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Martingale Problems and Path Properties of Solutions

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Abstract

Existence and uniqueness of solutions of martingale problems, not only in the class of r.c.l.l. or continuous process, but also in the class of progressively measurable processes has become important. In this article we give several examples which show that existence or uniqueness in any one class need not imply the same in another class. We also give an example of a well-posed martingale problem where the operator is not a core for the generator of the corresponding Markov process.

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1 Introduction

Beginning with the seminal work of Stroock and Varadhan on multidimensional diffusions, martingale problems have proved to be an important tool in the construction and analysis of Markov processes complementing the theory of semi-groups and their generators. Martingale problems were used to construct and study properties of multidimensional diffusions (Stroock and Varadhan (1979)), Infinite particle systems and Icing models (Holley and Stroock (1976)), processes associated with Boltzman equation (Tanaka (1978), Horowitz and Karandikar (1990)). In each of these cases, the processes were constructed directly on an appropriate path space, either the space of continuous functions or the space of r.c.l.l. (right continuous with left limits) functions. Thus the martingale problem was posed on a suitable path space and its existence and uniqueness was studied.

Martingale problems for general operators were considered in Ethier and Kurtz (1986). They considered solutions to the martingale problem for an abstract operator A requring only that the paths are measurable. They showed (Theorem IV.3.6) that if the state space is compact then under some fairly general conditions (or complete and separable with further conditions on A) every measurable solution has an r.c.l.l. modification. They used this result repeatedly to construct r.c.l.l. solutions in specific cases.

In Bhatt and Karandikar (1993a) it was pointed out that when a martingale problem is well posed in the class of r.c.l.l. processes and, in addition, when uniqueness also holds in the class of progressively measurable solutions, then there are interesting implications. In particular the following result is true: Echeverria (1982) proved a criterion for a probability measure to be invariant for the Markov process arising as a unique solution to a well posed martingale problem when the state space is a compact (or a locally compact separable) metric space. In Bhatt and Karandikar (1993a) it was shown that one can do away with the assumption of compactness (or local compactness) in Echeverria's result if one assumes that the martingale problem has a unique solution in the class of all progressivley measureable solutions. Thus it appears that the role of compactness in Echeverria's result was to ensure that every solution has a r.c.l.l. modification in which case well posedness in the class of r.c.l.l. solutions also implies well-posedness in the class of measurable solutions.

Thus it is important to study measurable solutions of martingale problems. As remarked in Bhatt and Karandikar (1993a, 1993b), in most cases of interest, where one has well posedness in r.c.l.l. solutions, one can easily prove that indeed uniqueness holds in the class of all mesurable solutions. This leads us to the following question. *Does well-posedness of a martingale problem in the class of r.c.l.l. solutions imply well-posedness in the class of measurable solutions?* There are related questions such as - *Can it happen that well-posedness holds in the class of mesurable solutions but there does not exist any r.c.l.l. solution?*

The interplay of path properties and martingale problems can be seen by the following simple and well known example: Let A be an operator on $C^2(I\!R)$ with domain consisting of just two functions $f(x) = x$ and $g(x) = x^2$. Let $Af = 0$ and $Ag = 1$. Then the martingale problem for A is well posed in the class of processes with continuous paths - Brownian motion is the only solution by the Levy's theorem, while there are several solutions if one allows r.c.l.l. paths - compenseted Poission process with unit intensity is one such process while difference of two independent Poisson processes with intensity $\frac{1}{2}$ is another. The next section has several examples which throw light on this interplay and which also answer questions raised in the previous paragraph.

When the martigale problem for an operator A is well posed, then under some general assumptions, it gives rise to a Markov process whose generator L is an extention of the operator A. This Markov process is uniquely determined by A and thus A contains all the information about the process. One may expect that A is in this case a core for L. The example in section 3 shows that this is not true.

