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February 21, 2003
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# Convergence of Lower Records and Infinite divisiblity

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# Convergence of Lower Records and Infinite divisiblity

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**Abstract**

We study the properties of sums of lower records from a distribution on $[0, \infty)$ which either is continuous, except possibly at the origin or has support contained in the set of non-negative integers. We find necessary and sufficient condition for the partial sums of lower records to converge almost surely to a proper random variable. Explicit formula for the Laplace transform of the limit is derived. This limit is infinite divisible and we show that all infinitely divisible random variables with continuous Levy measure on $[0, \infty)$ originate as infinite sums of lower records.

**Keywords:** Lower records, geometric distribution, infinite divisibility, asymptotic distributions, Levy measure, self-decomposable, generalised gamma convolutions.

**AMS 2000 Subject Classification:** 60E07, 60E10, 62E15, 62E20.

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# 1 Introduction

Let $F$ be any distribution function on $[0, \infty)$. Let $\{Z_n : n \geq 0\}$ be a sequence of independent and identically distributed (i.i.d.) observations from $F$. We say $Z_j$ is a *lower record value* if

$$Z_j \leq \min\{Z_1, Z_2, \ldots, Z_{j-1}\}.$$

By convention, $Z_0$ is a record. (The *upper records* may be defined likewise by reversing the inequality above). We note here that this definition is slightly different from the usual definition where the above is a strict inequality. However, when $F$ is continuous, the two definitions are equivalent. Further, if $F$ is continuous on $(0, \infty)$ and $F(0) > 0$, our lower records will be strictly decreasing, until it hits 0. Thereafter, with our definition, all subsequent records will assume value 0. While with the usual definition, there can be no new record. Since we are interested in the infinite sum of records, both the concepts yield same results.

Define $L_0 := 1$ and for $n \geq 1$,

$$L_n := \min\{j > L_{n-1} : Z_j \leq Z_{L_{n-1}}\}.$$

Define, $X_n = Z_{L_n}$. Thus, $\{X_n : n \geq 0\}$ is the sequence of lower records of i.i.d. observations from the distribution $F$. Note that $\mathbb{P}(x, du) = 1_{\{u \leq x\}} F(du)/F(x)$ defines a transition function. It may be observed that $\{X_n : n \geq 0\}$ is a Markov process with this transition function and the initial distribution $F$.

The interest in asymptotic behaviour of record statistics can be traced back to Gnedenko [5]. Later a thorough investigation was made by Resnick [8] who derived the class of all possible limit distributions of the $n$th upper record statistic as $n \to \infty$. His work has been followed by many others. In a recent work, Arnold and Villaseñor [1] derived some asymptotic properties of partial sums of the first $n$ upper records. Bose et. al. [2] in a subsequent article also dealt with sums of upper records and settled some of the questions raised in Arnold and Villaseñor (1999). In this paper we study the properties of sums of lower records.

In Section 2, $F$ is assumed to be continuous on $(0, \infty)$. In this case we obtain a necessary and sufficient condition for almost sure convergence of the sum of the records. The limit turns out to be infinitely divisible. We obtain an explicit relation between the Levy measure of the limit and the parent distribution $F$. This has several interesting consequences. First, we can show how specific classes of infinitely divisible distributions such as the self decomposable class, the class of generalised gamma convolutions etc arise from different classes of $F$. Further, any infinitely divisible distribution on $[0, \infty)$, under a mild restriction, arises as a limit of the sums of lower records from a suitable $F$. It thus gives a method of simulating observations from a given i.d. distribution which is known only through its Levy measure. We are also able to derive conditions on $F$ which guarantee absolutely continuity of the limit with respect to the Lebesgue measure.

In Section 3, the underlying distribution $F$ is assumed to be discrete with support in $\mathbb{N}_0$ and results similar to those in Section 2 are derived.

## 2 The continuous case

In this section we assume that $F$ is continuous on $(0, \infty)$. Note that 0 is the only possible point of discontinuity of $F$. The main results are Theorem 1 and Theorem 2 in Section 2.1. They provide necessary and sufficient conditions for the finiteness of $\sum_{n=0}^{\infty} X_n$ and establishes the Laplace transform of the limit. Further properties of the limit, including its connection to the class of i.d. distributions are given in Section 2.2.

### 2.1 Convergence Results

Obviously, $X_n$ decreases as $n$ increases, so has a limit, almost surely, which can be easily shown to be degenerate at the infimum of the support of $F$. This is stated in the following Proposition. Its proof is a rather simple consequence of the Borel-Cantelli lemma.

**Proposition 2.1** *Let $c_0 = \inf\{u : u \in \text{supp}(F)\}$. Then as $n \to \infty$, $X_n \downarrow c_0$ almost surely.*

We are interested in the convergence of infinite sum of the sequence of records $\{X_n : n \geq 0\}$. Therefore, we shall require in the sequel that $c_0 = 0$. Define $T_n = \sum_{k=0}^{n} X_k$. Since $X_n \geq 0$, for all $n \geq 0$, $T_n$ is non-decreasing as $n$ increases.

We now derive a necessary and sufficient condition for $T_n \to T$ such that $T < \infty$ almost surely. If $F(0) > 0$, define $N := \min\{n \geq 0 : X_n = 0\}$. Then, $N$ will be dominated by a geometric random variable with probability of success being $F(0)$. Thus, $N$ is finite almost surely. Therefore, the infinite sum becomes a finite sum and hence $T_n \to T$ where $T < \infty$ almost surely. Therefore, the more interesting case is when $F(0) = 0$ but $F(x) > 0$ for all $x > 0$. *Henceforth, we assume this.*

First, we note that since $T_n$ is non-decreasing in $n$, it must have a limit, albeit $\infty$. Therefore, $T_n \to T$ almost surely where $T$ may take value $+\infty$ with positive probability. If we show that $T_n \Rightarrow V_F$ where $V_F < \infty$ almost surely, we will have that $T \stackrel{d}{=} V_F$ and therefore $T < \infty$ almost surely.

