Under H_0, $\Delta_2(S_1, S) = 0$ and under H_3, $\Delta_2(S_1, S) \ge 0$. Define the kernel

$$\psi_2^*(T_i, \delta_i, T_j, \delta_j, T_k, \delta_k, T_l, \delta_l) = \begin{cases} 1 & \text{if } T_k > T_j > T_l > T_i,\ \delta_i = \delta_k = 1, \delta_j = \delta_l = 0 \\ & \text{if } T_k > T_i > T_l > T_j,\ \delta_i = \delta_k = 1, \delta_j = \delta_l = 0 \\ -1 & \text{if } T_l > T_j > T_k > T_i,\ \delta_i = \delta_k = 1, \delta_j = \delta_l = 0 \\ & \text{or } T_l > T_i > T_k > T_j,\ \delta_i = \delta_k = 1, \delta_j = \delta_l = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then the U-statistic corresponding to $\Delta_2(S_1, S)$ is given by

$$U_2 = \frac{1}{\binom{n}{4}} \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} \psi_2(T_{i_1}, \delta_{i_1}, T_{i_2}, \delta_{i_2}, T_{i_3}, \delta_{i_3}, T_{i_4}, \delta_{i_4}),$$

where ψ_2 is the symmetric version of ψ_2^*. Note that

$$\begin{aligned} E(U_2) &= 24\Delta_2(S_1, S) \\ &= \phi^2(1-\phi)^2/4 - \phi(1-\phi)\int_0^\infty S_0(t)dF_1(t) + \int_0^\infty S_1(t)S_0^2(t)dF_1(t). \end{aligned} \tag{3.5}$$

U_2 can be expressed as a function of ranks, following the arguments for such a representation for U_1. We have

$$\begin{aligned}\binom{n}{4}U_2 &= \sum_{i=1}^{n_1}(n_1-i)\binom{R_{(i)}-i}{2} \\ &- \sum_{i=1}^{n_1}(n_1-i)(R_{(i)}-i)(n_2-R_{(i)}+i) \\ &+ \sum_{j=1}^{n_2}(S_{(j)}-j)(n_1-S_{(j)}+j)(j-1) \\ &- \sum_{j=1}^{n_2}(n_2-j)\binom{(S_{(j)}-j}{2}. \end{aligned} \tag{3.6}$$

In terms of $X's$ and $Y's$, the above statistic is the same as another one proposed by Kochar (1979) to test for equality of failure rates with n_1 and n_2 fixed.

Theorem 3.5 *As n tends to ∞, under H_0, $n^{1/2}\{U_2 - E(U_2)\}$ converges in distribution to $N(0, \sigma_2^2)$, where $\sigma_2^2 = (384/35)\phi^3(1-\phi)^3$.*

We reject the null hypothesis for large value of $n^{1/2}U_2/\hat{\sigma_2}$ where $\hat{\sigma_2^2} = (384/35)\hat{\phi^3}(1-\hat{\phi})^3$.

Tests proposed in this section will help in discriminating between the constant or proportional warning-constant inspection and random sign censoring models and also to determine whether the corresponding mode of failure becomes more likely with increasing age.

3.4 Testing independence against $LTD(\delta \mid T)$

Here, we consider testing H_0 against H_4, where H_4 can equivalently be stated as

$$H_4 : \Phi_0^*(t) \downarrow t, t > 0.$$

A. Test I - U_1^*

$\Phi_0^*(t) \downarrow t$ is equivalent to $\Phi_0^*(t_1) \geq \Phi_0^*(t_2)$, whenever $t_1 \leq t_2$. This gives $\delta(t_1, t_2) = F_0(t_1)F(t_2) - F_0(t_2)F(t_1) \geq 0, t_1 \leq t_2$ with strict inequality for some (t_1, t_2). Define

$$\begin{aligned}\Delta_1(F_0, F) &= \int\int_{t_1 \leq t_2} \delta(t_1, t_2) dF_0(t_1) dF_0(t_2) \\ &= \int_0^\infty [F_0^2(t) - \phi_0^2/2] F(t) dF_0(t). \end{aligned} \tag{3.7}$$

Under H_0, $F_0(t)/F(t) = \phi_0$. This implies that $\Delta_1(F_0, F) = 0$. Under H_4, $\Delta_1(F_0, F) \geq 0$. Define the kernel

$$\psi_1^*(T_i, \delta_i, T_j, \delta_j, T_k, \delta_k, T_l, \delta_l) = \begin{cases} 1 & \text{if } T_k < T_j < T_l < T_i,\ \delta_i = \delta_j = \delta_k = 0, \delta_l = 1 \\ -1 & \text{if } T_l < T_j < T_k < T_i,\ \delta_i = \delta_j = \delta_k = 0, \delta_l = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the U-statistic corresponding to $\Delta_1(F_0, F)$ is given by

$$U_1^* = \frac{1}{\binom{n}{4}} \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} \psi_1(T_{i_1}, \delta_{i_1}, T_{i_2}, \delta_{i_2}, T_{i_3}, \delta_{i_3}, T_{i_4}, \delta_{i_4}),$$

where ψ_1 is the symmetric version corresponding to ψ_1^*. Note that $E(U_1^*) = 24\Delta_1(F_0, F)$. Under H_0, $E(U_1^*) = 0$ and under H_4, $E(U_1^*) \ge 0$.

