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Wavelet Linear Density Estimation for Associated Sequences

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Abstract: We develop a wavelet based linear density estimator for the estimation of the probability density function for a sequence of associated random variables with a common one-dimensional probability density function and obtain bounds on L_p-losses for such estimators.

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables. A finite family $\{X_1, \ldots, X_N\}$ of random variables is said to be *associated* if

$$Cov(h(X_1, \ldots, X_N), g(X_1, \ldots, X_N)) \geq 0$$

for any componentwise nondecreasinfg functions h and g on R^n such that the covariance exists. An infinite family of random variables is said to be *associated* if every finite subfamily is associated.

Associated random variables are of considerable interest in reliability studies, percolation theory and statistical mechanics. For a review of several probabilistic and statistical results for associated sequences, see Prakasa Rao and Dewan (2001).

Suppose that $\{X_n, n \geq 1\}$ is a sequence of associated random variables with a common one-dimensional marginal probability density function f. The problem of inerest is the estimation of probability density function f based on the observations $\{X_1, \ldots, X_N\}$. Kernel method of density estimation has been investigated in this context by Bagai and Prakasa Rao (1991, 1995) and Roussas (1991). A general method of density estimation using delta sequences was discussed in Dewan and Prakasa Rao (1999). We now propose an estimator based on wavelets and obtain bounds on the L_p-losses for the propsed estimator.

2 Preliminaries

Let $\{X_i, i \geq 1\}$ be a sequence of associated random variables with common one-dimensional marginal probability density function f. Suppose f is bounded and compactly supported. The problem is to estimate the probability density function f based on the observations $X_1, \ldots, X_n$.

Any function $f \in L_2(R)$ can be expanded in the form

$$\begin{aligned} f &= \sum_{k=-\infty}^{\infty} \alpha_{j_0,k}\phi_{j_0,k} + \sum_{j=j_0}^{\infty}\sum_{k=-\infty}^{\infty} \beta_{j,k}\psi_{j,k} \\ &= P_{j_0}f + \sum_{j=j_0}^{\infty} D_j f \end{aligned}$$

for any integer $j_0 \geq 1$ where the functions

$$\phi_{j_0,k}(x) = 2^{j_0/2}\phi(2^{j_0}x - k)$$

and

$$\psi_{j_0,k}(x) = 2^{j_0/2}\psi(2^{j_0}x - k)$$

constitute an orthonormal basis of $L_2(R)$ (Daubechies (1988)). The functions $\phi(x)$ and $\psi(x)$ are the scale function and the orthogonal wavelet function respectively. Observe that

$$\alpha_{j_0,k} = \int_{-\infty}^{\infty} f(x)\phi_{j_0,k}(x)dx$$

and

$$\beta_{j,k} = \int_{-\infty}^{\infty} f(x)\psi_{j,k}(x)dx.$$

We suppose that the function ϕ and ψ belong to C^{r+1} for some $r \geq 1$ and have compact support contained in an interval $[-\delta, \delta]$. It follows from the Corollary 5.5.2 in Daubechies (1988) that the function ψ is orthogonal to a polynomial of degree less than or equal to r. In particular

$$\int_{-\infty}^{\infty} \psi(x)x^{\ell}dx = 0, \ell = 0, 1, \ldots, r.$$

We assume that the following conditions hold.

(A1) The sequence $\{X_n, n \geq 1\}$ is a sequence of associated random variables with

$$u(n) = \sup_{i\geq 1} \sum_{|j-i|\geq n} Cov(X_i, X_j) \leq Cn^{-\alpha}$$

for some $C > 0$ and $\alpha > 0$.

(A2) Suppose the density function f belongs to the Besov class (cf. Meyer (1990))

$$F_{s,p,q} = \{f \in B^s_{p,q}, ||f||_{B^s_{p,q}} \leq M\}$$

for some $0 < s < r + 1, p \geq 1$ and $q \geq 1$, where

$$||f||_{B^s_{p,q}} = ||P_0 f||_p + [\sum_{j\geq 0}(||D_j f||_p 2^{js})^q]^{1/q}.$$

(For properties of Besov spaces, see Triebel (1992) (cf. Leblanc(1996)).

Define

$$\hat{f}_N = \sum_{k\in K_{j_0}} \hat{\alpha}_{j_0,k}\phi_{j_0,k} \tag{2. 1}$$

where

$$\hat{\alpha}_{j_0,k} = \frac{1}{N}\sum_{i=1}^{N} \phi_{j_0,k}(X_i) \tag{2. 2}$$

and K_{j_0} is the set of all k such that the the intersection of the support of f and the support of $\phi_{j_0,k}$ is nonempty. Since the function ϕ has a compact support by assumption, it follows that the cardinality of the set K_{j_0} is $O(2^{j_0})$.

We now study the properties of the estimator $\hat{f}_N$ as an estimator of the probability density function f.

3 Main Result

Let $p' \geq \max(2,p)$. We will now obtain bounds on $E_f||\hat{f}_N - f||^2_{p'}$.