For fixed $t > 0$, let us define for $x > 0$,

$$\psi_t^{(n)}(x) = \mathbb{E}(\exp(-tT_n) \mid X_0 \leq x) = \mathbb{E}(\exp(-t\sum_{k=0}^{n} X_k) \mid X_0 \leq x).$$

We first show that $\psi_t^{(n)}(x)$ is decreasing in $x$ for $x > 0$ for fixed $n \geq 0$. First, note that for $x_1 > x_2 > 0$, we have,

$$\begin{aligned}
& F(x_1)F(x_2)\left[\psi_t^{(0)}(x_1) - \psi_t^{(0)}(x_2)\right] \\
= \; & \left[F(x_2)\int_0^{x_1} \exp(-tu)F(du) - F(x_1)\int_0^{x_2} \exp(-tu)F(du)\right] \\
= \; & \left[F(x_2)\int_{x_2}^{x_1} \exp(-tu)F(du) - (F(x_1) - F(x_2))\int_0^{x_2} \exp(-tu)F(du)\right] \\
\leq \; & \left[F(x_2)\exp(-tx_2)\int_{x_2}^{x_1} F(du) - (F(x_1) - F(x_2))\exp(-tx_2)\int_0^{x_2} F(du)\right] \\
\leq \; & 0.
\end{aligned}$$

Thus, $\psi_t^{(0)}$ is decreasing. To apply induction, assume that $\psi_t^{(n)}$ is decreasing. Then, we have for $x_1 > x_2 > 0$,

$$\begin{aligned}
& F(x_1)F(x_2)\left[\psi_t^{(n+1)}(x_1) - \psi_t^{(n+1)}(x_2)\right] \\
= \; & \left[F(x_2)\int_0^{x_1} \exp(-tu)\psi_t^{(n)}(u)F(du) - F(x_1)\int_0^{x_2} \exp(-tu)\psi_t^{(n)}(u)F(du)\right] \\
= \; & \Big[F(x_2)\int_{x_2}^{x_1} \exp(-tu)\psi_t^{(n)}(u)F(du) \\
& \quad -(F(x_1) - F(x_2))\int_0^{x_2} \exp(-tu)\psi_t^{(n)}(u)F(du)\Big] \\
\leq \; & \Big[F(x_2)\exp(-tx_2)\psi_t^{(n)}(x_2)\int_{x_2}^{x_1} F(du) \\
& \quad -(F(x_1) - F(x_2))\exp(-tx_2)\psi_t^{(n)}(x_2)\int_0^{x_2} F(du)\Big] \\
\leq \; & 0.
\end{aligned}$$

Hence, $\psi_t^{(n)}$ is a decreasing function of $x$ for each $n$.

Since $X_n \geq 0$, we have that for every fixed $x$, $\psi_t^{(n)}(x)$ is decreasing in $n$. Hence, for fixed $x > 0$,

$$\psi_t(x) = \lim_{n\to\infty} \psi_t^{(n)}(x)$$

exists and lies between 0 and 1. Further, since each $\psi_t^{(n)}$ is decreasing in $x$, $\psi_t(x)$ is also a decreasing function of $x$. Define

$$\psi_t(\infty) = \lim_{x\uparrow\infty} \psi_t(x).$$

Next, we claim that

$$\lim_{n\to\infty} \mathbb{E}(\exp(-tT_n)) = \psi_t(\infty).$$

To prove this, fix any $\epsilon > 0$ and choose $y$ so large that $F(y) > 1-\epsilon$ and $|\psi_t(\infty) - \psi_t(y)| < \epsilon$. Then, we have,

$$\begin{aligned} & |\mathbb{E}(\exp(-tT_n)) - \psi_t(\infty)| \\ \leq \; & |\mathbb{E}(\exp(-tT_n)1_{\{X_0 \leq y\}}) - \psi_t(y)| + \mathbb{P}(X_0 > y) + |\psi_t(\infty) - \psi_t(y)| \\ \leq \; & F(y)|\psi_t^{(n)}(y) - \psi_t(y)| + \psi_t(y)|1 - F(y)| + 2\epsilon. \end{aligned}$$

Now, for this fixed $y$, we choose $N$ so large that $|\psi_t^{(n)}(y) - \psi_t(y)| \leq \epsilon$ for all $n \geq N$. Therefore, for all $n \geq N$, we have, $|\mathbb{E}(\exp(-tT_n)) - \psi_t(\infty)| \leq 4\epsilon$, proving the claim.

Now, conditioning on $X_0$,

$$\begin{aligned} \psi_t^{(n)}(x) &= \int_0^x \mathbb{E}\big(\exp(-t\sum_{j=0}^n X_j) \mid X_0 = u\big) F(du)/F(x) \\ &= \int_0^x \mathbb{E}\big(\exp(-tu - t\sum_{j=1}^n X_j) \mid X_1 \leq u\big) F(du)/F(x) \\ &= \int_0^x \exp(-tu)\psi_t^{(n-1)}(u) F(du)/F(x). \end{aligned} \tag{1}$$

Letting $n \to \infty$ and applying dominated convergence theorem on the right side, we see that $\psi_t$ satisfies the integral equation,

$$F(x)\xi(x) = \int_0^x \exp(-tu)\xi(u)F(du). \tag{2}$$

Since $F$ is continuous on $(0,\infty)$ and $\psi_t$ is bounded, from the above equation it easily follows that $\psi_t$ is continuous on $(0,\infty)$. Further, by setting

$$\psi_t(0) := \lim_{x \downarrow 0} \psi_t(x),$$

we see that $\psi_t$ is a continuous function on $[0,\infty)$ taking values in $[0,1]$. Henceforth, when we talk about solutions to this and similar integral equations, we shall always restrict to *bounded continuous* solutions.

Now, we are in a position to establish the following result.

**Theorem 1** *Suppose that $F(0) = 0$, $F(x) > 0$ for $x > 0$ and $F$ is continuous on $(0,\infty)$ and*

$$\int_0^1 \frac{uF(du)}{F(u)} < \infty. \tag{3}$$

*Then*

$$\psi_t(x) = \exp\Big(\int_0^x \frac{\big(\exp(-tu) - 1\big)F(du)}{F(u)}\Big).$$

*Further, $T_n \Rightarrow V_F$ where $V_F$ is a non-negative random variable whose Laplace transform for all $t > 0$ is given by*

$$\phi_F(-t) = \mathbb{E}(\exp(-tV_F)) = \exp\Big(\int_0^\infty \frac{\big(\exp(-tu) - 1\big)F(du)}{F(u)}\Big) \tag{4}$$

**Proof**: To prove the result, it suffices to show that:

(i) $\psi_t(\infty)$ is given by the above equation (4),

(ii) as a function of $t > 0$, $\psi_t(\infty)$ is the Laplace transform of a non-negative random variable whose distribution is proper; that is, without mass at infinity.