A rank representation of U_1^* is

$$\binom{n}{4} U_1^* = \sum_{j=1}^{n_1} \binom{R_{(j)} - j}{2} (n_2 + j - R_{(j)}) - \sum_{j=1}^{n_1} \binom{n_2 - R_{(j)} + j}{3}.$$

Theorem 3.6 *As n tends to ∞, under H_0, $n^{1/2}\{U_1^* - E(U_1^*)\}$ converges in distribution to $N(0, \sigma_1^{*2})$, where $\sigma_1^{*2} = (96/35)\phi_0^5(1 - \phi_0) = (96/35)\phi(1 - \phi)^5$.*

We reject the null hypothesis for large values of $n^{1/2}U_1^*/\hat{\sigma_1^*}$, where $\hat{\sigma_1^{*2}} = (96/35)\hat{\phi}_0^5(1 - \hat{\phi}_0) = (96/35)\hat{\phi}(1 - \hat{\phi})^5$.

B. Test II - U_2^*

In this section, we propose another test procedure for testing H_0 against H_4 using the TP_2 property of the subdistribution functions of (T, δ). Note that H_4 is equivalent to $F_i(t)$ being TP_2. Under TP_2, $F_1(t_2)F_0(t_1) - F_1(t_1)F_0(t_2) > 0$ for $t_1 < t_2$. Consider

$$\Delta_2(F_0, F) = \int_{t_1 < t_2} [F_1(t_2)F_0(t_1) - F_1(t_1)F_0(t_2)][dF_1(t_1)dF_0(t_2) + dF_1(t_2)dF_0(t_1)].$$

Under H_0, we have $\Delta_2(F_0, F) = 0$, and under H_4, $\Delta_2(F_0, F) \ge 0$. Define the kernel

$$\psi_2^*(T_i, \delta_i, T_j, \delta_j, T_k, \delta_k, T_l, \delta_l) = \begin{cases} 1 & \text{if } T_k < T_j < T_l < T_i,\ \delta_i = \delta_k = 0, \delta_j = \delta_l = 1 \\ & \text{if } T_k < T_i < T_l < T_j,\ \delta_i = \delta_k = 0, \delta_j = \delta_l = 1 \\ -1 & \text{if } T_l < T_j < T_k < T_i,\ \delta_i = \delta_k = 0, \delta_j = \delta_l = 1 \\ & \text{or } T_l < T_i < T_k < T_j,\ \delta_i = \delta_k = 0, \delta_j = \delta_l = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the U-statistic corresponding to $\Delta_2(F_0, F)$ is given by

$$U_2^* = \frac{1}{\binom{n}{4}} \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} \psi_2(T_{i_1}, \delta_{i_1}, T_{i_2}, \delta_{i_2}, T_{i_3}, \delta_{i_3}, T_{i_4}, \delta_{i_4}),$$

where ψ_2 is the symmetric version of ψ_2^*. Note that

$$\begin{aligned} E(U_2^*) &= 24\Delta_2(F_0, F) \\ &= 24[\phi_0^2(1 - \phi_0)^2/4 - \phi_0(1 - \phi_0) \int_0^\infty F_1(t)dF_0(t) \\ &\quad + \int_0^\infty F_0(t)F_1^2(t)dF_0(t)]. \end{aligned} \tag{3.8}$$

U_2^* can be expressed as a function of ranks, following the arguments for such a representation for U_1^*. We have

$$\begin{aligned}
\binom{n}{4} U_2^* &= \sum_{i=1}^{n_1} (n_1 - i) \binom{R_{(i)} - i}{2} \\
&+ \sum_{i=1}^{n_1} (n_1 - i)(R_{(i)} - i)(n_2 - R_{(i)} + i) \\
&- \sum_{j=1}^{n_2} (S_{(j)} - j)(n_1 - S_{(j)} + j)(j-1) \\
&- \sum_{j=1}^{n_2} (n_2 - j) \binom{(S_{(j)} - j}{2}. \qquad (3.9)
\end{aligned}$$

Theorem 3.7 *As n tends to ∞, under H_0, $n^{1/2}\{U_2^* - E(U_2^*)\}$ converges in distribution to $N(0, \sigma_2^{*2})$, where $\sigma_2^{*2} = (384/35)\phi_0^3(1-\phi_0)^3 = (384/35)\phi^3(1-\phi)^3$.*

We reject the null hypothesis for large values of $n^{1/2}U_2^*/\hat{\sigma_2^*}$, where $\hat{\sigma_2^{*2}} = (384/35)\hat{\phi_0^3}(1-\hat{\phi_0})^3 = (384/35)\hat{\phi^3}(1-\hat{\phi})^3$.

