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Sequential Estimation for Fractional Ornstein-Uhlenbeck Type Process

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Abstract

We investigate the asymptotic properties of the sequential maximum likelihhod estimator of the drift parameter for fractional Ornstein-Uhlenbeck type process satisfying a linear stochastic differential equation driven by fractional Brownian motion.

Keywords and phrases: fractional Ornstein-Uhlenbeck process; fractional Brownian motion; Sequential maximum likelihood estimation.

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1 Introduction

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999a). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion. Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. In a recent paper, Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in (1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0. \tag{1. 1}$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the obsrevation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

Parametric estimation for more general classes of stochastic processes satsfying linear stochastic differential equations driven fractional Brownian motion is studied in Prakasa Rao (2003a,b). Novikov (1972) investigated asymptotic properties of a sequential maximum likelihood estimator for the drift parameter in the Ornstein-Uhlenbeck process. We now discuss analogous results for fractional Ornstein-Uhlenbeck process.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions and the processes discussed in the following are $(\mathcal{F}_t)$-adapted. Further the natural fitration of a process is understood as the P-completion of the filtration generated by this process.

Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, s \geq 0. \tag{2. 1}$$

Let us consider a stochastic process $\{X_t, t \geq 0\}$ defined by the stochastic integral equation

$$X_t = \theta \int_0^t X(s)ds + \sigma W_t^H, t \geq 0 \tag{2. 2}$$

where θ and σ^2 are unknown constant drift and diffusion parameters respectively. For convenience we write the above integral equation in the form of a stochastic differential equation

$$dX_t == \theta X(t)dt + \sigma dW_t^H, t \geq 0 \tag{2. 3}$$

driven by the fractional Brownian motion W^H. Even though the process X is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration $(\mathcal{Z}_t)$ of the process Z coincides with the natural filtration $(\mathcal{X}_t)$ of the process X (Kleptsyna et al. (2000)). Define, for $0 < s < t$,

$$k_H = 2H\Gamma\,(\frac{3}{2} - H)\Gamma(H + \frac{1}{2}), \tag{2. 4}$$

$$k_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}, \tag{2. 5}$$

$$\lambda_H = \frac{2H\,\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}, \tag{2. 6}$$

$$w_t^H = \lambda_H^{-1} t^{2-2H}, \tag{2. 7}$$

and

$$M_t^H = \int_0^t k_H(t,s)dW_s^H, t \geq 0. \tag{2. 8}$$

The process M^H is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al. (1999)) and its quadratic variance $< M_t^H >= w_t^H$. Further more the natural filtration of the martingale M^H coincides with the natural fitration of the fBM W^H. Let

$$K_H(t,s) = H(2H-1)\frac{d}{ds}\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr, 0 \leq s \leq t. \tag{2. 9}$$

The sample paths of the process $\{X_t, t \geq 0\}$ are smooth enough so that the process Q defined by

$$Q(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)X_s ds, t \in [0,T] \tag{2. 10}$$

is welldefined where w^H and k_H are as defined in (2.7) and (2.5) respectively and the derivative is understood in the sense of absolute continuity with respect to the measure generated by w^H. More over the sample paths of the process Q belong to $L^2([0,T], dw^H)$ a.s. [P]. The following theorem due to Kleptsyna et al. (2000) associates a *fundamental semimartingale* Z associated with the process X such that the natural filtration $(\mathcal{Z}_t)$ coincides with the natural filtration $(\mathcal{X}_t)$ of X.

Theorem 2.1: Let the process $Z = (Z_t, t \in [0,T])$ be defined by

$$Z_t = \int_0^t k_H(t,s)dX_s \tag{2. 11}$$

where the function $k_H(t,s)$ is as defined in (2.5). Then the following results hold:
(i) The process Z is an $(\mathcal{F}_t)$ -semimartingale with the decomposition

$$Z_t = \theta \int_0^t Q(s)dw_s^H + \sigma M_t^H \tag{2. 12}$$

where M^H is the gaussian martingale defined by (2.8),
(ii) the process X admits the representation

$$X_t = \int_0^t K_H(t,s)dZ_s \tag{2. 13}$$

where the function K_H is as defined in (2.9), and
(iii) the natural fitrations of $(\mathcal{Z}_t)$ and $(\mathcal{X}_t)$ coincide.