We first assume (i) and show (ii). By Bondesson [3], page 8 for instance, it suffices to show that:

(a) $\psi_t(\infty)$ is completely monotone, that is, $(-1)^n \frac{d^n}{dt^n}\psi_t(\infty) \geq 0$ for $n \geq 1$,

(b) $\lim_{t\to 0}\psi_t(\infty) = 1$.

To show (a), differentiating under the integral, we have

$$\frac{d}{dt}\psi_t(\infty) = -\psi_t(\infty)\int_0^\infty \frac{\exp(-tu)uF(du)}{F(u)}.$$

Note that this is permissible when the integral $\int_0^\infty F(du)\exp(-tu)u/F(u) < \infty$. Since, $\exp(-tu)u \to 0$ as $u \to \infty$, we can say that $\exp(-tu)u \leq C_1 \min(u, 1)$ for some $C_1 > 0$. Thus, using condition (3), we have $\int_0^\infty F(du)\exp(-tu)u/F(u) < \infty$.

Now, consider the function $s(t) = \int_0^\infty \exp(-tu)uF(du)/F(u)$. It is easy to see that the function $s(t)$ is infinitely differentiable using similar arguments. Further, we have for all $n \geq 0$

$$(-1)^n \frac{d^n}{dt^n}s(t) = \int_0^\infty \frac{\exp(-tu)u^{n+1}F(du)}{F(u)} \geq 0.$$

Now assume that $(-1)^k \frac{d^k}{dt^k}\psi_t(\infty) \geq 0$ for all $k = 1, 2, \ldots, n$. Therefore,

$$\begin{aligned}
&(-1)^{n+1}\frac{d^{n+1}}{dt^{n+1}}\psi_t(\infty) \\
&= (-1)^n \frac{d^n}{dt^n}\psi_t(\infty)s(t) \\
&= (-1)^n \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dt^k}\psi_t(\infty)\frac{d^{n-k}}{dt^{n-k}}s(t) \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{d^k}{dt^k}\psi_t(\infty)(-1)^{n-k}\frac{d^{n-k}}{dt^{n-k}}s(t) \\
&\geq 0.
\end{aligned}$$

We now show (b). Clearly, as $t \to 0$, we have $(1 - \exp(-tu))/F(u) \to 0$. For $t < 1$, we have $1 - \exp(-tu) \leq \min(1, u)$. From the condition (3), we get that $\int_0^\infty \min(1, u)F(du)/F(u) < \infty$. Therefore, by dominated convergence theorem, we obtain $\psi_t(\infty) \to 1$ as $t \to 0$.

Finally we show (i). Define, for all $x \geq 0$,

$$g(x) := \exp\Big(-\int_0^x \frac{\big(1 - \exp(-tu)\big)F(du)}{F(u)}\Big).$$

Note that to show (i), it is enough to show that $\psi_t = g$. Towards this end, we show that:

(A) all solutions of (2) which are continuous on $[0,\infty)$ and assume values in $[0,1]$, are multiples of $g$,

(B) $\psi_t$ is the *largest* among all such solutions.

It is easier to establish (B). If $\eta$ is any such solution of (2) then we have

$$\begin{aligned}
\psi_t^{(0)}(x) &= \frac{1}{F(x)}\int_0^x \exp(-tu)F(du) \\
&\geq \frac{1}{F(x)}\int_0^x \exp(-tu)\eta(u)F(du) \\
&= \eta(x).
\end{aligned}$$

Using equation (1) and repeating the above arguments, it follows that $\psi_t^{(n)} \geq \eta$ for every $n \geq 0$. Thus, $\psi_t \geq \eta$.

We now establish (A). By using (3) it is easy to see that the non-negative expression $g$ is continuous, decreasing in $x$ and takes values in $[0,1]$. Note that $g(du) = -F(du)g(u)(1 - \exp(-tu))/F(u)$. Now, using the integration by parts formula, we have

$$\begin{aligned}
&\frac{1}{F(x)}\int_0^x \exp(-tu)g(u)F(du) \\
=\ & \frac{1}{F(x)}\Big[\int_0^x F(u)\frac{\big(\exp(-tu)-1\big)g(u)F(du)}{F(u)} + \int_0^x g(u)F(du)\Big] \\
=\ & \frac{1}{F(x)}\Big[\int_0^x F(u)g(du) + \int_0^x g(u)F(du)\Big] \\
=\ & \frac{1}{F(x)}(g(x)F(x) - g(0)F(0)) \\
=\ & g(x)
\end{aligned}$$

since $F(0) = 0$. Thus $g$ is a solution of the equation (2).

Now, we prove that all bounded continuous solutions of (2) are given by constant multiples of $g$. Indeed, it is easy to see that all constant multiples of $g$ are solutions of (2). Conversely, suppose that $\eta$ is a solution of (2). First assume that $\eta(0) = 0$. Now fix any $0 < \epsilon < 1$. Using the continuity of $\eta$ at 0, choose $\delta > 0$ such that $|\eta(x)| < \epsilon||\eta||$ for all $0 \leq x \leq \delta$ where $||\eta|| = \sup\{|\eta(x)| : x \in [0,\infty)\}$. Now, we have for any $x > \delta$,

$$\begin{aligned}
&|\eta(x)| \\
=\ & \frac{1}{F(x)}\left|\int_0^x \exp(-tu)(\eta(u))F(du)\right| \\
\leq\ & \frac{1}{F(x)}\int_0^x \exp(-tu)|\eta(u)|F(du) \\
=\ & \frac{1}{F(x)}\Big[\int_0^\delta \exp(-tu)|\eta(u)|F(du) + \int_\delta^x \exp(-tu)|\eta(u)|F(du)\Big] \\
\leq\ & \frac{1}{F(x)}\big[\epsilon||\eta||F(\delta) + \exp(-t\delta)||\eta||(F(x) - F(\delta))\big] \\
\leq\ & \max(\epsilon, \exp(-t\delta))||\eta||.
\end{aligned}$$

Thus, we have,

$$||\eta|| = \sup\{|\eta(x)| : x \in [0,\infty)\} \leq \max(\epsilon, \exp(-t\delta))||\eta||.$$

Since $\max(\epsilon, \exp(-t\delta)) < 1$, this yields that $||\eta|| = 0$.

