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Parametric estimation for linear stochastic differential equations driven by fractional Brownian Motion

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Parametric Estimation for Linear Stochastic Differential Equations Driven by Fractional Brownian Motion

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Abstract

We investigate the asymptotic properties of the maximum likelihhod estimator and Bayes estimator of the drift parameter for stochastic processes satisfying a linear stochastic differential equations driven by fractional Brownian motion. We obtain a Bernstein-von Mises type theorem also for such a class of processes.

Keywords and phrases: Linear stochastic differential equations ; fractional Ornstein-Uhlenbeck process; fractional Brownian motion; Maximum likelihgood estimation; Bayes estimation; Consistency; Asymptotic normality; Bernstein - Von Mises theorem.

AMS Subject classification (2000): Primary 62M09, Secondary 60G15.

1 Introduction

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999a). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion. Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. In a recent paper, Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

(1. 1) $$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0.$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the obsrevation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

We now discuss more general classes of stochastic processes satsfying linear stochastic differential equations driven fractional Brownian motion and study the asymptotic properties of the maximum likelihood and the Bayes estimators for parameters involved in such processes.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions and the processes discussed in the following are $(\mathcal{F}_T)$-adapted. Further the natural fitration of a process is understood as the P-completion of the filtration generated by this process.

Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with Hurst parameter $H \in (0,1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, s \geq 0. \tag{2. 1}$$

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$Y_t = \int_0^t C(s)ds + \int_0^t B(s)dW_s^H, t \geq 0 \tag{2. 2}$$

where $C = \{C(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process and $B(t)$ is a nonvanishing nonrandom function. For convenience we write the above integral equation in the form of a stochastic differential equation

$$dY_t = C(t)dt + B(t)dW_t^H, t \geq 0 \tag{2. 3}$$

driven by the fractional Brownian motion W^H. The integral

$$\int_0^t B(s)dW_s^H \tag{2. 4}$$

is not a stochastic integral in the Ito sense but one can define the integral of a deterministic function with respect to the fBM in a natural sense(cf. Norros et al. (1999).) Even though the process Y is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration $(\mathcal{Z}_t)$ of the process Z coincides with the natural filtration $(\mathcal{Y}_t)$ of the process Y (Kleptsyna et al. (2000)). Define, for $0 < s < t$,

$$k_H = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(H + \frac{1}{2}\right), \tag{2. 5}$$

$$k_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}, \tag{2. 6}$$

$$\lambda_H = \frac{2H\,\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}, \tag{2. 7}$$

$$w_t^H = \lambda_H^{-1} t^{2-2H}, \tag{2. 8}$$

and

$$M_t^H = \int_0^t k_H(t,s)dW_s^H, t \geq 0. \tag{2. 9}$$

The process M^H is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al. (1999)) and its quadratic variance $< M_t^H >= w_t^H$. Further more the natural filtration of the martingale M^H coincides with the natural fitration of the fBM W^H. In fact the stochastic integral

$$\int_0^t B(s)dW_s^H \tag{2. 10}$$

can be represented in terms of the stochastic intehral with respect to the martingale M^H. For a measurable function f on $[0,T]$, let

$$K_H^f(t,s) = -2H\frac{d}{ds}\int_s^t f(r)r^{H-\frac{1}{2}}(r-s)^{H-\frac{1}{2}}dr, 0 \leq s \leq t \tag{2. 11}$$

when the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure(see Samko et al. (1993) for sufficient conditions). The following result is due to Kleptsyna et al. (2000).

Therorem 2.1 Let M^H be the fundamental martingale associated with the fBM W^H dfined by (2.9). Then

$$\int_0^t f(s)dW_s^H = \int_0^t K_H^f(t,s)dM_s^H, t \in [0,T] \tag{2. 12}$$

a.s $[P]$ whenever both sides are well defined.