Kleptsyna et al. (2000) derived a Girsanov type formula for fractional Brownian motions. As an application , it follows that the Radon-Nikodym Derivative of the measure P_θ^T, generated by the stochastic process X when θ is the true parameter, with respect to the measure generated by the process X when $\theta = 0$, is given by

$$\frac{dP_\theta^T}{dP_0^T} = \exp[-\theta \int_0^T Q(s)dZ_s + \frac{1}{2}\theta^2 \int_0^T Q^2(s)dw_s^H]. \tag{2. 14}$$

From the representation (2.12), it follows that the quadratic variation $< Z >_T$ of the process Z on $[0,T]$ is equal to $\sigma^2 w_T^H$ a.s. and hence the parameter σ^2 can be estimated by the relation

$$\lim_n \Sigma[Z_{t_{i+1}^{(n)}} - Z_{t_i^{(n)}}]^2 = \sigma^2 w_T^H a.s. \tag{2. 15}$$

where $(t_i^{(n)})$ is an appropriate partition of $[0,T]$ such that

$$\sup_i |t_{i+1}^{(n)} - t_i^{(n)}| \to 0$$

as $n \to \infty$. Hence we can estimate σ^2 almost surely from any small interval as long as we have a continuous observation of the process. For further discussion, we assume that $\sigma^2 = 1$.

3 Maximum likelihood estimation

We consider the problem of estimation of the parameter θ based on the observation of the process $X = \{X_t, 0 \leq t \leq T\}$ for a fixed time T and study its asymptotic properties as $T \to \infty$. These results are due to Kleptsyna and Le Breton (2002) and Prakasa Rao (2003a,b).

Theorem 3.1: The maximum likelihood estimator θ from the observation $X = \{X_t, 0 \leq t \leq T\}$ is given by

$$\hat{\theta}_T = \{\int_0^T Q^2(s) dw_s^H\}^{-1} \int_0^T Q(s) dZ_s \tag{3. 1}$$

where the processes Q and Z are as defined by (2.10) and (2.11) respectively. Further more the estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$, that is,

$$\lim_{T\to\infty} \hat{\theta}_T = \theta \ \text{ a.s. } [P_\theta] \tag{3. 2}$$

for every $\theta \in R$.

We now discuss the limiting distribution of the MLE $\hat{\theta}_T$ as $T \to \infty$.

Theorem 3.2: Let

$$R_T = \int_0^T Q(s) dZ_s. \tag{3. 3}$$

Assume that there exists a norming function $I_t, t \geq 0$ such that

$$I_T^2 \int_0^T Q(t)^2 dw_t^H \to \eta^2 \ \text{ in probability as } T \to \infty \tag{3. 4}$$

where $I_T \to 0$ as $T \to \infty$ and η is a random variable such that $P(\eta > 0) = 1$. Then

$$(I_T R_T, I_T^2 < R_T >) \to (\eta Z, \eta^2) \text{ in law as } T \to \infty \tag{3. 5}$$

where the random variable Z has the standard normal distribution and the random variables Z and η are independent.

Proof: This theorem follows as a consequence of the central limit theorem for martingales (cf. Theorem 1.49 ; Remark 1.47 , Prakasa Rao(1999b), p. 65).

Observe that

$$I_T^{-1}(\hat{\theta}_T - \theta_0) = \frac{I_T R_T}{I_T^2 < R_T >} \tag{3. 6}$$

Applying the Theorem 3.2, we obtain the following result.

Theorem 3.3: Suppose the conditions stated in the Theorem 3.2 hold. Then

$$I_T^{-1}(\hat{\theta}_T - \theta_0) \to \frac{Z}{\eta} \text{ in law as } t \to \infty \tag{3. 7}$$

where the random variable Z has the standard normal distribution and the random variables Z and η are independent.

Remarks: If the random variable η is a constant with probability one, then the limiting distribution of the maximum likelihood estimator is normal with mean 0 and variance η^{-2}. Otherwise it is a mixture of the normal distributions with mean zero and variance η^{-2} with the mixing distribution as that of η.

4 Sequential maximum likelihood estimation

We now consider the problem of sequential maximum likelihood estimation of the parameter θ. Let h be a nonnegative number. Define the stopping rule $\tau(h)$ by the rule

$$\tau(h) = \inf\{t : \int_0^t Q^2(s)dw_s^H \geq h\}. \tag{4. 1}$$

Kletptsyna and Le Breton (2002) have shown that

$$\lim_{t\to\infty} \int_0^t Q^2(s)dw_s^H = +\infty \ \text{ a.s. } [P_\theta] \tag{4. 2}$$

for every $\theta \in R$. Then it can be shown that $P_\theta(\tau(h) < \infty) = 1$. If the process is observed up to a previuosly determined time T , we have observed that the maximum likelihood estimator is given by

$$\hat{\theta}_T = \{\int_0^T Q^2(s)dw_s^H\}^{-1} \int_0^T Q(s)dZ_s. \tag{4. 3}$$

The estimator

$$\begin{aligned} \hat{\theta}(h) &\equiv \hat{\theta}_{\tau(h)} \\ &= \{\int_0^{\tau(h)} Q^2(s)dw_s^H\}^{-1} \int_0^{\tau(h)} Q(s)dZ_s \\ &= h^{-1} \int_0^{\tau(h)} Q(s)dZ_s \end{aligned} \tag{4. 4}$$

is called the *sequential maximum likelihood estimator* of θ. We now study the asymptotic properties of the estimator $\hat{\theta}(h)$.