Now, assume that $\eta$ is a solution with $\eta(0) \neq 0$, then $\eta - \eta(0)g$ is also a solution of (2) and $(\eta - \eta(0)g)(0) = 0$. Therefore, we must have from the previous argument, $\eta - \eta(0)g \equiv 0$. In other words, $\eta = \eta(0)g$. This establishes the result completely. ∎

**Remark**: So far we have assumed that $F(0) = 0$. Suppose now that $F(0) > 0$. We have already remarked that $T < \infty$ almost surely in this case. Also, note that condition (3) is also satisfied. Denoting, the conditional Laplace transformation of $T$ given that $X_0 \leq x$ by $\psi_t(x)$, we note that $\psi_t(0) = 1$ and for any $x > 0$,

$$\psi_t(x) = \frac{1}{F(x)}\Big[\int_{(0,x]} \exp(-tu)\psi_t(u)F(du) + F(0)\psi_t(0)\Big].$$

Following the same method, we can also solve this integral equation to show that

$$\psi_t(x) = \exp\Big(-\int_{(0,x]} \frac{\big(1-\exp(-tu)\big)F(du)}{F(u)}\Big) = \exp\Big(-\int_0^x \frac{\big(1-\exp(-tu)\big)F(du)}{F(u)}\Big).$$

**Remark**: Let $\xi(t) = E(\exp(itT))$ be the characteristic function of $T$. It is easy now to prove that if (3) holds, then

$$\xi(t) = \exp\Big(\int_0^\infty \frac{\big(\exp(itu)-1\big)F(du)}{F(u)}\Big). \tag{5}$$

Now, we prove the "converse" of Theorem 1.

**Theorem 2** *Suppose that $F(0) = 0, F(x) > 0$ for $x > 0$ and $F$ is continuous on $(0,\infty)$ and*

$$\int_0^1 \frac{uF(du)}{F(u)} = \infty. \tag{6}$$

*Then*

$$T_n \to \infty \text{ as } n \to \infty \text{ almost surely.}$$

**Proof**: As discussed earlier, it is enough to show that $T_n \Rightarrow \infty$. For this it is enough to show that the only bounded continuous solution of the integral equation (2) is zero. We follow arguments similar to that in the previous Theorem and use similar integral equations.

Let $\eta$ be any bounded solution of (2). Now, fix $\delta > 0$ and choose $0 < x_0 < \delta$. Consider the integral equation: for $x > x_0$,

$$\xi(x) = \frac{1}{F(x)}\Big[\int_{x_0}^x \exp(-tu)\xi(u)F(du) + F(x_0)\xi(x_0)\Big] \tag{7}$$

with the boundary condition $\xi(x_0) = \eta(x_0)$. Clearly, $\eta$ restricted to $[x_0,\infty)$ is a solution of (7).

The uniqueness argument in the previous theorem can be easily modified to prove that (7) has a unique solution.

Now consider the continuous function $g_{x_0} : [x_0, \infty) \to [0, 1]$

$$g_{x_0}(x) := \eta(x_0) \exp\Big( - \int_{x_0}^{x} \frac{(1 - \exp(-tu))F(du)}{F(u)} \Big).$$

It is easy, following arguments given before, to verify that $g_{x_0}$ satisfies (7). Therefore, we have $\eta(x) = g_{x_0}(x)$ for all $x > x_0$. Thus, we have,

$$\eta(\delta) = \eta(x_0) \exp\Big( - \int_{x_0}^{\delta} \frac{(1 - \exp(-tu))F(du)}{F(u)} \Big).$$

This is true for every $0 < x_0 < \delta$. Now, let $x_0 \to 0$. Then by condition (6), $\int_{x_0}^{\delta} \frac{(1-\exp(-tu))F(du)}{F(u)} \to \infty$ while $\eta(x_0)$ remains bounded. Hence the right side of the above equation tends to zero. This implies that $\eta(\delta) = 0$. But $\delta$ was arbitrary. Hence $\eta \equiv 0$, proving the theorem. ■

The condition (3) is quite easy to verify for a large class of distributions. Here, we consider a class of examples where the distribution function at $u$ shows a decay as a power of $u$.

**Example 1** *If $F$ admits a density $f$ in a neighbourhood of the origin such that for $u > 0$, $C_2 u^{\gamma-1} \le f(u) \le C_3 u^{\gamma-1}$ where $0 < C_2 \le C_3 < \infty$ and $\gamma > 0$, then $T < \infty$.*

**Proof**: Clearly $F(u) = \int_0^u f(x)dx \ge C_2 u^{\gamma}/\gamma$ for $0 \le u \le \delta$ for some $\delta > 0$. Thus, for any $\beta > 0$, we have

$$\begin{aligned}
\int_0^1 \frac{uF(du)}{F(u)} &= \int_0^{\delta} \frac{uf(u)du}{F(u)} + \int_{\delta}^1 \frac{uF(du)}{F(u)} \\
&\le \int_0^{\delta} \frac{C_3 \gamma u u^{\gamma-1} du}{C_2 u^{\gamma}} + \frac{1}{F(\delta)} \int_{\delta}^1 F(du) \\
&= C_3\gamma/C_2 \int_0^{\delta} du + \frac{F(1) - F(\delta)}{F(\delta)} \\
&= (C_3\gamma/C_2)\delta + \frac{F(1) - F(\delta)}{F(\delta)} < \infty
\end{aligned}$$

This proves the result. ■

In the next example, we consider where the decay of the distribution function near 0 is faster than any polynomial.

**Example 2** *Suppose that $F(x) = \exp(-C_4 x^{-\gamma})$ for $x > 0$ with $F(0) = 0$ where $C_4$ and $\gamma > 0$. Then $T < \infty$ if and only if $\gamma < 1$.*

**Proof**: Clearly $f(u) = C_4\gamma \exp(-C_4 u^{-\gamma}) u^{-\gamma-1}$ for $u > 0$. Therefore, we have,

$$\int_0^1 \frac{uf(u)du}{F(u)} = C_4\gamma \int_0^1 u^{(1-\gamma)-1} du.$$

Clearly, the above integral is finite if and only if $(1 - \gamma) > 0$, i.e., $\gamma < 1$. This proves the result. ■

**Remark** : Monotone transforms will preserve the records. In other words, if $h$ is a continuous, strictly increasing function on $[0, \infty)$ with $h(0) = 0$, then the sequence $\{h(X_n) : n \geq 0\}$ will represent the sequence of records from the distribution $F \circ h^{-1}$. Using this identification, we may obtain that, $\sum_{n=0}^{\infty} h(X_n) < \infty$ almost surely if and only if

$$\int_0^1 \frac{h(u)F(du)}{F(u)} < \infty.$$

## 2.2 Properties of the infinite sum

In this subsection, we will consider the properties of the distribution of $T$. So, we will assume throughout the section that the convergence criteria is satisfied, i.e., $\int_0^1 uF(du)/F(u) < \infty$. From (2.3.1) of Bondesson [3], a random variable taking values in $\mathbb{R}_+$, is infinitely divisible (i.d.) if and only if the Laplace transform can be expressed as

$$m(-t) = \exp\Big(-at + \int_{(0,\infty)} (\exp(-tu) - 1)L(du)\Big)$$

where $t > 0, a \geq 0$ and $L$, the Levy measure, is non-negative and satisfies

$$\int_{(0,\infty)} \min(1, u)L(du) < \infty.$$

Setting $a = 0$ and $L(du) = F(du)/F(u) = \log F(du)$, we can write the Laplace transform of $T$ in (4), in the above form. Thus, we have the following result.