Suppose the sample paths of the process $\{\frac{C(t)}{B(t)}, t \geq 0\}$ are smooth enough (see Samko et al. (1993)) so that

$$Q_H(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{C(s)}{B(s)}ds, t \in [0,T] \tag{2. 13}$$

is welldefined where w^H and k_H are as defined in (2.8) and (2.6) respectively and the derivative is understood in the sense of absoulute continuity. The following theorem due to Kleptsyna et al. (2000) associates a *fundamental semimartingale* Z associated with the process Y such that the natural filtration $(\mathcal{Z}_t)$ coincides with the natural filtration $(\mathcal{Y}_t)$ of Y.

Theorem 2.2: Suppose the sample paths of the process Q_H defined by (2.13) belong P-a.s to $L^2([0,T], dw^H)$ where w^H is as defined by (2.8). Let the process $Z = (Z_t, t \in [0,T])$ be defined by

$$Z_t = \int_0^t k_H(t,s)B^{-1}(s)dY_s \tag{2. 14}$$

where the function $k_H(t,s)$ is as defined in (2.6). Then the following results hold:
(i) The process Z is an $(\mathcal{F}_t)$ -semimartingale with the decomposition

$$Z_t = \int_0^t Q_H(s)dw_s^H + M_t^H \tag{2. 15}$$

where M^H is the fundamental martingale defined by (2.9),
(ii) the process Y admits the representation

$$Y_t = \int_0^t K_H^B(t,s)dZ_s \tag{2. 16}$$

where the function K_H^B is as defined in (2.11), and
(iii) the natural fitrations of $(\mathcal{Z}_t)$ and $(\mathcal{Y}_t)$ coincide.

Kleptsyna et al. (2000) derived the following Girsanov type formula as a consequence of the Theorem 2.2.

Theorem 2.3: Suppose the assumptions of Theorem 2.2 hold. Define

$$\Lambda_H(T) = \exp\{-\int_0^T Q_H(t)dM_t^H - \frac{1}{2}\int_0^t Q_H^2(t)dw_t^H\}. \tag{2. 17}$$

Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^* = \Lambda_H(T)P$ is a probability measure and the probability measure of the process Y under P^* is the same as that of the process V defined by

$$V_t = \int_0^t B(s)dW_s^H, 0 \leq t \leq T. \tag{2. 18}$$

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3 Main Results

Let us consider the stochastic differential equation

$$dX(t) = [a(t, X(t)) + \theta\, b(t, X(t))]dt + \sigma(t)dW_t^H, t \geq 0 \tag{3. 1}$$

where $\theta \in \Theta \subset R, W = \{W_t^H, t \geq 0\}$ is a fractional Brownian motion with Hurst parameter H and $\sigma(t)$ is a positive nonvanishing function on $[0, \infty)$. In other words $X = \{X_t, t \geq 0\}$ is a stochastic process satisfying the stochastic integral equation

$$X(t) = X(0) + \int_0^t [a(s, X(s)) + \theta\, b(s, X(s))]ds + \int_0^t \sigma(s)dW_s^H, t \geq 0. \tag{3. 2}$$

Let

$$C(\theta, t) = a(t, X(t)) + \theta\, b(t, X(t)), t \geq 0 \tag{3. 3}$$

and assume that the sample paths of the process $\{\frac{C(\theta,t)}{\sigma(t)}, t \geq 0\}$ are smooth enough so that the the process

$$Q_{H,\theta}(t) = \frac{d}{dw_t^H}\int_0^t k_H(t, s)\frac{C(\theta, s)}{\sigma(s)}ds, t \geq 0 \tag{3. 4}$$

is welldefined where w_t^H and $k_H(t, s)$ are as defined in (2.8) and (2.6) respectively. Suppose the sample paths of the process $\{Q_{H,\theta}, 0 \leq t \leq T\}$ belong almost surely to $L^2([0, T], dw_t^H)$. Define

$$Z_t = \int_0^t \frac{k_H(t, s)}{\sigma(s)}dX_s, t \geq 0. \tag{3. 5}$$

Then the process $Z = \{Z_t, t \geq 0\}$ is an $(\mathcal{F}_t)$-semimartingale with the decomposition

$$Z_t = \int_0^t Q_{H,\theta}(s)dw_s^H + M_t^H \tag{3. 6}$$

where M^H is the fundamental martingale defined by (2.9) and the process X admits the representation