2 Examples

To begin with we introduce some terminology to differentiate between the path properties of the solutions of martingale problems. Let A be an operator on $C_b(E)$ with domain $D(A)$. Here E denotes the state space and will be assumed to be a complete and separable metric space. $C_b(E)$ denotes the space of all bounded continuous real valued functions on E. Let $\mu \in \mathcal{P}(E)$ where $\mathcal{P}(E)$ denotes the space of probability measures on E. As usual, δ_x denotes the probability measure given by $\delta_x(F) = 1_F(x)$ for any Borel subset of E.

Definition 2.1. An E valued process $(X_t)_{t\in[0,T]}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ is said to be a solution to the $\mathcal{G}$- martingale problem for (A,μ) with respect to a filtration $\{\mathcal{F}_t : t \in [0,T]\}$ if

(i) X is $\{\mathcal{F}_t\}$ - progressively measurable,

(ii) $\mathcal{L}(X_0) = \mu$

(iii) $E\int_0^T |Af(X_s)|ds < \infty \;\; \forall f \in D(A), t \in [0,T]$

and

(iv) for every $f \in D(A)$

$$f(X_t) - \int_0^t Af(X_s)ds, \quad t \in [0,T]$$

is a $\{\mathcal{F}_t\}$ - martingale.

Definition 2.2. An E valued process $(X_t)_{t\in[0,T]}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ is said to be a solution to the $\mathcal{D}$ - martingale problem for (A,μ) with respect to a filtration $\{\mathcal{F}_t : t \in [0,T]\}$ if X is a solution to the $\mathcal{G}$ - martingale problem for (A,μ) (as in Definition 2.1) and X has r.c.l.l. paths.

Definition 2.3. An E valued process $(X_t)_{t\in[0,T]}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ is said to be a solution to the $\mathcal{C}$ - martingale problem for (A,μ) with respect to a filtration

$\{\mathcal{F}_t : t \in [0,T]\}$ if X is a solution to the $\mathcal{G}$ - martingale problem for (A,μ) (as in Definition 2.1) and X has continuous paths.

Definition 2.4. The $\mathcal{G}$ martingale problem is said to be well-posed if for all $x \in E$, there exists a solution to the $\mathcal{G}$ martingale problem for (A, δ_x) and any two solutions have the same finite dimensional distributions. The well-posedness of the $\mathcal{D}$ and $\mathcal{C}$ martingale problems is defined similarly.

In the following examples, E is going to be either $(0,1]$ or $[0,1)$ with the topology inherited from $\mathbb{R}$. In either case, E becomes a complete separable metric space (for a suitable metric). Let $C_b^1(E)$ denote the class of bounded continuously differentiable functions on E. We will use the following lemma in the examples below.

Lemma 2.1. *Let X be a solution of the martingale problem for A. Suppose that f, f^2 belong to $D(A)$ and that*

$$Af^2 - 2f\,Af = 0. \tag{2.1}$$

Then

$$f(X(t)) = f(X(0)) + \int_0^t Af(X(s))ds \;\; a.s. \;\; \forall t \tag{2.2}$$

Proof. : Let $\tilde{U}_t = f(X_t)$ and $V_t = \int_0^t Af(X_s)ds$. Then $\tilde{M}_t = \tilde{U}_t - V_t$ is a martingale and in view of (2.1), $\tilde{N}_t = \tilde{U}_t^2 - \int_0^t 2\tilde{U}_s dV_s$ is also a martingale. The martingale $\tilde{M}$ has a r.c.l.l. modification, which we denote by M. Then $U_t = M_t + V_t$ is a r.c.l.l. modicfication of $\tilde{U}$ and $N_t = U_t^2 - \int_0^t 2U_s dV_s$ is a martingale (r.c.l.l. modification of $\tilde{N}$). Using Ito's formula, one has

$$\begin{aligned}
M_t^2 &= U_t^2 + V_t^2 - 2U_tV_t \\
&= \left(N_t + 2\int_0^t U_s dV_s\right) + V_t^2 - \left(2\int_0^t U_s dV_s + 2\int_0^t V_s dU_s\right) \\
&= N_t + V_t^2 - 2\int_0^t V_s dU_s \\
&= N_t + V_t^2 - 2\int_0^t V_s dV_s + 2\int_0^t V_s dM_s \\
&= N_t + 2\int_0^t V_s dM_s + (V_t^2 - 2\int_0^t V_s dV_s)
\end{aligned}$$