**Proposition 2.2** *The distribution of $T$ is infinitely divisible.*

One question that arises now is the following: what is the class of infinitely divisible distributions which arise in this way? To answer this question, we observe first that for any $x > 0$, the function $h_L : (0, \infty) \to \mathbb{R}_+$ defined by

$$h_L(x) = L([x, \infty)) = \int_x^{\infty} F(du)/F(u) = -\log F(x) \tag{8}$$

is a continuous function in $x$.

Now, suppose that a given infinitely divisible distribution has $L$ as its Levy measure. Then it satisfies the condition that $\int_{(0,\infty)} \min(1, u)L(du) < \infty$. This implies that $L([x, \infty)) < \infty$ for any $x > 0$. Further suppose that $L$ satisfies the property that $h_L(x) = L([x, \infty))$ is continuous in $x$. Given any such $L$, define $F : [0, \infty) \to [0, 1]$ by the above relation (8), i.e., $F(0) = \exp(-L((0, \infty)))$ and for any $x > 0$

$$F(x) = \exp(-h_L(x)) = \exp(-L([x, \infty))). \tag{9}$$

Define, $F(x) = 0$ for all $x < 0$.

Clearly, $F$ is a non-decreasing function. Since $h_L$ is continuous on $(0, \infty)$, so is $F$ on $(0, \infty)$. Further, $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \exp(-h_L(x)) = 1$. Also, note that $F$ is right-continuous at

0 since $\lim_{x\downarrow 0} L([x,\infty)) = L((0,\infty))$. The only possible point of discontinuity of $F$ is at 0. Clearly if the mass of $L$ is infinite, i.e., $L((0,\infty)) = \infty$, $F(0) = 0$, hence $F$ is continuous. However, if $L((0,\infty)) < \infty$, we have that $F(0) > 0$ and hence $F$ admits a point of discontinuity at 0.

Now, suppose that we have a sequence of records $\{X_n : n \geq 0\}$ from the distribution $F$. First, we note that from (9), $L(du) = F(du)/F(u)$ on the set $\{u > 0\}$. Thus,

$$\begin{aligned}
\int_0^1 \frac{uF(du)}{F(u)} &= \int_{(0,1)} u \log F(du) \\
&\leq \int_{(0,1]} u \log F(du) + \int_1^\infty \log F(du) \\
&= \int_{(0,\infty)} \min(1,u) \log F(du) \\
&= \int_{(0,\infty)} \min(1,u) L(du) \\
&< \infty.
\end{aligned}$$

Thus, $\sum_{n=0}^\infty X_n < \infty$ almost surely. Further, it is obvious from above that the Laplace transform of $\sum_{n=0}^\infty X_n$ and the given infinitely divisible random variable match. Thus, we have proved the following characterization theorem.

**Theorem 3** *If $K$ is an infinitely divisible random variable on $[0,\infty)$ such that its Levy measure $L$ has the property that $h_L(x) = L([x,\infty))$ is a continuous function of $x$ on $(0,\infty)$. Then, there exists a distribution $F$ with 0 as its only possible point of discontinuity such that*

$$\sum_{n=0}^{\infty} X_n \stackrel{d}{=} K \tag{10}$$

*where $X_n$ is the $(n+1)$-th record from the distribution $F$. Moreover, $F$ is given by the equation (9).*

**Remark** : Equation (10) provides a way of *simulating* an infinite divisible random variables $K$ with a given Levy measure $L$. Simply define $F$ as in (9) and approximate $K$ by $\sum_{n=0}^N X_n$ where $\{X_n : n \geq 0\}$ are the lower records from $F$ and $N$ is a sufficiently large integer. If the Levy measure has finite mass, we can obtain the exact distribution by choosing $N$ as a random integer with $N = \min\{n \geq 0 : X_n = 0\}$. In the case when $L$ has infinite mass it will be interesting to study the rate of convergence of the sum $\sum_{n=0}^N X_n$ to $\sum_{n=0}^\infty X_n$ as $N \to \infty$.

Observe that if $F$ has density $f$, then the Levy density of $T$ is given by $f(y)/F(y)$. A natural question is under what conditions on $F$ will the distribution of $T$ belong to specified *subclasses* of the class of infinitely divisible laws. For a comprehensive description of interesting subclasses of i.d. laws, see Bondesson ([3]). We give below three such classes:

*The class $\mathcal{L}$*: This is the class of *self decomposable* laws, or the so called class $\mathcal{L}$. It consists of distributions of those random variables $X$ for which for every $c \in (0,1]$, there exists a random variable $\epsilon_c$ independent of $X$ such that $X \stackrel{d}{=} cX + \epsilon_c$. The proof of the following Proposition is straightforward from Bondesson ([3], page 18) and is omitted.

**Proposition 2.3** *If $F$ admits a density $f$ such that $\int_0^1 (uf(u)/F(u))du < \infty$ and $uf(u)/F(u)$ is decreasing in $u$, then $T$ is self-decomposable. Conversely, for any self-decomposable random variable $T$, there exists a density $f$ with $\int_0^1 (uf(u)/F(u)du < \infty$ and $uf(u)/F(u)$ is decreasing in $u$ such that $\sum_{n=0}^{\infty} X_n \stackrel{d}{=} T$ where $\{X_n : n \geq 0\}$ is the sequence of lower records from the distribution with density $f$.*

*The subclass $\mathcal{T}_2$*: This class consists of *generalized mixtures of exponentials*, arising as weak limits of mixtures of exponentials and is characterized by complete monotonicity of the Levy density (see Bondesson [3], page 138). Using this and noting that $-\log(F(\cdot))$ is completely monotone if and only if $f(y)/F(y)$ is, we derive the following proposition.