$$X_t = \int_0^t K_H^\sigma(t,s)dZ_s \tag{3. 7}$$

where the function K_H^σ is as defined by (2.11). Let P_θ^T be the measure induced by the process $\{X_t, 0 \leq t \leq T\}$ when θ is the true parameter. Following Theorem 2.3, we get that the Radon-Nikodym derivative of P_θ^T with respect to P_0^T is given by

$$\frac{dP_\theta^T}{dP_0^T} = \exp[-\int_0^T Q_{H,\theta}(s)dZ_s + \frac{1}{2}\int_0^T Q_{H,\theta}^2(s)dw_s^H]. \tag{3. 8}$$

Maximum likelihood estimation

We now consider the problem of estimation of the parameter θ based on the observation of the process $X = \{X_t, 0 \leq t \leq T\}$ and study its asymptotic properties as $T \to \infty$.

Strong consistency:

Let $L_T(\theta)$ denote the Radon-Nikodym derivative $\frac{dP_\theta^T}{dP_0^T}$. The maximum likelihood estimator (MLE) is defined by the relation

$$L_T(\hat{\theta}_T) = \sup_{\theta \in \Theta} L_T(\theta). \tag{3. 9}$$

We assume that there exists a measurable maximum likelihood estimator. Sufficient conditions can be given for the existence of such an estimator (cf. Lemma 3.1.2, Prakasa Rao (1987)).

Note that

$$\begin{aligned} Q_{H,\theta}(t) &= \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{C(\theta,s)}{\sigma(s)}ds \\ &= \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{a(s,X(s))}{\sigma(s)}ds + \theta\frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{b(s,X(s))}{\sigma(s)}ds \\ &= J_1(t) + \theta J_2(t).(say) \end{aligned} \tag{3. 10}$$

Then

$$\log L_T(\theta) = -\int_0^T (J_1(t)+\theta J_2(t))dZ_t + \frac{1}{2}\int_0^T (J_1(t)+\theta J_2(t))^2 dw_t^H \tag{3. 11}$$

and the likelihood equation is given by

$$-\int_0^T J_2(t)dZ_t + \int_0^T (J_1(t)+\theta J_2(t))J_2(t)dw_t^H = 0. \tag{3. 12}$$

Hence the MLE $\hat{\theta}_T$ of θ is given by

$$\hat{\theta}_T = \frac{\int_0^T J_2(t)dZ_t + \int_0^T J_1(t)J_2(t)dw_t^H}{\int_0^T J_2^2(t)dw_t^H}. \tag{3. 13}$$

Let θ_0 be the true parameter. Using the fact that

$$dZ_t = (J_1(t)+\theta_0 J_2(t))dw_t^H + dM_t^H, \tag{3. 14}$$

it can be shown that

$$\frac{dP_\theta^T}{dP_{\theta_0}^T} = \exp[(\theta_0-\theta)\int_0^T J_2(t)dM_t^H - \frac{1}{2}(\theta_0-\theta)^2\int_0^T J_2^2(t)dw_t^H. \tag{3. 15}$$

Following this representation of the Radon-Nikodym Derivative, we obtain that

$$\hat{\theta}_T - \theta_0 = \frac{\int_0^T J_2(t)dM_t^H}{\int_0^T J_2^2(t)dw_t^H}. \tag{3. 16}$$

Note that the quadratic variation $< Z >$ of the process Z is the same as the quadratic variation $< M^H >$ of the martingale M^H which in turn is equal to w^H. This follows from the relations (2.15) and (2.9). Hence we obtain that

$$[w_T^H]^{-1}\lim_n \Sigma[Z_{t_{i+1}^{(n)}} - Z_{t_i^{(n)}}]^2 = 1 a.s[P_{\theta_0}]$$

where $(t_i^{(n)}$ is a partition of the interval $[0,T]$ such that $\sup|t_{i+1}^{(n)} - t_i^{(n)}|$ tends to zero as $n\to\infty$. If the function $\sigma(t)$ is an unknown constant σ, the above property can be used to obtain a strongly consistent estimator of σ^2 based on the continuous observation of the process X over the interval $[0,T]$. Here after we assume that the nonrandom function $\sigma(t)$ is known.