Now N_t and $2\int_0^t V_s dM_s$ are local martingales. Further, $V_t^2 - 2\int_0^t V_s dV_s = 0$ since V is a process with absolutely continuous paths. Hence M_t^2 is a local martinagle. Since f and hence M is bounded, it follows that M_t^2 is a martingale. This also implies that $M_t - f(X_0)$ and $(M_t - f(X_0))^2$ are martingales with $M_0 - f(X_0) = 0$. Hence we get $M_t = f(X_0)$ a.s. for all t. which is same as $\tilde{M}_t = f(X_0)$ *a.s.* This completes the proof. □

It can be easily checked in all the examples given below that $D(A) \subset C_b(E)$ is an algebra that separates points on E. *i.e.* given two distinct points $x, y \in E$, there exists $f \in D(A)$ such that $f(x) \neq f(y)$. It also does not vanish on E. *i.e.* Given $x \in E$, there exists $f \in D(A)$ such that $f(x) \neq 0$. Further, in all the examples the operator A is the restriction of the derivative operator to a suitable domain. Using the fact that

$$(f^2)' = 2f(f') \quad \text{for all differentiable functions } f,$$

it follows that $Af^2 = 2fAf$ holds for all $f \in D(A)$ and hence that (2.2) holds.

Example 2.1. Let $E = (0,1]$, $D(A_1) = \{f : f \in C_b^1(E)\}$. Let A_1 be defined on $D(A_1)$ by $Af = f'$. Let $x \in E$ and let X be a solution to the martingale problem for (A_1, δ_x). Note that the function $f(y) = y$ belongs to $D(A_1)$. Hence using (2.2) for this f we get

$$X_t = x + t \;\; a.s. \;\; \forall t. \tag{2.3}$$

Since $E = (0,1]$, it is clear that (2.3) is impossible. Hence, none of the $\mathcal{G}, \mathcal{D}$ and $\mathcal{C}$ martingale problems for A_1 admit any solution.

Example 2.2. Let $E = (0,1]$, $D(A_2) = \{f : f \in C_b^1(E), f'(1) = 0\}$. Let A_2 be defined on $D(A_2)$ by $A_2 f = f'$.

The function $f(y) = y$ is not in $D(A_2)$ however, for $x \in (0,1)$ and $0 < \epsilon < 1 - x$, we can get a $f \in D(A_2)$ such that $f(y) = y$ for $y \in [x, 1-\epsilon]$ and hence using (2.2) for this f we get

$$X_t = x + t \;\; a.s. \;\; \forall t \text{ with } 0 < x + t < 1 - \epsilon.$$

Thus the solution moves to the right with uniform velocity till it reaches 1 and unlike in example 2.1 it is allowed to stay there since $f'(1) = 0$ for all $f \in D(A_2)$.

Thus $X_2(t) = [(x+t) \wedge 1]$ is the only solution of the martingale problem for (A_2, δ_x). The solution is continuous. Thus the $\mathcal{G}$-martinagle problem, $\mathcal{D}$-martingale problem as well as $\mathcal{C}$-martinagle problem are all well posed (and any solution is continuous).

Example 2.3. Let $E = (0,1]$, $D(A_3) = \{f : f \in C_b^1(E), f'(1) = 0, \text{ and } \lim_{x \to 0} f(x) = f(1)\}$. Let A_3 be defined on $D(A_3)$ by $A_3 f = f'$.

Since A_3 is a restriction of A_2, X_2 defined in Example 2.2 above continues to be a solution of the martingale problem for (A_3, δ_x). In fact it is the only solution with r.c.l.l. paths. The additional boundary condition on the domain namely $\lim_{x \to 0} f(x) = f(1)$ allows a possible solution to "jump" once it reaches 1 and move as if it is starting from 0. However, since 0 is not in E, such a solution cannot be r.c.l.l.

