isid/ms/2002/18
July 22, 2002
http://www.isid.ac.in/~statmath/eprints

Zakai equation of nonlinear filtering with Ornstein-Uhlenbeck noise: Existence and Uniqueness

Abhay Bhatt
Balram Rajput
and
Jie Xiong

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

Zakai equation of nonlinear filtering with Ornstein-Uhlenbeck noise: Existence and Uniqueness

Abhay Bhatt[1,2], Balram Rajput[2] and Jie Xiong[2,3]

Abstract

We consider a filtering model where the noise is an Ornstein-Uhlenbeck process independent of the signal X. The signal is assumed to be a Markov difusion process. We derive the (analogue of) Zakai equation in this setup. It is a system of two measure valued equations satisfied by the unnormalised conditional distribution. We also prove uniqueness of solution to these equations.

1 Introduction

The process of interest - the *system process* X - is unobservable. We can observe the (*observation*) process Y - a (known) function h of X - which in addition is corrupted by noise N. We want to filter out the noise N from the observations Y and get an estimate of the process X. This is *filtering theory.* The filtering model can be written as

$$Y_t = \int_0^t h(X_s)ds + N_t, \qquad 0 \leq t \leq T. \tag{1.1}$$

The best estimate of X is the conditional distribution of X_t given the observations upto time t - $\{Y_s; 0 \leq s \leq t\}$. This is called the *optimal filter* and is denoted by π_t.

In the classical theory of filtering the noise N is assumed to be a Brownian motion. In this case π_t is known to satisfy a measure valued stochastic differential equation called the Fujisaki-Kallianpur-Kunita (FKK) equation. See [3] and [5].

The unnormalised conditional distribution μ_t of X_t given $\{Y_s; 0 \leq s \leq t\}$ has also been studied extensively in the literature. μ also satisfies a stochastic differential equation called the Zakai equation - which has the added advantage of being linear in μ and is driven by the observation process Y. See [11]. Uniqueness of solution to the measure valued equations of filtering under fairly general conditions on the observation function and on the signal process X, when the noise is a Brownian motion has been established in [1].

Recently, interest has developed in filtering theory when the noise is a general Gaussian process. See *e.g.* [7], [9] and [2]. Here the authors, under differing conditions, derive a Bayes' formula for the optimal filter π - anologous to the classical case ([6]) - for the filtering problem when the noise is a general Gaussian process. In [4] the authors use the result of [9] to derive

[1]Indian Statistical Institute, Delhi

[2]Department of Mathematics, University of Tennessee, Knoxville

[3]Research supported partially by NSA

a Zakai equation for μ when the noise process is connected to a Brownian motion via a certain kernel.

In this article we consider the filtering problem when the noise is a Ornstein-Uhlenbeck (velocity) process independent of the signal. In the next section we introduce the filtering model. We derive the equations of filtering for the unnormalised conditional distribution when the signal is a diffusion Markov process with smooth diffusion and drift coefficients. We need to consider the pair $(\mu_t, \sigma_{s,t})$ (see (2.11)-(2.12)). Here $\sigma_{s,t}$ is the analogous *two* - parameter unnormalised conditional distribution of X given Y. The interesting point is that the (analogue of the) Zakai equation - unlike in the classical case - is a system of two measure valued SDE's. We derive these SDE's via a particle representation proof based on the lines of Kurtz and Xiong [8].

Uniqueness is proved in Section 3. We show that the pair $(\mu_t, \sigma_{s,t})$ is the unique solution of this system of equations under the additional assumption that the law of X_0 has a density.

For the sake of notational simplicity we consider the one dimensional signal case. The results are true for a general d-dimensional diffusion as well.

2 Zakai equation

Fix a probability space $(\Omega, \mathcal{F}, P)$. We will assume that the signal process, X, is a $\mathbb{R}$ valued diffusion process governed by the SDE

$$dX_t = b(X_t)dt + c(X_t)dB_t \tag{2.1}$$

where $b,\ c$ are bounded Lipschitz continuous real valued functions and B is a standard Brownian motion independent of X_0. It is well-known that SDE (2.1) admits a unique solution. Furthermore the paths of this solution are continuous.