We now discuss the probelem of estimation of the parameter θ on the basis of the observation of the process X or equivalently the process Z on the interval $[0,T]$.

Theorem 3.1: The maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent, that is,

$$\hat{\theta}_T \to \theta_0 \ \text{ a.s } [P_{\theta_0}] \text{ as } T\to\infty \tag{3. 17}$$

provided

$$\int_0^T J_2^2(t)dw_t^H \to \infty \ \text{ a.s } [P_{\theta_0}] \text{ as } T\to\infty. \tag{3. 18}$$

Proof: This theorem follows by observing that the process

$$R_t \equiv \int_0^T J_2(t)dM_t^H, t\geq 0 \tag{3. 19}$$

is a local martingale with the quadratic variation process

$$< R_T >= \int_0^T J_2^2(t)dw_t^H \tag{3. 20}$$

and applying the Strong law of large numbers (cf. Liptser (1980); Prakasa Rao(1999b), p. 61) under the condition (30) stated above.

Remark: For the case fractional Ornstein-Uhlenbeck process investigated in Kleptsyna and Le Breton (2002), it can be checked that the condition stated in equation (3.18) holds and hence the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T\to\infty$.

Limiting distribution:

We now discuss the limiting distribution of the MLE $\hat{\theta}_T$ as $T \to \infty$.

Theorem 3.2: Assume that the functions $b(t,s)$ and $\sigma(t)$ are such that the process $\{R_t, t \geq 0\}$ is a local continuous martingale and that there exists a norming function $I_t, t \geq 0$ such that

$$I_T^2 < R_T >= I_T^2 \int_0^T J_2^2(t) dw_t^H \to \eta^2 \text{ in probability as } T \to \infty \tag{3. 21}$$

where $I_T \to 0$ as $T \to \infty$ and η is a random variable such that $P(\eta > 0) = 1$. Then

$$(I_T R_T, I_T^2 < R_T >) \to (\eta Z, \eta^2) \text{ in law as } T \to \infty \tag{3. 22}$$

where the random variable Z has the standard normal distribution and the random variables Z and η are independent.

Proof: This theorem follows as a consequence of the central limit theorem for martingales (cf. Theorem 1.49 ; Remark 1.47 , Prakasa Rao(1999b), p. 65).

Observe that

$$I_T^{-1}(\hat{\theta}_T - \theta_0) = \frac{I_T R_T}{I_T^2 < R_T >} \tag{3. 23}$$

Applying the Theorem 3.2, we obtain the following result.

Theorem 3.3: Suppose the conditions stated in the Theorem 3.2 hold. Then

$$I_T^{-1}(\hat{\theta}_T - \theta_0) \to \frac{Z}{\eta} \text{ in law as } t \to \infty \tag{3. 24}$$

where the random variable Z has the standard normal distribution and the random variables Z and η are independent.

Remarks: If the random variable η is a constant with probability one, then the limiting distribution of the maximum likelihood estimator is normal with mean 0 and variance η^{-2}. Otherwise it is a mixture of the normal distributions with mean zero and variance η^{-2} with the mixing distribution as that of η.

Bayes estimation

Suppose that the parameter space Θ is open and Λ is a prior probability measure on the parameter space Θ. Further suppose that Λ has the density $\lambda(.)$ with respect to the Lebesgue measure and the density function is continuous and positive in an open neighbourhood of θ_0, the true parameter. Let

$$\alpha_T \equiv I_T R_T = I_T \int_0^T J_2(t) dM_t^H \tag{3. 25}$$

and

$$\beta_T \equiv I_T^2 < R_T >= I_T^2 \int_0^T J_2^2(t) dw_t^H. \tag{3. 26}$$

We have seen earlier that the maximum likelihood estimator satisfies the relation

$$\alpha_T = (\hat{\theta}_T - \theta_0) I_T^{-1} \beta_T. \tag{3. 27}$$

The posterior density of θ given the observation $X^T \equiv \{X_s, 0 \leq s \leq T\}$ is given by

$$p(\theta|X^T) = \frac{\frac{dP_\theta^T}{dP_{\theta_0}^T}\lambda(\theta)}{\int_\Theta \frac{dP_\theta^T}{dP_{\theta_0}^T}\lambda(\theta)d\theta}. \tag{3. 28}$$