Observe that

$$E_f||\hat{f}_N - f||^2_{p'} \leq 2(||f - P_{j_0}f||^2_{p'} + E_f||\hat{f}_N - P_{j_0}f||^2_{p'}). \tag{3. 1}$$

We now estimate the terms on the right side of the above equation.

Lemma 3.1 For any $f \in F_{s,p,q}, s \geq \frac{1}{p}$, there exists a constant C_1 such that

$$||f - P_{j_0}f||^2_{p'} \leq C_1 2^{-2s'j_0} \tag{3. 2}$$

where

$$s' = s + \frac{1}{p'} - \frac{1}{p}. \tag{3. 3}$$

Proof: See Leblanc (1996), p.83.

We will now estimate the second term in the equation (3.1). Note that

$$\begin{aligned} E_f||\hat{f}_N - P_{j_0}f||^2_{p'} &= E_f||\sum_{k\in K_{j_0}}(\hat{\alpha}_{j_0,k} - \alpha_{j_0,k})\phi_{j_0,k}||^2_{p'} \\ &\leq C_2 E_f\{||\hat{\alpha}_{j_0,.} - \alpha_{j_0,.}||^2_{\ell_{p'}(Z)}\}2^{2j_0(\frac{1}{2}-\frac{1}{p'})} \end{aligned}$$

for some constant $C_2 > 0$ by Lemma 1 in Leblanc (1996), p.82 (cf. Meyer (1990)). Here Z is the set of all integers $-\infty < k < \infty$ and the norm

$$||\lambda||_{\ell_p(Z)} = (\sum_{k\in Z}|\lambda_k|^p)^{1/p}.$$

Hence

$$E_f||\hat{f}_N - P_{j_0}f||^2_{p'} \leq C_2 2^{2j_0(\frac{1}{2}-\frac{1}{p'})}\{\sum_{k\in K_{j_0}} E_f|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^{p'}\}^{2/p'}. \tag{3. 4}$$

Let

$$W_i = \eta(X_i) = \phi_{j_0,k}(X_i) - E_f(\phi_{j_0,k}(X_i)), 1 \leq i \leq N.$$

Then

$$\hat{\alpha}_{j_0,k} - \alpha_{j_0,k} = \frac{1}{N}\sum_{i=1}^{N} W_i$$

and

$$E_f|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^{p'} = N^{-p'} E_f|\sum_{i=1}^{N} W_i|^{p'}.$$

Observe that the random variables $W_i, 1 \leq i \leq N$ are funcions of associated random variables $X_i, 1 \leq i \leq N$. We will now estimate the term

$$E_f|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^{p'}$$

by applying Rosenthal type inequality for functions of associated random variables due to Shao and Yu (1996), p.210. Note that the sequence of random variables $\eta(X_i), 1 \leq i \leq N$ are identically distributed with mean zero. Further more the function $\eta(x)$ is differentiable with

$$\text{(3. 5)} \qquad \begin{aligned} \sup_{-\infty<x<\infty} |\eta'(x)| &= \sup_{-\infty<x<\infty} |\phi'_{j_0,k}(x)| \\ &\leq 2^{3j_0/2} \sup_{-\infty<x<\infty} |\phi'(2^{j_0}x - k)| \\ &\leq 2^{3j_0/2} \sup_{-\infty<x<\infty} |\phi'(x)| \\ &\leq B_0 2^{3j_0/2} \end{aligned}$$

for some constant $B_0 > 0$ since $\phi \in C^{r+1}$ for some $r \geq 1$. In addition, for any $d \geq 0$,

$$\text{(3. 6)} \qquad \begin{aligned} E_f[|\eta(X_1)|^d] &\leq 2^d (E_f|\phi_{j_0,k}(X_1)|^d + B_1^d) \\ &\leq 2^d 2^{j_0 d/2} (E_f[|\phi(2^{j_0}X_1 - k)|^d] + B_1^d) \\ &\leq 2^{d+1} 2^{j_0 d/2} B_1^d \\ &= B_2 2^{j_0 d/2} \end{aligned}$$

where B_1 is a bound on ϕ following the assumption that it has compact support and that $\phi \in C^{r+1}$ and B_2 is a positive constant independent of j_0.

(A3) Suppose that $\max(2, p) \leq p' < d < \infty$.

Applying Theorem 4.2 in Shao and Yu (1996), it follows that for any $\varepsilon > 0$, there exists a constant G_0 depending only on ε, d, p' and α such that

$$\text{(3. 7)} \qquad \begin{aligned} E_f|\sum_{i=1}^{N} W_i|^{p'} &\leq D_0(N^{1+\varepsilon} E|\eta(X_1)|^{p'} + \\ &\quad (N \max_{1\leq i\leq N} \sum_{\ell=1}^{N} |Cov(\eta(X_i), \eta(X_\ell))|)^{p'/2} + \\ &\quad N^{(d(p'-1)-p'+\alpha(p'-d)/(d-2)\vee(1+\varepsilon)} \\ &\quad \times ||\eta(X_1)||_d^{d(p'-2)/(d-2)} (B_0^2 2^{3j_0} C)^{(d-p')/(d-2)}) \end{aligned}$$

where B_0 is as defined above. Note that the constants D_0 and B_0 are independent of $k \in K_{j_0}$ and j_0. Applying Newman's inequality (Newman (1984)), we obtain that

$$(3.\ 8) \qquad |Cov(\eta(X_i), \eta(X_\ell)| \leq \{\sup_{-\infty<x<\infty} |\eta'(x)|\}^2 Cov(X_i, X_\ell) \leq B_0^2 2^{3j_0}.$$