Let W be a Standard Brownian motion independent of X. We will investigate the nonlinear filtering model where the observation process Y is given by

$$Y_t = \int_0^t h(X_s)ds + O_t. \tag{2.2}$$

Here, the observation function h is assumed to be a bounded continuous function and the noise process O_t is an Ornstein-Uhlenbeck process satisfying the SDE

$$dO_t = -\beta O_t dt + dW_t \tag{2.3}$$

with $\beta > 0$. The optimal filter π_t is given by

$$\pi_t f = \mathbb{E}(f(X_t)|\mathcal{F}_t^Y), \quad \forall f \in C_b(\mathbb{R})$$

where $\mathcal{F}_t^Y = \sigma\{Y_s : 0 \le s \le t\}$ is the σ-field generated by all observations upto time t.

We will recast (2.2) in a form which helps us in deriving an equation for the filter. For this purpose we proceed as follows. Let $C([0,T],\mathbb{R})$ denote the space of all continuous functions from $[0,T]$ to $\mathbb{R}$. Let $E = [0,T] \times C([0,T],\mathbb{R})$. Define $H : E \to \mathbb{R}$ by

$$\begin{aligned} H(t,\mathbf{x}) &= \frac{d}{dt}\left[e^{\beta t}\int_0^t h(\mathbf{x}(r))dr\right] \\ &= \beta e^{\beta t}\int_0^t h(\mathbf{x}(r))dr + e^{\beta t}h(\mathbf{x}(t)). \end{aligned} \tag{2.4}$$

For any $\mathbf{x} \in C([0,T],\mathbf{R})$, let $\mathbf{x}^t \in C([0,T],\mathbf{R})$ denote the path $\mathbf{x}$ stopped at t. *i.e.* $\mathbf{x}^t(r) = \mathbf{x}(t \wedge r), r \in [0,1]$.

Define an operator A on $C_b(\mathbf{R})$ as follows. Let $D(A) = C_b^2(\mathbf{R})$, the space of twice continuously differentiable functions on $\mathbf{R}$ with bounded derivatives. Let

$$Af(x) = \frac{1}{2}c^2(x)f''(x) + b(x)f'(x). \tag{2.5}$$

Then A uniquely determines the Markov processes X as a solution of its martingale problem. (See Stroock and Varadhan [10].)

Define $S_t \equiv (t, X^t)$. It is well-known that S is an E-valued Markov process. Also S is a unique solution of the martingale problem for an operator $\bar{A}$ on $C_b(E)$ which can be defined as follows. Let $D(\bar{A})$ be the algebra generated by functions of the form $\{F : E \to \mathbf{R} : F(t,\mathbf{x}) = g(t)f(\mathbf{x}_t), g \in C[0,T] \cap C^1(0,T), f \in D(A)\}$. Define

$$\bar{A}F(t,\mathbf{x}) = g'(t)f(\mathbf{x}_t) + g(t)Af(\mathbf{x}(t)). \tag{2.6}$$

Now let $M_t = e^{\beta t}O_t$. Then (2.3) implies that

$$M_t = \int_0^t e^{\beta s}dW_s.$$

Clearly M is a $\mathcal{F}_t^W$ martingale. Correspondingly, let $Z_t = e^{\beta t}Y_t$. The filtering model (2.2) can now be rewritten as

$$\begin{aligned} Z_t &= e^{\beta t}\int_0^t h(X_s)ds + M_t \\ &= \int_0^t H(S_u)du + M_t \end{aligned} \tag{2.7}$$

where H is as in (2.4). Let

$$\Lambda_t^{-1} \equiv \exp\left\{-\int_0^t e^{-2\beta u}H(S_u)dM_u - \frac{1}{2}\int_0^t e^{-2\beta u}|H(S_u)|^2du\right\}$$

Note that independence of X and W implies that S is independent of M. Hence Λ_t^{-1} is a P-martingale. Morevoer P_0 defined by

$$\frac{dP_0}{dP} = \Lambda_T^{-1}$$

is a probability measure on $(\Omega, \mathcal{F})$. Also, Girsanov's theorem implies that under P_0, Z is a martingale independent of S and that the law of S under the two measures P and P_0 remains unchanged. In particular, under P_0, S is a Markov process and is the unique solution of the martingale problem for $\bar{A}$.