For example, X_3 defined by

$$X_3(t) = \begin{cases} (x+t) \pmod 1 & \text{if } (x+t) \pmod 1 \neq 0 \\ 1 & \text{if } (x+t) \pmod 1 = 0 \end{cases}$$

is also a solution of the martingale problem for (A_3, δ_x). X_3 is a left continuous process with right limits and does not have a r.c.l.l. modification. This process X_3 is akin to uniform motion on a circle.

In fact we can construct many more solutions to the martingale problem for A_3. A process which moves uniformly (with speed 1) till it reaches 1, stops for an arbitrary amount of time there and then begins moving (uniformly) as if it is starting from 0 will also be a solution.

Thus uniqueness holds for the martingale problem in the class of r.c.l.l. (and continuous) solutions but not in the class of all progressively solutions. *i.e.* The $\mathcal{D}$ and the $\mathcal{C}$ martingale problems for A_3 are well-posed but the $\mathcal{G}$ martingale problem for A_3 is not.

Example 2.4. Let $E = (0, 1]$, $D(A_4) = \{f : f \in C_b^1(E), \lim_{x\to 0} f(x) = f(1)\}$. Let A_4 be defined on $D(A_4)$ by $A_4 f = f'$.

The process X_3, defined in Example 2.3 above, is a solution of the martingale problem for (A_4, δ_x). However, unlike in Example 2.3 the solution cannot stay at 1 since there exist functions f in $D(A_4)$ such that $f'(1)$ is not equal to zero. Thus X_3 is the only solution and uniqueness holds in the class of all progressively measurable solutions. It is clear that there is no r.c.l.l. solution of the martingale problem for A_4. Thus the $\mathcal{G}$ martingale problem for A_4 is well-posed. However, the $\mathcal{D}$ and $\mathcal{C}$ martingale problems for A_4 are not well posed.

Example 2.5. Let $E = [0, 1)$, $D(A_5) = \{f : f \in C_b^1(E), \lim_{x\to 1} f(x) = f(0)\}$. Let A_5 be defined on $D(A_5)$ by $A_5 f = f'$.

Let $X_5(t) = (x + t)(\text{mod } 1)$. Arguing as in Example 2.4 it can be seen that X_5 is the unique solution of the martingale problem for (A_5, δ_x) which clearly is r.c.l.l. Thus uniqueness holds in the class of all r.c.l.l. solutions and also in the class of progressively measurable solutions but there is no continuous solution of the martingale problem for A_5. Thus the $\mathcal{G}$ and $\mathcal{D}$ martingale problems for A_5 are well-posed but the $\mathcal{C}$ martingale problem for A_5 is not.

Example 2.6. Let $E = [0, 1)$, $D(A_6) = \{f : f \in C_b^1(E), \lim_{x\to 1} f(x) = f(0), f'(0) = 0\}$. Let A_6 be defined on $D(A_6)$ by $A_6 f = f'$.

Then X_5 (defined above in Example 2.5) is a r.c.l.l. solution of the martingale problem for (A_6, δ_x). Once again as in Example 2.3 we can see that there are many other r.c.l.l. solutions - any process which jumps to 0 as soon as it "reaches" 1, stays at 0 for an arbitrary amount of time and then starts moving with speed 1 towards 1.

None of the $\mathcal{G}$, $\mathcal{D}$ or $\mathcal{C}$ martingale problems for A_6 are well-posed. The only continuous solution of the martingale problem is the process constant at 0. No initial distribution, other than δ_0 admits a continuous solution.

The above six examples emphasize the point that the martingale problem may be well-posed in one class and not well posed in some other class. We summarize all the examples in Table 2.1.

In examples 2.2, 2.3 and 2.6, there was a point x such that $Af(x) = 0$ for all $f \in D(A)$. (x=1 for first two and x=0 in the third case). It may appear that perhaps the odd behaviour regarding the solutions of martingale problems is due to this. However, the following simple example tells us that that is not so.