Let us write $t = I_T^{-1}(\theta - \hat{\theta}_T)$ and define

$$p^*(t|X^T) = I_T \;\; p(\hat{\theta}_T + tI_T|X^T). \tag{3. 29}$$

Then the function $p^*(t|X^T)$ is the posterior density of the transformed variable $t = I_T^{-1}(\theta - \hat{\theta}_T)$. Let

$$\begin{aligned} \nu_T(t) \;\; &\equiv \;\; \frac{dP_{\hat{\theta}_T + tI_T}/dP_{\theta_0}}{dP_{\hat{\theta}_T}/dP_{\theta_0}} \\ &= \;\; \frac{dP_{\hat{\theta}_T + tI_T}}{dP_{\hat{\theta}_T}} a.s. \end{aligned} \tag{3. 30}$$

and

$$C_T = \int_{-\infty}^{\infty} \nu_T(t)\lambda(\hat{\theta}_T + tI_T)dt. \tag{3. 31}$$

It can be checked that

$$p^*(t|X^T) = C_T^{-1}\nu_T(t)\lambda(\hat{\theta}_T + tI_T). \tag{3. 32}$$

Further more, the equations (3.15) and (3.27)-(3.32) imply that

$$\begin{aligned} \log \nu_T(t) \;\; &= \;\; I_T^{-1}\alpha_T[(\hat{\theta}_T + tI_T - \theta_0) - (\hat{\theta}_T - \theta_0)] \\ &\quad -\frac{1}{2}I_T^{-2}\beta_T[(\hat{\theta}_T + tI_T - \theta_0)^2 - (\hat{\theta}_T - \theta_0)^2] \\ &= \;\; t\alpha_T - \frac{1}{2}t^2\beta_T - t\beta_T I_T^{-1}(\hat{\theta}_T - \theta_0) \\ &= \;\; -\frac{1}{2}\beta_T t^2 \end{aligned} \tag{3. 33}$$

in view of equation (3.27).

Suppose that the convergence in the condition in the equation (3.21) holds almost surely under the measure P_{θ_0} and the limit is a constant $\eta^2 > 0$ with probability one. For convenience, we write $\beta = \eta^2$. Then

$$\beta_T \to \beta \;\; \text{a.s} \;\; [P_{\theta_0}] \;\; \text{as} \;\; T \to \infty. \tag{3. 34}$$

Then it is obvious that

$$\lim_{T \to \infty} \nu_T(t) = \exp[-\frac{1}{2}\beta t^2] \;\; \text{a.s.} \;\; [P_{\theta_0}] \tag{3. 35}$$

and for any $0 < \varepsilon < \beta$,

$$\log \nu_T(t) \le -\frac{1}{2}t^2(\beta - \varepsilon) \tag{3. 36}$$

for every t for T sufficiently large. Further more , for every $\delta > 0$, there exists $\varepsilon' > 0$ such that

$$\sup_{|t|>\delta I_T^{-1}} \nu_T(t) \le \exp[-\frac{1}{4}\varepsilon' I_T^{-2}] \tag{3. 37}$$

for T sufficiently large.

Suppose that $H(t)$ is a nonnegative measurable function such that, for some $0 < \varepsilon < \beta$,

$$\int_{-\infty}^{\infty} H(t) \exp[-\frac{1}{2}t^2(\beta - \varepsilon)]dt < \infty. \tag{3. 38}$$

Suppose the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent, that is,

$$\hat{\theta}_T \to \theta_0 \ \text{ a.s } \quad [P_{\theta_0}] \text{ as } \ T \to \infty. \tag{3. 39}$$