Now, the optimal filter $\bar{\pi}_t$ for the model (2.7) satisfies $\forall F \in C_b(E)$,

$$\begin{aligned}\bar{\pi}_t F &\equiv \mathbf{E}(F(S_t)|\mathcal{F}_t^Z) \\ &= \mathbf{E}(F(S_t)|\mathcal{F}_t^Y) \\ &= \mathbf{E}_{P_0}(F(S_t)\Lambda_t|\mathcal{F}_t^Y)/\mathbf{E}_{P_0}(\Lambda_t|\mathcal{F}_t^Y) \\ &\equiv \bar{\mu}_t F/\bar{\mu}_t \mathbf{1},\end{aligned}$$

where $\bar{\mu}_t$ is the unnormalised conditional distribution of S_t given $\mathcal{F}_t^Z$. We now derive the Zakai equation for $\bar{\mu}_t$.

Proposition 2.1. *$\bar{\mu}_t$ satisfies the equation*

$$\bar{\mu}_t F = \bar{\mu}_0 F + \int_0^t \bar{\mu}_s(\bar{A}F)ds + \int_0^t e^{-2\beta s}\bar{\mu}_s(HF)dZ_s. \qquad \forall F \in D(\bar{A})$$

Proof: Fix $F \in D(\bar{A})$. Consider independent copies S^i of S. Let

$$N_t^i = F(S_t^i) - F(S_0^i) - \int_0^t \bar{A}F(S_u^i)du.$$

Then $\{N^i, i \geq 1\}$ are independent P_0-martingales that are also independent of the P_0-martingale Z_t. Let

$$d\Lambda_t^i = e^{-2\beta t}\Lambda_t^i H(S_t^i)dZ_t.$$

Then it is easy to see that

$$\Lambda_t^i = \exp\left\{\int_0^t e^{-2\beta u}H(S_u^i)dZ_u - \frac{1}{2}\int_0^t e^{-2\beta u}|H(S_u^i)|^2 du\right\}.$$

By Itô's formula, we have

$$d(F(S_t^i)\Lambda_t^i) = e^{-2\beta t}F(S_t^i)H(S_t^i)\Lambda_t^i dZ_t + \Lambda_t^i[\bar{A}F(S_t^i)]dt + \Lambda_t^i dN_t^i. \tag{2.8}$$

It is clear that the sequence of processes $\{(\Lambda^i, F(S^i)) : i \geq 1\}$ is exchangable. Thus the limit $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \Lambda_t^i F(S_t^i)$ exists under P_0 and the ergodic theorem implies that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \Lambda_t^i F(S_t^i) = \mathbf{E}_{P_0}(\Lambda_t F(S_t)|\mathcal{I}) \tag{2.9}$$

where $\mathcal{I}$ is the invariant σ-field of the stationary sequence $\{(S^i, N^i, Z) : i \geq 1\}$. As in Kurtz and Xiong ([8, Theorem 2.3]) we use the independence of (S^i, N^i) to note that $\mathcal{I}$ is contained in the completion of the σ-field generated by Z. Now (2.9) implies

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\Lambda_t^i F(S_t^i) &= \mathbf{E}_{P_0}(\Lambda_t F(S_t)|\mathcal{F}_T^Z)\\ &= \mathbf{E}_{P_0}(\Lambda_t F(S_t)|\mathcal{F}_t^Z)\\ &= \bar{\mu}_t F. \end{aligned} \tag{2.10}$$

Arguing similarly as above, using (2.8), (2.9), (2.10) and the fact that N^1 and Z are independent under P_0 we get

$$\bar{\mu}_t F = \bar{\mu}_0 F + \int_0^t e^{-2\beta s}\bar{\mu}_s(HF)dZ_s + \int_0^t \bar{\mu}_s(\bar{A}F)ds.$$

This completes the proof. □

Remark 2.2. *The above Proposition can also be proved along the lines of the proof of the classical Zakai equation. Here we have given a different* **particle-representation** *proof. See Kurtz and Xiong [8].*

Let μ_t denote the unnormalized conditional distribution of X_t given $\mathcal{F}_t^Y$. *i.e.*

$$\mu_t f = \mathbf{E}_{P_0}\left[f(X_t)\Lambda_t|\mathcal{F}_t^Y\right]. \tag{2.11}$$

Also for $0 \le s, t \le T$, let $\sigma_{s,t}$ be defined by

$$\sigma_{s,t} f = \mathbf{E}_{P_0}\left[h(X_s)f(X_t)\Lambda_t|\mathcal{F}_t^Y\right]. \tag{2.12}$$

Note that $\sigma_{t,t}f = \mu_t(hf)$. We will now use Proposition 2.1 to derive the analogue of the Zakai equation for (μ_t) which involves $(\sigma_{s,t})$.