**Proposition 2.4** *The random variable $T$ belongs to $\mathcal{T}_2$ if and only if $-\log(F(\cdot))$ is completely monotone.*

As a specific case, note that the Levy density $l(y) = \beta y^{-1} \exp(-ty) 1_{\{y>0\}}$ characterizes the Gamma distribution $\mathbf{G}(\beta, t)$ where $\beta,\ t > 0$. This will arise as the distribution of $T$ if

$$F(x) = \exp\Big(-\beta \int_x^{\infty} y^{-1} \exp(-ty) dy\Big).$$

*The class $\mathcal{T}$*: This is a subclass of both $\mathcal{L}$ and $\mathcal{T}_2$ and consists of *generalized gamma convolutions* defined as weak limits of finite convolutions of Gamma distributions. By Theorem 3.1.1 of [3], a distribution $G$ which gives full mass to the set of nonnegative real numbers, belongs to the class $\mathcal{T}$ if and only if its Levy density $l(\cdot)$ satisfies

$$yl(y) = \int_0^{\infty} \exp(-yt)\, U(dt)$$

for all $y > 0$, and for some measure $U$ on $(0, \infty)$ with $\int_0^1 |\log t| U(dt) < \infty$ and $\int_1^{\infty} t^{-1} U(dt) < \infty$. Equivalently, if $yl(y)$ is completely monotone on $(0, \infty)$. Thus, we have the following result.

**Proposition 2.5** *Suppose that $F$ is a distribution so that $\int_0^1 uf(u)/F(u)du < \infty$ and $uf(u)/F(u)$ is completely monotone. Then, the random variable $T$ belongs to $\mathcal{T}$. Conversely, if $T$ is a random variable in $\mathcal{T}$, there exists $F$ such that $T = \sum_{n=0}^{\infty} X_n$ where $\{X_n : n \geq 0\}$ is the sequence of lower records from $F$. Further, $F$ admits a density which satisfies the above conditions.*

The proof of the above proposition is straight forward, by noting that $l(y)$ is completely monotone if $yl(y)$ is completely monotone. As example of the above, the positive stable distributions have $l(y) \propto y^{-\alpha-1}$ for $y > 0$ where $0 < \alpha < 1$. These arise from $F(x) = \exp(-cx^{-\alpha}),\ x > 0$.

When does the distribution of $T$ admit a density? Using the infinite divisibility, we may obtain answers to this question.

**Proposition 2.6** *Suppose that $F(0) = 0$, $F$ admits a density $f$ on the whole of $(0,\infty)$ such that $\int_0^1 x \log F(dx) < \infty$ and $\int_0^\infty xF(dx) < \infty$. Then $T$ is absolutely continuous with respect to the Lebesgue measure.*

**Proof**: Hudson and Tucker [6] obtained sufficient conditions for an infinitely divisible distribution to be equivalent to the Lebesgue measure. Using Theorem 1 of Hudson and Tucker [6], it is enough to verify that the Levy measure is absolutely continuous with respect to the Lebesgue measure, has infinite mass and finite first moment.

In our case, we have the Levy measure as $f(x)/F(x)1_{\{x>0\}}dx$ and satisfies the properties of absolute continuity and the infinite mass (see the discussion before Theorem 3). It is clear that

$$\begin{aligned}\int_0^\infty \frac{xf(x)}{F(x)}dx &\leq \int_0^1 \frac{xf(x)}{F(x)}dx + \int_1^\infty \frac{xf(x)}{F(x)}dx \\ &\leq \int_0^1 \frac{xf(x)}{F(x)}dx + \frac{1}{F(1)}\int_1^\infty xf(x)dx \\ &< \infty.\end{aligned}$$

This proves the result. ∎

From our discussion so far about the behaviour of $T$, it is clear that the behaviour of $F$ near the origin plays a crucial role. Therefore, one might expect that if $F$ admits a density only near the origin, $T$ might still admit a density. We give a partial answer to the above question using sufficient conditions available in terms of the characteristic function.

**Proposition 2.7** *Suppose that $F$ admits a density $f$ in a neighbourhood of the origin such that for some $\gamma > 0$, we have*

$$\liminf_{u\to 0} \frac{f(u)}{u^{\gamma-1}} = C_5 \text{ and } \limsup_{u\to 0} \frac{f(u)}{u^{\gamma-1}} = C_7$$

*and $2C_5\gamma > C_7$. Then $T$ admits a density.*

**Proof**: Fix an $\epsilon \in (0, C_5)$ such that $c(\epsilon) := (C_5 - \epsilon)\gamma/(C_6 + \epsilon) > 1/2$ and choose $\delta > 0$ so that $(C_5 - \epsilon)x^{\gamma-1} \leq f(x) \leq (C_6 + \epsilon)x^{\gamma-1}$ for all $x \leq \delta$. Then, we have, for all $x \leq \delta$,

$$F(x) \leq (C_6 + \epsilon)x^\gamma/\gamma.$$

Let $\xi(t) = \mathbb{E}(\exp(itT))$ be the characteristic function of $T$. Hence, for any $t \in \mathbb{R}$, we have

$$\begin{aligned}\mid \xi(t) \mid &= \mid \exp\Big(\int_0^\infty \frac{(\cos(tu) - 1)}{F(u)}F(du) + i\int_0^\infty \frac{\sin(tu)}{F(u)}F(du)\Big) \mid \\ &= \exp\Big(-\int_0^\infty \frac{(1-\cos(tu))}{F(u)}F(du)\Big) \\ &\leq \exp\Big(-\int_0^\delta \frac{(1-\cos(tu))}{F(u)}F(du)\Big).\end{aligned}$$

Now we have, for any $t > 0$,

$$
\begin{aligned}
\int_0^\delta \frac{(1-\cos(tu))}{F(u)} F(du) &\geq \int_0^\delta \frac{(1-\cos(tu))(C_5-\epsilon)u^{\gamma-1}\gamma}{(C_6+\epsilon)u^\gamma} du \\
&= c(\epsilon)\int_0^\delta \frac{(1-\cos(tu))du}{u} \\
&= c(\epsilon)\int_0^{t\delta} \frac{(1-\cos(u))du}{u} \\
&\geq c(\epsilon)\sum_{j=0}^{[t\delta/\pi]-1}\int_{j\pi}^{(j+1)\pi} \frac{(1-\cos(u))du}{u} \\
&\geq c(\epsilon)\sum_{j=0}^{[t\delta/\pi]-1}\frac{1}{(j+1)\pi}\int_{j\pi}^{(j+1)\pi} (1-\cos(u))du \\
&\geq c(\epsilon)\sum_{j=0}^{[t\delta/\pi]-1}\frac{1}{j+1} \\
&\geq c(\epsilon)\log(1+[t\delta/\pi]) \\
&\geq c(\epsilon)\log(t\delta/\pi).
\end{aligned}
$$