4 Asymptotic relative efficiency

To compare alternative tests proposed in this paper for testing H_0 against H_2, H_0 against H_3 and H_0 against H_4, we compute asymptotic relative efficiency of the tests within a semiparametric family of distributions proposed in Deshpande (1990). The semiparametric family considered here is $F_1(t) = pF^a(t)$, $F_0(t) = F(t) - pF^a(t)$, where $1 \leq a \leq 2$, $0 \leq p \leq 0.5$ and $F(t)$ is a proper distribution function. Note that $\phi = p$ and

$$\Phi_1(t) = \frac{p(1 - F^a(t))}{1 - F(t)}$$

which is an increasing function of t. Also,

$$\Phi_0^*(t) = 1 - pF^{a-1}(t)$$

which is a decreasing function of t. H_0 corresponds to $a = 1$, and other alternative hypotheses correspond to $1 < a \leq 2$. By the limiting theorem of U-statistics, all the U-statistics proposed here have asymptotic normal distribution under both null and the alternative hypothesis. The asymptotic relative efficiency of test U_1 with respect to test U_2 is then defined as $eff(U_1, U_2) = e(U_2)/e(U_1)$ where $e(U) = \mu'^2(1)/var(U \mid H_0)$ and $\mu'(1)$ is the derivative of expected value of U with respect to a evaluated at $a = 1$, and $var(U \mid H_0)$ is the asymptotic variance of $n^{1/2}U$ under H_0. Tests U_1 and U_2 are equally efficient and the same is true for tests U_1^* and U_2^*. Tests U_3 and U_3^* are equally efficient but the general test U_k is four times more efficient compared to these tests. This indicates the superiority of U_k as it is consistent for the alternative H_2.

For this particular family of distributions, the other alternative tests are equally efficient. But this need not be true in general.

5 Illustrations

We consider two real data sets here, one where the empirical $\Phi_1(t)$ is nondecreasing and the empirical $\Phi_0^*(t)$ is nonincreasing. In the other example, both of these seem to be fairly constant.

Example 1: Nair (1993)

Consider the data on the times to failure, in millions of operations, and modes of failure of 37 switches, obtained from a reliability study conducted at AT&T, given in Nair (1993). There are two possible modes of failure, denoted by A ($\delta = 1$) and B ($\delta = 0$), for these switches. Figure 1 shows the empirical estimates of the conditional probabilities corresponding to failure modes A and B, respectively. The empirical Φ_1 function corresponding to failure mode A is clearly increasing and the empirical Φ_0^* function corresponding to B is decreasing, indicating that the failure mode A becomes more likely with increase in the age of the switch.

Table 1 gives the values of the test statistics. The value of Z corresponding to U_k is 2.70 and hence we may conclude that the failure time and the type of failure are dependent. The nonlinearity of the plot in Figure 1 supports this conclusion. Both the tests for PQD accepts the null hypothesis of independence of T and δ. However, U_1 accepts H_0 and U_2 rejects it in favour of the alternative hypothesis that $\Phi_1(t)$ is increasing. The test for checking whether $\Phi_0^*(t)$ is decreasing, rejects the null hypothesis and hence we may conclude that $\Phi_0^*(t)$ is a nonincreasing function of t.

Example 2: Hoel (1972)

Consider the data set obtained from a laboratory experiment on male mice which had received a radiation dose of 300 rads at an age of 5 to 6 weeks given in Hoel (1972). The death occurred due to cancer ($\delta = 1$), or other causes ($\delta = 0$). Figure 2 shows the empirical conditional probabilities and in this case, the empirical conditional probability $\Phi_1(t)$ seen to be almost flat and the curve corresponding to $\Phi_0^*(t)$ is not so flat.

Table 2 gives the values of the test statistics. All the proposed tests accept the null hypothesis of independence of T and δ.

6 Concluding remarks

It is now a common practice to model the competing risks in terms of (T, δ). Hence, it is of prime importance to check whether T and δ are independent. We have proposed tests based on U-statistics to check whether T and δ are independent or not. It is clear that the tests perform satisfactorily in distinguishing between the hypotheses. If the hypothesis of independence is accepted then one can simplify the model and study the failure time and cause of failure separately. If the hypothesis is rejected then one can think of a suitable model under specific dependence between T and δ in terms of the incidence functions.