Proposition 2.3. *μ_t and $\sigma_{s,t}$ satisfy the system of equations*

$$\mu_t f = \mu_0 f + \int_0^t \mu_s(Af)ds + \int_0^t e^{-\beta s}\left(\beta\int_0^s \sigma_{u,s} f du + \mu_s(hf)\right)dZ_s, \tag{2.13}$$

$$\sigma_{s,t} f = \mu_s(hf) + \int_s^t \sigma_{s,u}(Af)du + \int_s^t e^{-2\beta(u-s)}\sigma_{s,u}(hf)dZ_u, \qquad \forall f \in D(A). \tag{2.14}$$

Proof: Fix $f \in D(A)$. Let $F \in D(\bar{A})$ be defined by $F(t, \mathbf{x}) = f(\mathbf{x}_t)$. Note that $Af(\mathbf{x}_t) = \bar{A}F(t, \mathbf{x}^t)$. It follows from (2.1) that

$$\mu_t f = \mu_0 f + \int_0^t \mu_s(Af)ds + \int_0^t e^{-2\beta s}\bar{\mu}_s(HF)dZ_s.$$

As

$$\begin{aligned}(HF)(s, \mathbf{x}^s) &= \left(\beta e^{\beta s}\int_0^s h(\mathbf{x}_u)du + e^{\beta s}h(\mathbf{x}_s)\right)f(\mathbf{x}_s)\\ &= \beta e^{\beta s}\int_0^s h(\mathbf{x}_u)f(\mathbf{x}_u)du + e^{\beta s}h(\mathbf{x}_s)f(\mathbf{x}_s),\end{aligned}$$

(2.13) follows.

(2.14) follows from the same arguments as in the proof of Proposition 2.1 by noting that

$$N_{s,t} = h(X_s)\left[f(X_t) - f(X_s) - \int_s^t Af(X_u)du\right]$$

is a martingale for $t \geq s$. □

3 Uniqueness

In this section we will show uniqueness of solution to the system of equations (2.13)-(2.14). For $f \in C_b(\mathbf{R})$, let $F(t, x^t) = f(x_t)$. Then $\mu_t f = \bar{\mu}_t F$. For $\delta > 0$, let p_δ denote the density kernel of a normal random variable with variance δ. For a measure ν on $\mathbf{R}$, let $T_\delta \nu$ denote the function defined by $T_\delta \nu(x) = \int p_\delta(x-y)\nu(dy)$. Define

$$\sigma^\delta_{s,t} = T_\delta \sigma_{s,t} \text{ and } \mu^\delta_t = T_\delta \mu_t.$$

Then $\sigma^\delta_{s,t}$ and μ^δ_t are $H_0 \equiv L^2(\mathbf{R})$-valued processes. With an abuse of notation, for $f \in C_b(\mathbf{R})$, $T_\delta f$ will denote the function $\int p_\delta(x-y)f(y)dy$. Note that $T_\delta f \in D(A)$. Thus recalling the definition of the operator A (see (2.5)) and using (2.14) we get

$$\begin{aligned}
\left\langle \sigma^\delta_{s,t}, f\right\rangle_0 &= \sigma_{s,t}(T_\delta f) \\
&= \mu_s(hT_\delta f) + \int_s^t \sigma_{s,u}(\frac{c^2}{2}(T_\delta f)'' + b(T_\delta f)')du + \int_s^t e^{-2\beta(u-s)}\sigma_{s,u}(hT_\delta f)dZ_u \\
&= \langle T_\delta(h\mu_s), f\rangle_0 + \int_s^t \left\langle (T_\delta(\frac{c^2}{2}\sigma_{s,u}))'' - (T_\delta(b\sigma_{s,u})', f\right\rangle_0 du \\
&\qquad + \int_s^t e^{-2\beta(u-s)} \left\langle (T_\delta(h\sigma_{s,u}), f\right\rangle_0 dZ_u.
\end{aligned}$$

By Itô's formula, we have

$$\begin{aligned}
\left\langle \sigma^\delta_{s,t}, f\right\rangle_0^2 &= \langle T_\delta(h\mu_s), f\rangle_0^2 + \int_s^t 2\left\langle \sigma^\delta_{s,u}, f\right\rangle_0 \left\langle (T_\delta(\frac{c^2}{2}\sigma_{s,u}))'' - (T_\delta(b\sigma_{s,u})', f\right\rangle_0 du \\
&\quad + \int_s^t 2\left\langle \sigma^\delta_{s,u}, f\right\rangle_0 \left\langle (T_\delta(h\sigma_{s,u}), f\right\rangle_0 e^{-2\beta(u-s)}dZ_u \\
&\quad + \int_s^t e^{-2\beta(u-s)} \left\langle (T_\delta(h\sigma_{s,u}), f\right\rangle_0^2 du. \qquad (3.1)
\end{aligned}$$