Thus, we obtain, for $t > 0$,

$$
\mid \xi(t) \mid \leq \exp(-c(\epsilon)\log(t\delta/\pi)) = C_7 t^{-c(\epsilon)} \tag{11}
$$

where $C_7 = 1/(\delta/\pi)^{c(\epsilon)}$. Again, for $t < 0$, since cosine is an even function, we have the same estimate. Now, noting that $c(\epsilon) > 1/2$, we have that $\int_{-\infty}^{\infty} |\xi(t)|^2 dt < \infty$. Therefore, using exercise 11, page 159 of Chung [4], we have that $T$ admits a density. ∎

## 3 The discrete case

In this section, we will derive similar results for the case when $F$ is concentrated on non-negative integers. In this case, with our definition, we may have repetition of records. Let $\mathbb{N}_0 = \{0, 1, \dots\}$. Suppose that $F$ is a distribution with mass only on $\mathbb{N}_0$. Then, define $p_k = F(k) - F(k-)$ where $F(k-)$ is the left limit as $u \uparrow k$. Then, we have $0 \leq p_k \leq 1$ and $\sum_{k=0}^{\infty} p_k = 1$ and $F(k) = \sum_{j=0}^{k} p_j$ for $k = 0, 1, \dots$.

Let $X_0 \sim F$. Then $\{X_n : n \geq 0\}$ is a sequence of *lower record values* if it is a Markov chain with the stationary transition probabilities given by the truncated distribution

$$
\mathbb{P}(X_{n+1} = j | X_n = i) = p_j / \sum_{l=0}^{i} p_l \quad \text{if} \quad j \leq i.
$$

Again, it is clear that $X_n$ converges almost surely to the constant $\min\{j : p_j > 0\}$. In case this minimum is 0, one can then naturally raise the question of the convergence of the sequence of partial sums of these records. As we have already discussed in Section 2.1, the minimum is actually hit in a finite time and the sum is finite with probability 1.

Let $T = \sum_{n=0}^{\infty} X_n$. It is easy to see that $T < \infty$ almost surely if and only if $p_0 > 0$. *Henceforth, we will assume that* $p_0 > 0$. Now, we turn to distributional properties of $T$. First, let us obtain a formula for its characteristic function by using our familiar conditioning argument. This formula readily yields infinite divisibility of $T$.

Let us fix any $t \in \mathbb{R}$, and define

$$\phi_k(t) = \mathbb{E}(\exp(itT)|X_0 \le k). \tag{12}$$

Clearly if $k = 0$, $T = 0$ almost surely. Thus, $\phi_0(t) = 1$ for all $t \in \mathbb{R}$.

**Proposition 3.1** *The characteristic function of* $T$ *is given by*

$$\phi(t) = \lim_{k\to\infty} \phi_k(t) = \prod_{j=1}^{\infty} \frac{F(j-1)}{F(j) - p_j \exp(itj)}. \tag{13}$$

(Note that the product on the right side is non-zero since $\sum_{j=1}^{\infty} |1 - F(j-1)/(F(j) - p_j \exp(itj))| < C_8 \sum_{j=1}^{\infty} p_j < \infty$ where $C_8 > 0$ is a constant).

**Proof**: For the first equality, note that, $\phi_k(t) = \mathbb{E}(\exp(itT)1_{\{X_0 \le k\}})/F(k)$. Now, as $k \to \infty$, $F(k) \to 1$; and also $1_{\{X_0 \le k\}} \to 1$ almost surely. Thus, by dominated convergence theorem, we have $\phi_k(t) \to \phi(t)$.

Now, we have, for any $k \ge 0$,

$$\begin{aligned}
\phi_k(t) &= \mathbb{E}(\exp(itT)|X_0 \le k) = \sum_{j=0}^{k} \mathbb{E}(\exp(itT)1_{\{X_0=j\}})/F(k) \\
&= \sum_{j=0}^{k} \mathbb{E}(\exp(it(j + \sum_{n=1}^{\infty} X_n))1_{\{X_0=j\}})/F(k) \\
&= \sum_{j=0}^{k} \exp(itj) p_j \mathbb{E}(\exp(it\sum_{n=1}^{\infty} X_n)|X_1 \le j)/F(k) \\
&= \sum_{j=0}^{k} \exp(itj) p_j \phi_j(t)/F(k)
\end{aligned}$$

Thus, we have

$$\phi_k(t)F(k) = \sum_{j=0}^{k} \exp(itj) p_j \phi_j(t).$$

Therefore, for $k \ge 1$,

$$\phi_k(t)F(k) - \phi_{k-1}(t)F(k-1) = \exp(itk) p_k \phi_k(t).$$

This implies that for $k \geq 1$,

$$\phi_k(t) = \frac{\phi_{k-1}(t)}{1 + p_k(1 - \exp(itk))/F(k-1)}. \tag{14}$$

Thus, from the fact that $\phi_0(t) = 1$ for all $t \in \mathbb{R}$, using the above recursion formula, we have

$$\phi_k(t) = \prod_{j=1}^{k} \frac{1}{1 + p_j(1 - \exp(itj))/F(j-1)}.$$

Hence, letting $k \to \infty$ in the above expression, the characteristic function of $T$ is given

$$\phi(t) = \prod_{j=1}^{\infty} \frac{1}{1 + p_j(1 - \exp(itj))/F(j-1)},$$

which simplifies to (13). ■

Towards obtaining the representation for $T$ that yields infinite divisibility, we start with a definition.

**Definition 1** *A random variable $G$ taking values in $\mathbb{N}_0 = \{0, 1, \ldots,\}$ will be said to have a geometric distribution with multiplicity $k$ and parameter $\alpha$, (denoted by $G \sim \mathrm{Geo}(k, \alpha)$) if*

$$\mathbb{P}(G = n) = \begin{cases} (1-\alpha)\alpha^j & \text{if } n = jk \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where $k \geq 1$ and $0 < \alpha < 1$.