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Table 1: Values of the test statistics for Nair's data (1993)

U-statistics	Expectation	Variance	Z	Conclusion
$U_k = 0.26$	0	0.33	2.70	Reject H_0
$U_1 = 0.04$	0	0.03	1.45	Accept H_0
$U_2 = 0.15$	0	0.17	2.26	Reject H_0
$U_1^* = 0.06$	0	0.06	2.29	Reject H_0
$U_2^* = 0.15$	0	0.17	2.18	Reject H_0
$U_3 = 0.59$	0.46	0.33	1.35	Accept H_0
$U_3^* = 0.67$	0.54	0.33	1.35	Accept H_0

Table 2: Values of the test statistics for Hoel's data (1972)

U-statistics	Expectation	Variance	Z	Conclusion
$U_k = 0.11$	0	0.32	1.86	Accept H_0
$U_1 = 0.04$	0	0.09	1.50	Accept H_0
$U_2 = 0.06$	0	0.15	1.63	Accept H_0
$U_1^* = 0.01$	0	0.02	1.14	Accept H_0
$U_2^* = 0.05$	0	0.15	1.38	Accept H_0
$U_3 = 0.66$	0.61	0.32	0.93	Accept H_0
$U_3^* = 0.45$	0.39	0.32	0.53	Accept H_0

Figure 1: Time versus empirical $\Phi_1(t)$, $\Phi_1(0)$, $\Phi_0^*(t)$ and $\Phi_0^*(\infty)$ for the data given in Nair (1993). Solid squares denote $\Phi_1(t)$, dashed line denotes $\Phi_1(0)$, pluses denotes $\Phi_0^*(t)$ and solid line denotes $\Phi_0^*(\infty)$.

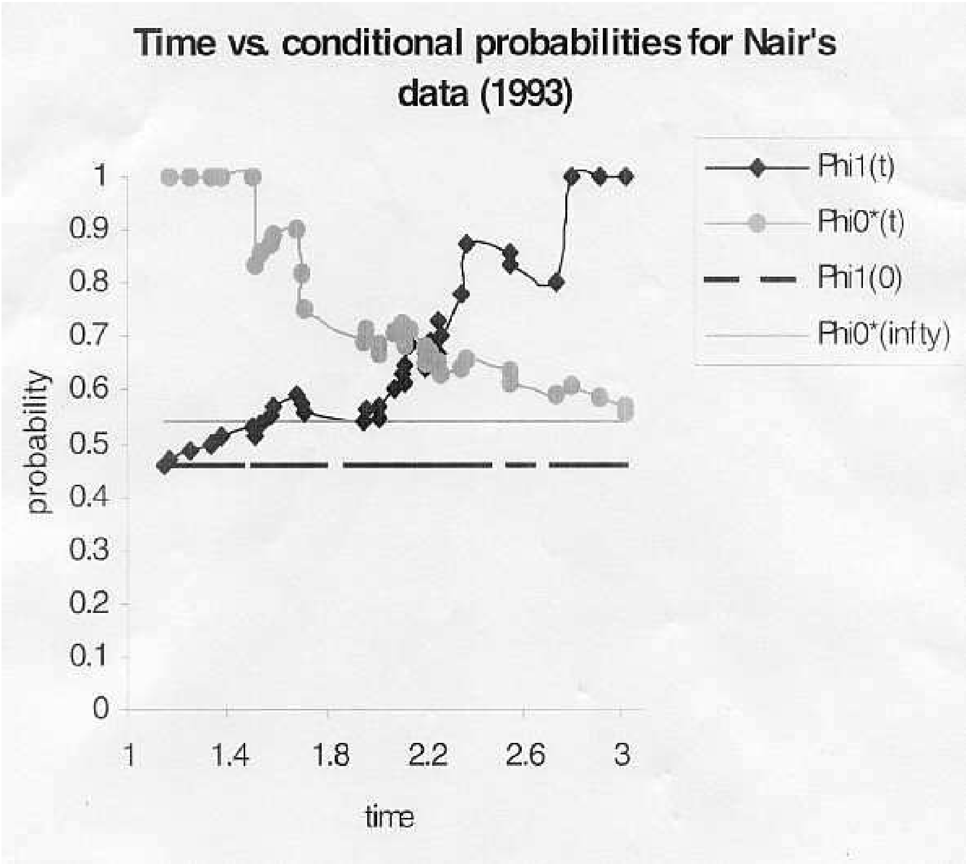

Figure 2: Time versus empirical $\Phi_1(t)$, $\Phi_1(0)$, $\Phi_0^*(t)$ and $\Phi_0^*(\infty)$ for the data given in Hoel (1972). Solid squares denote $\Phi_1(t)$, dashed line denotes $\Phi_1(0)$, pluses denote $\Phi_0^*(t)$ and solid line denotes $\Phi_0^*(\infty)$.