Let $\{f_i : i \geq 1\}$ be a CONS in H_0. Equation (3.1) holds for each f_i. Adding over i and using Lemmas 3.2 and 3.3 in Kurtz and Xiong [8], we get that there exists a constant K such that

$$\begin{aligned}
\mathbf{E}\|\sigma^\delta_{s,t}\|_0^2 &= \mathbf{E}\|T_\delta(h\mu_s)\|_0^2 + \mathbf{E}\int_s^t 2\left\langle \sigma^\delta_{s,u}, (T_\delta(\frac{c^2}{2}\sigma_{s,u}))'' - (T_\delta(b\sigma_{s,u})'\right\rangle_0 du \\
&\quad + \mathbf{E}\int_s^t e^{-2\beta(u-s)}\|T_\delta(h\sigma_{s,u})\|_0^2 du \\
&\leq K\mathbf{E}\|\mu^\delta_s\|_0^2 + K\mathbf{E}\int_s^t \|T_\delta(|\sigma_{s,u}|)\|_0^2 du. \qquad (3.2)
\end{aligned}$$

We cannot directly use Gronwall's inequality here. Hence we proceed as follows. Let $\xi_t^\pm$ be independent Markov processes which are solutions of the martingale problem for $(A, \mu_s(h^\pm \cdot)/\mu_s(h^\pm))$ respectively. Then

$$\begin{aligned}\sigma_{s,t} f &= \mathbf{E}\left(f(\xi_t^+) \exp\left(\int_s^t e^{-2\beta(u-s)} h(\xi_u^+) \circ dZ_u\right) \middle| \mathcal{F}_u^Z \right) \mu_s(h^+) \\ &\quad - \mathbf{E}\left(f(\xi_t^-) \exp\left(\int_s^t e^{-2\beta(u-s)} h(\xi_u^-) \circ dZ_u\right) \middle| \mathcal{F}_u^Z \right) \mu_s(h^-) \\ &\equiv \sigma_{s,t}^+ f - \sigma_{s,t}^- f.\end{aligned}$$

As in (2.13), we have

$$\sigma_{s,t}^\pm f = \mu_s(h^\pm f) + \int_s^t \sigma_{s,u}^\pm (Af) du + \int_s^t e^{-2\beta(u-s)} \sigma_{s,u}^\pm (hf) dZ_u.$$

Since $\sigma_{s,t}^\pm$ are positive measures, arguing as in (3.2) we get

$$\begin{aligned}\mathbf{E}\|\sigma_{s,t}^{\pm,\delta}\|_0^2 &\leq K\mathbf{E}\|\mu_s^\delta\|_0^2 + K\mathbf{E}\int_s^t \|T_\delta(|\sigma_{s,u}^\pm|)\|_0^2 du \\ &= K\mathbf{E}\|\mu_s^\delta\|_0^2 + K\mathbf{E}\int_s^t \|\sigma_{s,u}^{\pm,\delta}\|_0^2 du\end{aligned}$$

Now Gronwall's inequality implies that

$$\mathbf{E}\|\sigma_{s,t}^{\pm,\delta}\|_0^2 \leq K_1 \mathbf{E}\|\mu_s^\delta\|_0^2.$$

Therefore

$$\mathbf{E}\|T_\delta(|\sigma_{s,t}|)\|_0^2 \leq K_2 \mathbf{E}\|\mu_s^\delta\|_0^2. \tag{3.3}$$

Applying the same kind of arguement to (2.14), similar to (3.2), we get

$$\mathbf{E}\|\mu_t^\delta\|_0^2 \leq \|\mu_0^\delta\|_0^2 + K\int_0^t \mathbf{E}\|\mu_s^\delta\|_0^2 ds. \tag{3.4}$$

As a result we have the following proposition.