Example Number	Existence $\mathcal{C}$	Existence $\mathcal{D}$	Existence $\mathcal{G}$	Uniqueness $\mathcal{C}$	Uniqueness $\mathcal{D}$	Uniqueness $\mathcal{G}$
2.1	$\times$	$\times$	$\times$	-	-	-
2.2	✓	✓	✓	✓	✓	✓
2.3	✓	✓	✓	✓	✓	$\times$
2.4	$\times$	$\times$	✓	-	-	✓
2.5	$\times$	✓	✓	-	✓	✓
2.6	$\times$	✓	✓	-	$\times$	$\times$

Table 2.1: An overview of the examples

Example 2.7. Let E and A_3 be as in Example 2.3. Let E' be any other complete separable metric space and let A' be an operator on $C_b(E')$ such that $D(A')$ is an algebra which separates points on E'. Assume that the martingale problem for A' is well posed in the class of all progressively measurable processes and that the unique solution Y has continuous paths.

Let $F = E \times E'$. Let an operator B be defined as follows. Let $D(B) =$ the algebra generated by $\{f \otimes g : f \in D(A_3), g \in D(A')\}$. And for $f \otimes g \in D(B)$, define

$$B(f \otimes g)(x, y) = (A_3 f)(x) g(y) + f(x)(A'g)(y).$$

B is extended to the entire $D(B)$ by linearity. Note that there is no $(x, y) \in F$ such that $B(\tilde{f})(x, y) = 0$ for all $\tilde{f} \in D(B)$.

It now follows from Theorem IV.10.1 of Ethier and Kurtz (1986) that (X, Z) is a solution of the martingale problem for B if and only if X and Z are independent, X is a solution of the martingale problem for A_3 and Z is a solution of the martingale problem for A'. In particular, it follows that (X_2, Y) (defined, say, on the product space) is the unique solution of the martingale problem for B with continuous paths. However, (X_3, Y) (again defined on the product space) is also a (non r.c.l.l.) solution of the martingale problem for B. Thus the $\mathcal{D}$ and $\mathcal{C}$ martingale problems for B are well posed but the $\mathcal{G}$ martingale problem is not.

Similar construction would work for examples 2.2 and 2.6 as well.

3 Well-posedness and Core

As in the previous section let E denote a complete separable metric space and let A be an operator on $C_b(E)$ with domain $D(A)$. For a subset V of $C_b(E)$, let bp-closure of V be the smallest set containing V which is closed under bounded pointwise convergence of a sequence of functions in V. We will denote this set by bp-closure(V).

In this section we will assume that there exists a countable subset $\{f_n : n \geq 1\}$ of $D(A)$ such that

$$\{(f, Af) : f \in D(A)\} \subseteq \text{bp-closure}\{(f_n, Af_n) : n \geq 1\}.$$

We also assume that the $\mathcal{D}$ martingale problem for A is well posed. We will denote the distribution of the unique (r.c.l.l.) solution of the martingale problem for (A, δ_x) by P_x (P_x is a probability measure on $D([0,T], E)$). In this framework, it follows that

$$x \to P_x(F) \text{ is Borel measurable for all Borel subsets } F \text{ of } D([0,T], E)).$$

Further, T_t defined by,

$$T_t f(x) = \int_{D([0,T],E)} f(\omega(t))\, dP_x(\omega) \quad f \in C_b(E)$$

is a semigroup and its generator L is an extension of A (i.e., domain of L denoted by $D(L)$ contains $D(A)$ and for $f \in D(A)$, $Lf = Af$). See Chapter 4 in Ethier and Kurtz (1986). See also Horowitz and Karandikar (1990)

In Bhatt and Karandikar(1993) the following criterion for an invariant measure was proved. This is an extention of a result due to Echeverria (1982). See also Ethier and Kurtz (1986). Let $D(A)$ be an algebra that separates points and the $\mathcal{G}$ martingale problem for A be well-posed (in addition to $\mathcal{D}$ martingale problem being well-posed). Then for any probability measure μ

$$\int_E (Af)(x) d\mu(x) = 0 \quad \text{for all } f \in \mathcal{D} \tag{3.4}$$

implies that μ is an invariant measure for the semigroup (T_t).