For any $\delta > 0$, consider

$$\begin{aligned}
&\int_{|t|\le\delta I_T^{-1}} H(t)|\nu_T(t)\lambda(\hat{\theta}_T + tI_T) - \lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2)|dt \\
&\le \int_{|t|\le\delta I_T^{-1}} H(t)\lambda(\theta_0)|\nu_T(t) - \exp(-\frac{1}{2}\beta t^2)|dt \\
&+ \int_{|t|\le\delta I_T^{-1}} H(t)\nu_T(t)|\lambda(\theta_0) - \lambda(\hat{\theta}_T + tI_T)|dt \\
&= A_T + B_T(say).
\end{aligned} \tag{3. 40}$$

It is clear that, for any $\delta > 0$,

$$A_T \to 0 \ \text{ a.s } \ [P_{\theta_0}] \ \text{ as } \ T \to \infty \tag{3. 41}$$

by the dominated convergence theorem in view of the inequality in (3.36), the equation (3.35) and the condition in the equation (3.38). On the other hand, for T sufficiently large,

$$0 \le B_T \le \sup_{|\theta-\theta_0|\le\delta} |\lambda(\theta) - \lambda(\theta_0)| \int_{|t|\le\delta I_T^{-1}} H(t)\exp[-\frac{1}{2}t^2(\beta - \varepsilon)]dt \tag{3. 42}$$

since $\hat{\theta}_T$ is strongly consistent and $I_T^{-1} \to \infty$ as $T \to \infty$. The last term on the right side of the above inequality can be made smaller than any given $\rho > 0$ by choosing δ sufficiently small in view of the continuity of $\lambda(.)$ at θ_0. Combining these remarks with the equations (3.41) and (3.42), we obtain the following lemma.

Lemma 3.3: Suppose the conditions (3.34), (3.38) and (3.39) hold. Then there exists $\delta > 0$ such that

$$\lim_{T\to\infty} \int_{|t|\le\delta I_T^{-1}} H(t)|\nu_T(t)\lambda(\hat{\theta}_T + tI_T) - \lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2)|dt = 0. \tag{3. 43}$$

For any $\delta > 0$, consider

$$\begin{aligned}(3.\ 44)\qquad & \int_{|t|>\delta I_T^{-1}} H(t)|\nu_T(t)\lambda(\hat{\theta}_T + tI_T) - \lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2)|dt \\ & \le \int_{|t|>\delta I_T^{-1}} H(t)\nu_T(t)\lambda(\hat{\theta}_T + tI_T)dt \\ & \quad + \int_{|t|>\delta I_T^{-1}} H(t)\lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2)dt \\ & \le \exp[-\frac{1}{4}\varepsilon' I_T^{-2}] \int_{|t|>\delta I_T^{-1}} H(t)\lambda(\hat{\theta}_T + tI_T)dt \\ & \quad + \lambda(\theta_0)\int_{|t|>\delta I_T^{-1}} H(t)\exp(-\frac{1}{2}\beta t^2)dt \\ & = U_T + V_T (say).\end{aligned}$$

Suppose the following condition holds for every $\varepsilon > 0$ and $\delta > 0$:

$$(3.\ 45)\qquad \exp[-\varepsilon I_T^{-2}]\int_{|u|>\delta} H(uI_T^{-1})\lambda(\hat{\theta}_T + u)du \to 0 \ \text{ a.s.}[P_{\theta_0}] \ \text{ as } \ T \to \infty.$$

It is clear that, for every $\delta > 0$,

$$(3.\ 46)\qquad V_T \to 0 \ \text{ as } \ T \to \infty$$

in view of the condition stated in (3.38) and the fact that $I_T \to \infty$ a.s. $[P_{\theta_0}]$ as $T \to \infty$. The condition stated in (3.45) implies that

$$(3.\ 47)\qquad U_T \to 0 \ \text{ a.s } [P_{\theta_0}] \ \text{ as } \ T \to \infty$$

for every $\delta > 0$. Hence we have the following lemma.