Combining the above estimates, we get that

$$(3.\ 9) \qquad E_f|\sum_{i=1}^{N} W_i|^{p'} \leq D_0(N^{1+\varepsilon}2^{(j_0/2)p'}B_2 + (N \max_{1\leq i\leq N} \sum_{\ell=1}^{N} Cov(X_i, X_\ell)B_0^2 2^{3j_0})^{p'/2} + N^{(d(p'-1)-p'+\alpha(p'-d)/(d-2)\vee(1+\varepsilon)} \times (B_0^{1/d}2^{j_0/2})^{d(p'-2)/(d-2)}(B_0^2 2^{3j_0}C)^{(d-p')/(d-2)}).$$

Since the above estimate holds for all $k \in K_{j_0}$ and the cardinality of K is $O(2^{j_0})$, it follows that

$$(3.\ 10) \quad E_f||\hat{f}_N - P_{j_0}f||_{p'}^2 \leq C_2 2^{2j_0(\frac{1}{2}-\frac{1}{p'})}2^{j_0}\{D_0(N^{1+\varepsilon}2^{(j_0/2)p'}B_2 + (N \max_{1\leq i\leq N} \sum_{\ell=1}^{N} Cov(X_i, X_\ell)B_0^2 2^{3j_0})^{p'/2} + N^{(d(p'-1)-p'+\alpha(p'-d)/(d-2)\vee(1+\varepsilon)} \times (B_0^{1/d}2^{j_0/2})^{d(p'-2)/(d-2)}(B_0^2 2^{3j_0}C)^{(d-p')/(d-2)})\}^{2/p'}.$$

Hence there exists a constant $C_3 > 0$ such that

$$(3.\ 11) \quad E_f||\hat{f}_N - f||_{p'}^2 \leq C_3[2^{2j_0(\frac{3}{2}-\frac{1}{p'})}2^{j_0}\{D_0(N^{1+\varepsilon}2^{j_0/2p'}B_2 + (N \max_{1\leq i\leq N} \sum_{\ell=1}^{N} Cov(X_i, X_\ell)B_0^2 2^{3j_0})^{p'/2} + N^{(d(p'-1)-p'+\alpha(p'-d)/(d-2)\vee(1+\varepsilon)} \times (B_0^{1/d}2^{j_0/2})^{d(p'-2)/(d-2)}(B_0^2 2^{3j_0}C)^{(d-p')/(d-2)})\}^{2/p'} + 2^{-2s'j_0}]$$

and we have the following main result.

Theorem 3.2: Suppose the conditions (A1)-(A3) hold. Let $\max(2, p) \leq p' < d < \infty$. Define the wavelet linear estimator $\hat{f}_N$ as defined by the relation (2.1). Then, for every $\varepsilon > 0$, there corresponds a constant $C > 0$ such that

$$(3.\ 12) \qquad E_f||\hat{f}_N - f||_{p'}^2 \leq C[2^{2j_0(\frac{3}{2}-\frac{1}{p'})}2^{j_0}\{(N^{1+\varepsilon}2^{(j_0/2)p'} +$$

$$(N \max_{1\le i\le N} \sum_{\ell=1}^{N} Cov(X_i, X_\ell) 2^{3j_0})^{p'/2} +$$
$$N^{(d(p'-1)-p'+\alpha(p'-d)/(d-2)\vee(1+\varepsilon)}$$
$$\times 2^{(j_0/2)d(p'-2)/(d-2)} 2^{3j_0(d-p')/(d-2)})\}^{2/p'} + 2^{-2s'j_0}].$$

4 Remarks

Suppose $1 \le p' \le 2$. One can get similar bounds as in Theorem 3.2 for the expected loss

$$E_f||\hat{f}_N - f||_{p'}^{p'}$$

observing that

$$E_f||\hat{f}_N - f||_{p'}^{p'} \le 2^{p'-1}(||f - P_{j_0}f||_{p'}^{p'} + E_f||\hat{f}_N - P_{j_0}f||_{p'}^{p'}), \tag{4. 1}$$

$$||f - P_{j_0}f||_{p'}^{p'} \le C_4 2^{-p's'j_0}, \tag{4. 2}$$

and

$$E_f||\hat{f}_N - P_{j_0}f||_{p'}^{p'} \le C' 2^{2j_0(\frac{p'}{2}-1)} \sum_{k\in K_{j_0}} E_f|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^{p'} \tag{4. 3}$$

for some positive constants C_4 and C'. We will not discuss the details.

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