If this result can be extended to signed measures, namely, that for a signed measure μ, (3.4) implies that (the signed measure) μ is an invariant measure for the semigroup (T_t), then it would imply that A is a core for L. Our example given below shows that this is not true.

Recall that a restriction B of L to $D(B)$ is said to be a core if

$$\{(f, Lf) : f \in D(L)\} \subseteq \text{bp-closure}\{(f, Bf) : f \in D(B)\}.$$

Here we would construct an example where the martingale problem is well posed and (3.4) holds for a signed measure but it is not an invariant measure.

Example 3.1. Let $E = (0,1] \times (0,1]$. Define an operator A on E as follows. Let $D(A) =$ linear span $\{f \otimes g : f, g \in C_b^1((0,1]), \lim_{x\to 0} f(x) = \lim_{x \to 0} g(x) = 0, f'(1) = g'(1) = 0\}$. And for $f \otimes g \in D(A)$, define

$$A(f \otimes g)(x,y) = f'(x)g(y) + f(x)g'(y).$$

Note that $D(A)$ is an algebra that separates points in E and vanishes nowhere. Using Example 2.2 and Theorem IV.10.1 in Ethier and Kurtz (1986), it follows that the $\mathcal{G}$, $\mathcal{D}$ and $\mathcal{C}$ martingale problems for A are well-posed. The unique continuous solution is given by

$$(X(t), Y(t)) = ([(x+t) \wedge 1], [(Y(0)+t) \wedge 1]) .$$

Since the solution will hit $(1,1)$ with probability 1, and since $(1,1)$ is an absorbing state, the only invariant measure for the Markov process is $\delta_1 \times \delta_1$.

Now let $\mu, \nu \in \mathcal{P}(E)$ be defined by $\mu = \lambda \times \delta_1$, $\nu = \delta_1 \times \lambda$ where λ denotes the Lebesgue measure on $(0,1]$ and δ_1 is the Dirac measure at 1. Then note that for $f \otimes g \in D(A)$,

$$\int_E A(f\otimes g)(x,y)\mu(dx,dy) = g(1)\int_0^1 f'(x)\lambda(dx) + g'(1)\int_0^1 f(x)\lambda(dx) = f(1)g(1).$$

Similarly $\int_E A(f\otimes g)(x,y)\nu(dx,dy) = f(1)g(1)$ for all $f \otimes g \in D(A)$. Recalling that $D(A)$ is the linear span of functions of the form $f \otimes g$, we get

$$\int AF d(\mu - \nu) = 0 \quad \forall F \in D(A).$$

However, $\mu - \nu$ is clearly not an invariant measure for the process (X, Y). This can be seen as follows.

Let $(T_t)_{t\geq 0}$ defined on $C_b(E)$ be the semigroup corresponding to the Markov process (X, Y). Then $(T_t)_{t\geq 0}$ is given by

$$T_t F(x,y) = F((x+t)\wedge 1, (y+t)\wedge 1).$$

Then for any $f, g \in C[(0,1]]$,

$$\int_E (f\otimes g)\, d(\mu-\nu) = [\tilde{f}(1) - \tilde{f}(0)]g(1) - f(1)[\tilde{g}(1) - \tilde{g}(0)] \tag{3.5}$$

where $\tilde{f}$ and $\tilde{g}$ are antiderivatives of f and g respectively. On the other hand, for $t = 1$, $T_1(f \otimes g)(x,y) \equiv f(1)g(1)$ which in turn implies that

$$\int_E T_1(f\otimes g)\, d(\mu-\nu) = 0.$$

Clearly, we can find examples of f, g such that RHS in (3.5) is not zero. Thus $\mu - \nu$ is not an invariant measure for T_t.

Thus A is not a core for the generator of $(T_t)_{t\geq 0}$.

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