We shall first prove a lemma which is an analogue of the Cramer-Rao inequality for sequential plans $(\tau(X), \hat{\theta}_\tau(X))$ for estimating the parameter θ satisfying the property

$$E_\theta\{\hat{\theta}_\tau(X)\} = \theta \tag{4. 5}$$

for all θ.

Lemma 4.1: Suppose that differentiation under the integral sign with respect to θ on the left side of the equation (4.5) is permissible. Further suppose that

$$E_\theta\{\int_0^{\tau(X)} Q^2(s)dw_s^H\} < \infty \tag{4. 6}$$

for all θ. Then

$$Var_\theta\{\hat{\theta}_\tau(X)\} \geq \{E_\theta[\int_0^{\tau(X)} Q^2(s)dw_s^H]\}^{-1} \tag{4. 7}$$

for all θ.

Proof: Let P_θ be the measure generated by the process $X(t), t \leq \tau(X)$ for given θ. It follows from the results discussed above that

$$\frac{dP_\theta}{dP_{\theta_0}} = \exp\{(\theta-\theta_0)\int_0^{\tau(X)} Q(s)dZ_s - \frac{1}{2}(\theta^2-\theta_0^2)\int_0^{\tau(X)} Q^2(s)dw_s^H\} \text{ a.s } [P_{\theta_0}]. \tag{4. 8}$$

Differentiating (4.5) with respect to θ under the integral sign, we get that

$$E_\theta[\hat{\theta}_\tau(X)\{\int_0^{\tau(X)} Q(s)dZ_s - \theta\int_0^{\tau(X)} Q^2(s)dw_s^H\}] = 1. \tag{4. 9}$$

Theorem 2.1 implies that

$$dZ_s = \theta Q_s dw_s^H + dM_s^H \tag{4. 10}$$

and hence

$$\int_0^T Q(s)dZ_s = \theta\int_0^T Q^2(s)dw_s^H + \int_0^T Q(s)dM_s^H. \tag{4. 11}$$

The above relation in turn implies that

$$E_\theta\{\int_0^{\tau(X)} Q(s)dZ_s - \theta\int_0^{\tau(X)} Q^2(s)dw_s^H\} = 0 \tag{4. 12}$$

and

$$E_\theta\{\int_0^{\tau(X)} Q(s)dZ_s - \theta\int_0^{\tau(X)} Q^2(s)dw_s^H\}^2 = E_\theta\{\int_0^{\tau(X)} Q^2(s)dw_s^H\} \tag{4. 13}$$

from the properties of the fundamental martingale M^H and the fact that the quadratic variation $< M^H >_t$ of the process M_t^H is w_t^H. Applying the Cauchy-Schwartz inequality to the left side of the equation (4.9), we obtain that

$$Var_\theta\{\hat{\theta}_\tau(X)\} \geq \{E_\theta[\int_0^{\tau(X)} Q^2(s)dw_s^H]\}^{-1} \tag{4. 14}$$

for all θ.

A sequential plan $(\tau(X), \hat{\theta}_\tau(X))$ is said to be *efficient* if there is equality in (4.7) for all θ.

We now prove the main result.

Theorem 4.2: Consider the fractional Ornstein-Uhlenbeck process governed by the stochastic differential equation (2.3) with $\sigma = 1$ driven by the fractional Brownian motion W^H with $H \in (\frac{1}{2}, 1)$. Then the sequential plan $(\tau(h), \hat{\theta}(h))$ defined by the equations (4.1) and (4.4) has the following properties for all θ.

(i) $\hat{\theta}(h) \equiv \hat{\theta}_{\tau(h)}$ is normally distributed with $E_\theta(\hat{\theta}(h)) = \theta$ and $Var_\theta(\hat{\theta}(h)) = h^{-1}$;

(ii) the plan is efficient; and

(iii) the plan is closed, that is, $P_\theta(\tau(h) < \infty) = 1$.