Lemma 3.4: Suppose that the conditions (3.34), (3.38) and (3.39) hold. Then for every $\delta > 0$,

$$(3.\ 48)\qquad \lim_{T\to\infty}\int_{|t|>\delta I_T^{-1}} H(t)|\nu_T(t)\lambda(\hat{\theta}_T + tI_T) - \lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2))|dt = 0.$$

Lemmas 3.3 and 3.4 together prove that

$$(3.\ 49)\qquad \lim_{T\to\infty}\int_{|t|>\delta I_T^{-1}} H(t)|\nu_T(t)\lambda(\hat{\theta}_T + tI_T) - \lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2)|dt = 0.$$

Let $H(t) \equiv 1$. It follows that

$$C_T \equiv \int_{-\infty}^{\infty} \nu_T(t)\lambda(\hat{\theta}_T + tI_T)dt.$$

Relation (3.49) implies that

$$(3.\ 50)\qquad C_T \to \lambda(\theta_0)\int_{-ity}^{\infty} \exp(-\frac{1}{2}\beta t^2)dt = \lambda_{\theta_0}(\frac{\beta}{2\pi})^{-1/2} \ \text{ a.s}[P_{\theta_0}]$$

as $T \to \infty$. Further more

$$\begin{aligned}(3.\ 51) \qquad & \int_{-\infty}^{\infty} H(t)|p^*(t|X^T) - (\frac{\beta}{2\pi})^{1/2}\exp(-\frac{1}{2}\beta t^2)|dt \\ & \leq \int_{-\infty}^{\infty} H(t)|\nu_T(t)\lambda(\hat{\theta}_T + tI_T) - \lambda(\theta_0)\exp(-\frac{1}{2}\beta t^2)|dt \\ & + \int_{-\infty}^{\infty} H(t)|C_T^{-1}\lambda(\theta_0) - (\frac{\beta}{2\pi})^{1/2}|\exp(-\frac{1}{2}\beta t^2)dt.\end{aligned}$$

The last two terms tend to zero almost surely $[P_{\theta_0}]$ by the equations (3.49) and (3.50). Hence we have the follwing theorem which is an analogue of the Bernstein - von Mises theorem proved in Prakasa Rao (1981) for a class of processes satisfying a linear stochastic differential equation driven by the standard Wiener process.

Theorem 3.5: Let the assumptions (3.34),(3.38),(3.39) and (3.45) hold where $\lambda(.)$ is a prior density which is continuous and positive in an open neighbourhood of θ_0, the true parameter. Then

$$(3.\ 52) \qquad \lim_{T\to\infty}\int_{-\infty}^{\infty} H(t)|p^*(t|X^T) - (\frac{\beta}{2\pi})^{1/2}\exp(-\frac{1}{2}\beta t^2)|dt = 0 \text{ a.s } [P_{\theta_0}].$$

As a consequence of the above theorem, we obtain the following result by choosing $H(t) = |t|^m$, for integer $m \geq 0$.

Theorem 3.6: Assume that the following conditions hold:

$$(3.\ 53) \qquad (C1)\ \ \hat{\theta}_T \to \theta_0 \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty,$$

$$(3.\ 54) \qquad (C2)\ \ \beta_T \to \beta > 0 \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty.$$

Further suppose that

(C3)$\lambda(.)$ is a prior probability density on Θ which is continuous and positive in an open neighbourhood of θ_0, the true parameter and

$$(3.\ 55) \qquad (C4)\ \ \int_{-\infty}^{\infty} |\theta|^m \lambda(\theta)d\theta < \infty$$

for some integer $m \geq 0$. Then

$$(3.\ 56) \qquad \lim_{T\to\infty}\int_{-\infty}^{\infty} |t|^m|p^*(t|X^T) - (\frac{\beta}{2\pi})^{1/2}\exp(-\frac{1}{2}\beta t^2)|dt = 0 \text{ a.s } [P_{\theta_0}].$$

In particular, choosing $m = 0$, we obtain that

$$(3.\ 57) \qquad \lim_{T\to\infty}\int_{-\infty}^{\infty} |p^*(t|X^T) - (\frac{\beta}{2\pi})^{1/2}\exp(-\frac{1}{2}\beta t^2)|dt = 0 \text{ a.s } [P_{\theta_0}]$$

whenver the conditions (C1), (C2) and (C3) hold. This is the analogue of the Bernstein-von Mises theorem for a class of diffusion processes proved in Prakasa Rao (1981) and it shows the asymptotic convergence in L_1-mean of the posterior density to the normal distribution.