In other words, $G \sim \mathrm{Geo}(k, \alpha)$ if and only if $G/k$ follows the usual geometric distribution with parameter $\alpha$. In this case, the characteristic function of $G$ can be written as

$$\begin{aligned} \phi_G(t) &= \frac{1-\alpha}{1 - \alpha \exp(itk)} \\ &= \exp\Big(\sum_{n=1}^{\infty} \big(\exp(itkn) - 1\big)\frac{\alpha^n}{n}\Big) \\ &= \exp\Big(\sum_{n=1}^{\infty} \big(\exp(itn) - 1\big)\mu'(n; k, \alpha)\Big) \end{aligned}$$

where

$$\mu'(n; k, \alpha) = \begin{cases} \frac{\alpha^j}{j} & \text{if } n = jk \\ 0 & \text{otherwise.} \end{cases}$$

It is clear from the above that this characteristic function is infinitely divisible. Indeed, we can write it down in the Lévy-Khintchine form (see e.g. Lukacs [7], Theorem 5.5.1) easily from the above. Taking logarithm of $\phi_G(t)$ and denoting it by $w_G(t)$, we have

$$w_G(t) = ib_G t + \sum_{n=1}^{\infty} \Big(\exp(itn) - 1 - \frac{itn}{1+n^2}\Big)\frac{1+n^2}{n^2}\mu(n; k, \alpha) \tag{16}$$

where

$$\mu(n;k,\alpha) = \frac{n^2}{1+n^2}\mu'(n;k,\alpha) = \begin{cases} \frac{k^2 j\alpha^j}{1+(jk)^2} & \text{if } n = jk,\ j \geq 1, \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

and

$$b_G = \sum_{n=1}^{\infty} \frac{n}{1+n^2}\mu'(n;k,\alpha) = k\sum_{j=1}^{\infty} \frac{\alpha^j}{1+(jk)^2} < \infty. \tag{18}$$

It is also clear that $\sum_{n=1}^{\infty} \mu(n;k,\alpha) < \infty$.

**Proposition 3.2** *Let $\{G_n : n \geq 1\}$ be a sequence of independent variables with $G_n$ having* $\text{Geo}(k_n, \alpha_n)$ *distribution. Then, $G = \sum_{n=1}^{\infty} G_n < \infty$ almost surely if and only if $\sum_{n=1}^{\infty} \alpha_n < \infty$ and in that case $G$ is infinitely divisble.*

**Proof**: Since each $G_n$ takes values in $\mathbb{N}_0$, we have $G < \infty$ almost surely if and only if $\mathbb{P}(\limsup_{n\to\infty}\{G_n > 0\}) = 0$. Indeed, if $\omega \in \limsup_{n\to\infty}\{G_n > 0\}$, then there exist a sequence $n_j \uparrow \infty$ such that $\omega \in \{G_{n_j} > 0\}$ for every $j \geq 1$. This implies, owing to the discrete nature of $G_n$'s, that $G = \infty$. Conversely, if $\omega \notin \limsup_{n\to\infty}\{G_n > 0\}$, then there exists $n_0$ such that $G_n = 0$ for all $n > n_0$, which implies that $G = \sum_{n=1}^{n_0} G_n < \infty$.

Now, for the "if" part, since $\sum_{n=1}^{\infty} \mathbb{P}(G_n > 0) = \sum_{n=1}^{\infty} \alpha_n < \infty$, by the first Borel-Cantelli lemma, we have $\mathbb{P}(\limsup_{n\to\infty}\{G_n > 0\}) = 0$. Conversely, if $\sum_{n=1}^{\infty} \alpha_n = \infty$, then from the independence of the events $\{G_n > 0\}$, we have, by the second Borel-Cantelli lemma, $\mathbb{P}(\limsup_{n\to\infty}\{G_n > 0\}) = 1$. This proves the "if" part. To show that $G$ is infinitely divisible, define $H_n = \sum_{j=1}^{n} G_j$. Since each $G_j$ is infinitely divisible, so is $H_n$. Now $H_n \to G$ almost surely, hence also in distribution. Thus, by Theorem 7.6.5 of Chung ([4], page 244), $G$ is infinitely divisible. ∎

**Remark** The Levy measure of $G$ can be written from (17) and (18). Indeed, the logarithm of the characteristic function of $G$, denoted by $w_{\sum_{n=1}^{\infty} G_n}(t)$ can be written as

$$w_{\sum_{n=1}^{\infty} G_n}(t) = ibt + \sum_{j=1}^{\infty}\Big(\exp(itj) - 1 - \frac{itj}{1+j^2}\Big)\frac{1+j^2}{j^2}\mu_j \tag{19}$$

where

$$b = \sum_{n=1}^{\infty} b_n \text{ and } \quad \mu_j = \sum_{n=1}^{\infty} \mu(j;k_n,\alpha_n). \tag{20}$$

It is fairly easy to verify that $b < \infty$ and $\sum_{j=1}^{\infty} \mu_j < \infty$.

We may now state the representation for $T$.

**Theorem 4 (Representation Theorem)** *For any $F$ having support in $\mathbb{N}_0$ with $F(0) > 0$, we have*

$$T \stackrel{d}{=} \sum_{k=1}^{\infty} G_k$$

*where $\{G_k : k \geq 1\}$ is a sequence of independent random variables with $G_k$ having the distribution* $\text{Geo}(k, \frac{p_k}{F(k)})$. *Hence $T$ is infinitely divisible. Conversely, given any infinite convolution of* $\{\text{Geo}(k, \alpha_k) : k \geq 1\}$ *such that $\sum_{k=1}^{\infty} \alpha_k < \infty$, there exists $F$ having support in $\mathbb{N}_0$ such that the above representation holds.*

**Proof**: From the fact that $\sum_{k=0}^{\infty} p_k = 1$ and $F(k) \to 1$ as $k \to \infty$, it readily follows that $\sum_{k=1}^{\infty} p_k/F(k) < \infty$. Hence from the Proposition 3.2, $G = \sum_{k=1}^{\infty} G_k$ is finite and infinitely divisible.

Now note that $T$ and $G$ have the same characteristic function. This proves the representation and that $T$ is infinitely divisible.

For the last part, given any $\{\alpha_k : k \geq 1\}$ such that $\sum_{k=1}^{\infty} \alpha_k < \infty$, define for any $k \geq 0$,

$$F(k) = \prod_{j=k+1}^{\infty} (1 - \alpha_j).$$

Since $\sum_{k=1}^{\infty} \alpha_k < \infty$, we have that $F(0) > 0$ and $F(k) \uparrow 1$ as $k \to \infty$. Clearly for this distribution function, we will have $\alpha_k = 1 - F(k-1)/F(k)$ for all $k \geq 1$. This proves the result.

■

**Acknowledgement**. The authors are thankful to Rajeeva L. Karandikar for his helpful suggestions on the manuscript.

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