Proposition 3.1. *If $\mu_0 \in H_0$, then $\mu_t \in H_0$ and $\sigma_{s,t} \in H_0$ a.s.*

Proof: Applying Gronwall's inequality to (3.4) we get

$$\mathbf{E}\|\mu_t^\delta\|_0^2 \leq \|\mu_0^\delta\|_0^2 e^{Kt}.$$

Letting $\delta \to 0$, we have

$$\mathbf{E}\|\mu_t\|_0^2 \leq \|\mu_0\|_0^2 e^{Kt} < \infty,$$

and hence $\mu_t \in H_0$ a.s. the assertion that $\sigma_{s,t} \in H_0$ a.s. also follows similarly from (3.3). $\square$

Now we are ready to state and prove the main theorem.

Theorem 3.2. *Suppose that μ_0 has a square integrable density. Then the measure valued processes $(\mu_t, \sigma_{s,t} :\ 0 \le s \le t \le T)$ are such that $\pi_t = \mu_t\langle\mu_t, 1\rangle^{-1}$ and are the unique solution to the system of equations* (2.13)–(2.14).

Proof: If there are two solutions, we use $\tilde{\sigma}_{s,t}$ and $\tilde{\mu}_t$ to denote the difference. then, $\tilde{\sigma}_{s,t},\ \tilde{\mu}_t \in H_0$ a.s. Similar to (3.2), we have

$$\mathbf{E}\|\tilde{\sigma}^{\delta}_{s,t}\|_0^2 \le K\mathbf{E}\|T_\delta(|\tilde{\mu}_s|)\|_0^2 + K\mathbf{E}\int_s^t \|T_\delta(|\tilde{\sigma}_{s,u}|)\|_0^2 du.$$

Taking $\delta \to 0$, we have

$$\begin{aligned} \mathbf{E}\|\tilde{\sigma}_{s,t}\|_0^2 &\le K\mathbf{E}\||\tilde{\mu}_s|\|_0^2 + K\mathbf{E}\int_s^t \||\tilde{\sigma}_{s,u}|\|_0^2 du \\ &= K\mathbf{E}\|\tilde{\mu}_s\|_0^2 + K\mathbf{E}\int_s^t \|\tilde{\sigma}_{s,u}\|_0^2 du. \end{aligned}$$

Similarly, applying (3.4), we have

$$\mathbf{E}\|\tilde{\mu}_t\|_0^2 \le K\int_0^t \mathbf{E}\|\tilde{\mu}_s\|_0^2 ds$$

and hence, $\mathbf{E}\|\tilde{\mu}_t\|_0^2 = 0$. This in turn implies that $\mathbf{E}\|\tilde{\sigma}_{s,u}\|_0^2 = 0$. □

References

[1] Bhatt, A. G., Kallianpur, G. and Karandikar, R. L. (1995). Uniqueness and robustness of solution of measure-valued equations of nonlinear filtering. *Ann. Probab.* **23(4)**, 1895–1938.

[2] Bhatt, A. G. and Karandikar, R. L. (2001). On filtering with Ornstein-Uhlenbeck process as noise. *Manuscript.*

[3] Fujisaki, M., Kallianpur, G. and Kunita, H. (1972). Stochastic dufferential equations for the nonlinear filtering problem. *Osaka J. Math.* **9**, 19–40.

[4] Gawarecki, L. and Mandrekar, V. (2000). On the Zakai equation of filtering with Gaussian noise. *Stochastics in finite and infinite dimensions, Volume in honour of Gopinath Kallianpur, eds. T. Hida et al*, 145–151.

[5] Kallianpur, G. (1980). Stochastic filtering theory. Springer, New York.

[6] Kallianpur, G. and Striebel, H. (1968). Estimations of stochastic processes: Arbitrary system process with additive white noise observation errors. *Ann. Math. Statist.* **39**, 785–801.

[7] Kunita, H. (1993). Representation and stability of nonlinear filters associated with Gaussian noises. *Stochastic Processes: A festschrift in honour of Gopinath Kallianpur, eds. Cambanis et al, Springer-Verlag, New York.* 201–210.

[8] Kurtz, Thomas G. and Xiong, Jie (1999). Particle representations for a class of nonlinear SPDEs. *Stochastic Process. Appl.* **83**, 103–126.

[9] Mandrekar, V. and Mandal, P. K. (2000). A bayes formula for Gaussian processes and its applications. *SIAM J. Control Optim.* **39**, 852–871.

[10] Stroock, D. and Varadhan, S. R. S. (1979) Multidimensional Diffusion Processes. Springer-Verlag, Berlin.

[11] Zakai, M. (1969). On the optimal filtering of diffusion processes. *Z. Wahrsch. Verw. Gebiete* **11**, 230–243.