Proof: Let

(4. 15)
$$J_T = \int_0^T Q(s)dM_s^H$$

From the results in Kartazas and Shreve (1988), Revuz and Yor (1991) and Ikeda and Watanabe (1981), it follows that there exists a standard Wiener process W such that

(4. 16)
$$J_T = W(< J >_T) \text{ a.s}$$

with respect to the filtration $\{\mathcal{F}_{\tau_t}, t \geq 0\}$ under P where $\tau_t = \inf\{s :< J >_s > t\}$. Hence the process

(4. 17)
$$\int_0^{\tau(h)} Q(s)dM_s^H$$

is a standard Wiener process. Observe that

(4. 18)
$$\begin{aligned}\hat{\theta}(h) &= h^{-1}\int_0^{\tau(h)} Q(s)dZ_s \\ &= h^{-1}\{\theta\int_0^{\tau(h)} Q^2(s)dw_s^H + \int_0^{\tau(h)} Q(s)dM_s^H\} \\ &= \theta + h^{-1}\int_0^{\tau(h)} Q(s)dM_s^H \\ &= \theta + h^{-1}J_{\tau(h)} \\ &= \theta + h^{-1}W(< J >_{\tau(h)})\end{aligned}$$

which proves that the estimator $\hat{\theta}(h)$ is normally distributed with mean θ and variance h^{-1}. Since

(4. 19)
$$E_\theta\{\int_0^{\tau(h)} Q^2(s)dw_s^H\} = h,$$

it follows that the plan is efficient by the Lemma 4.1. Since

(4. 20)
$$P_\theta(\tau(h) \geq T) = P_\theta\{\int_0^T Q^2(s)dw_s^H < h\}$$

for every $T \geq 0$, it follows that $P_\theta(\tau(h) < \infty) = 1$ from the observation

(4. 21)
$$P_\theta(\int_0^\infty Q^2(s)dw_s^H = \infty) = 1.$$

We now discuss some results on the probability distribution of the observation time $\tau(h)$ and the mean observation time $E_\theta\{\tau(h)\}$.

It follows from the definition of the stopping time $\tau(h)$ that

$$
\begin{aligned}
(4.\ 22) \qquad P_\theta(\tau(h) \geq T) &= P_\theta\left(\int_0^T Q^2(s)dw_s^H < h\right) \\
&= P_\theta\left(\exp\left(-\int_0^T Q^2(s)dw_s^H\right) > e^{-h}\right) \\
&\leq e^h E_\theta\left[\exp\left(-\int_0^T Q^2(s)dw_s^H\right)\right] \\
&= \psi_T^H(\theta,1)(say).
\end{aligned}
$$

Kleptsyna and Le Breton (2002) have proved that

$$
(4.\ 23) \qquad \psi_T^H(\theta,1) = \left[\frac{4(sin\pi H)\sqrt{\theta^2+2}e^{-\theta T}}{\pi T D_T^H(\theta,\sqrt{\theta^2+2})}\right]^{1/2}
$$

where

$$
\begin{aligned}
(4.\ 24) \qquad D_T^H(\theta,\alpha) &= [\alpha\ cosh(\frac{\alpha}{2}T) - \theta\ sinh(\frac{\alpha}{2}T)]^2 I_{-H}(\frac{\alpha}{2}T)I_{H-1}(\frac{\alpha}{2}T) - \\
&\quad -[\alpha\ sinh(\frac{\alpha}{2}T) - \theta\ cosh(\frac{\alpha}{2}T)]^2 I_{1-H}(\frac{\alpha}{2}T)I_H(\frac{\alpha}{2}T)
\end{aligned}
$$

where I_ν is the modified Bessel function of the first kind and order ν (cf. Watson (1995)). Kleptsyna and Le Breton (2002) have also proved that

$$
(4.\ 25) \qquad \psi_T^H(\theta,1) \simeq \left[\frac{4(sin\pi H)(\theta^2+2)}{2+(sin\pi H)(\sqrt{\theta^2+2}-\theta)^2}\right]^{1/2} \exp\{-((\theta+\sqrt{\theta^2+2})/2)T\}
$$

as $T \to \infty$. In particular it follows that for every θ,

$$
(4.\ 26) \qquad P_\theta(\tau(h) \geq T) = O(e^h \gamma_1(\theta)\exp\{-\gamma_2(\theta)T\})
$$

as $T \to \infty$ where $\gamma_i(\theta) > 0, i = 1, 2$. Further more

$$
\begin{aligned}
(4.\ 27) \qquad E_\theta(\tau(h)) &= \int_0^\infty P_\theta(\tau(h) \geq u)\ du \\
&= \int_0^\infty P_\theta\left(\int_0^u Q^2(s)dw_s^H < h\right)\ du \\
&= \int_0^\infty P_\theta\left(\exp\left(-\int_0^u Q^2(s)dw_s^H\right) > e^{-h}\right)\ du \\
&\leq \int_0^\infty e^h E_\theta\left[\exp\left(-\int_0^u Q^2(s)dw_s^H\right)\right]\ du \\
&= e^h \int_0^\infty \psi_u^H(\theta,1)\ du
\end{aligned}
$$

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