As a Corollory to the Theorem 3.6, we also obtain that the conditional expectation, under P_{θ_0}, of $[I_T^{-1}(\hat{\theta}_T - \theta)]^m$ converges to the corresponding m-th abosolute moment of the normal distribution with mean zero and variance β^{-1}.

We define a *regular Bayes estimator* of θ, corresponding to a prior probability density $\lambda(\theta)$ and the loss function $L(\theta,\phi)$, based on the observation X^T, as an estimator which minimizes the posterior risk

$$B_T(\phi) \equiv \int_{-ity}^{\infty} L(\theta,\phi)p(\theta|X^T)d\theta. \tag{3. 58}$$

over all the estimators ϕ of θ. Here $L(\theta,\phi)$ is a loss function defined on $\Theta \times \Theta$.

Suppose there exists a measurable regular Bayes estimator $\tilde{\theta}_T$ for the parameter θ (cf. Theorem 3.1.3, Prakasa Rao (1987).) Suppose that the loss function $L(\theta,\phi)$ satisfies the following conditions:

$$L(\theta,\phi) = \ell(|\theta-\phi|) \geq 0 \tag{3. 59}$$

and the function $\ell(t)$ is nondecreasing for $t \geq 0$. An example of such a loss function is $L(\theta,\phi) = |\theta - \phi|$. Suppose there exist nonnegative functions $R(t)$, $K(t)$ and $G(t)$ such that

$$(D1)\;\; R(t)\ell(tI_T) \leq G(t) \;\;\text{for all}\;\; T \geq 0, \tag{3. 60}$$

$$(D2)\;\; R(t)\ell(tI_T) \to K(t) \;\;\text{as}\;\; T \to \infty \tag{3. 61}$$

uniformly on bounded intervals of t. Further suppose that the function

$$(D3)\;\; \int_{-\infty}^{\infty} K(t+h)\exp[-\frac{1}{2}\beta t^2]dt \tag{3. 62}$$

has a strict minimum at $h = 0$, and
(D4)the function $G(t)$ satisfies the conditions similar to (3.38) and (3.45).

We have the following result giving the asymptotic properties of the Bayes risk of the estimator $\tilde{\theta}_T$.

Thoeren 3.7: Suppose the conditions (C1) to (C3) in the Theorem 3.6 and the conditions (D1) to (D4) stated above hold. Then

$$I_T^{-1}(\tilde{\theta}_T - \hat{\theta}_T) \to 0 \;\;\text{a.s}\;\; [P_{\theta_0}] \;\;\text{as}\;\; T \to \infty \tag{3. 63}$$

and

$$\lim_{T\to\infty} R(T)B_T(\tilde{\theta}_T) = \lim_{T\to\infty} R(T)B_T(\hat{\theta}_T) \tag{3. 64}$$
$$= (\frac{\beta}{2\pi})^{1/2}\int_{-\infty}^{\infty} K(t)\exp[-\frac{1}{2}\beta t^2]dt \;\;\text{a.s}\;\; [P_{\theta_0}]$$

We omit the proof of this theorem as it is similar to the proof of Theorem 4.1 in Borwanker et al. (1971).

We have observed earlier that

$$I_T^{-1}(\hat{\theta}_T - \theta_0) \to N(0, \beta^{-1}) \text{ in law as } T \to \infty. \tag{3. 65}$$

As a consequence of the Theorem 3.7, we obtain that

$$\tilde{\theta}_T \to \theta_0 \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty \tag{3. 66}$$

and

$$I_T^{-1}(\tilde{\theta}_T - \theta_0) \to N(0, \beta^{-1}) \text{ in law as } T \to \infty. \tag{3. 67}$$

In other words, the Bayes estimator is asymptotically normal and has asymptotically the same distribution as the maxiumum likelihood estimator. The asymptotic Bates risk of the estimator is given by the Theorem 3.7.

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