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Moment inequalities for supremum of empirical processes for ϕ-mixing sequences

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MOMENT INEQUALITIES FOR SUPREMUM OF EMPIRICAL PROCESSES FOR ϕ-MIXING SEQUENCES

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Abstract

Let $\{X_n, -\infty < n < \infty\}$ be a stationary ϕ-mixing process with the one-dimensional marginal distribution function F and the density function f. Let $F_n(x)$ be the empirical distribution function based on the observations $\{X_i, 1 \leq i \leq n\}$ and $W_n^* = \sup_{-\infty<x<\infty} \sqrt{n}|F_n(x) - F(x)|$. We obtain upper bounds for $E(W_n^*)$. We give an application to get bounds on the expectation of the supremum of the deviation of a kernel density estimator $\hat{f}_n(x)$ from true density function $f(x)$. Similar results were obtained for a kernel type estimator $\hat{F}_n(x)$ for the true distribution function $F(x)$.

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1 Introduction

Moment inequalities for the supremum of empirical processes with applications to kernel type estimation of a density function and a distribution function for identically distributed observations were investigated in Ahmad (2002). We obtain similar results for mixing procesess.

2 Preliminaries

Let $\{X_n, -\infty < n < \infty\}$ be a stationary ϕ-mixing sequence defined on a probability space $(\Omega, \mathcal{F}, P)$ with each X_i having a continuous distribution function $F(x)$ and density function $f(x)$. Let $Y_i = F(X_i), -\infty < i < \infty$. Define $G(t) = P(Y_i \leq t)$. Let $F_n(x)$ denote the empirical distribution function based on the observations $\{X_i, 1 \leq i \leq n\}$ and $G_n(t)$ denote the empirical distribution function based on the observations $\{Y_i, 1 \leq i \leq n\}$. It is easy to see that

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| \leq \tilde{D}_n = \sup_{0\leq t\leq 1} |G_n(t) - t|. \tag{2. 1}$$

The following result is due to Kim (1999).

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Theorem 2.1: Let $\{X_n, n \geq 1\}$ be a stationary and ϕ-mixing sequence of random variables such that

$$\sum_{i=1}^{\infty} \phi_i < \infty. \tag{2. 2}$$

Then, for every positive integer $k \geq 1$, there corresponds a constant $C_k > 0$ such that for any $\lambda \geq 1$,

$$\sup_n P(\sqrt{n}\tilde{D}_n \geq \lambda) \leq C_k \lambda^{-2k}. \tag{2. 3}$$

Let

$$D_n = \sqrt{n}\tilde{D}_n. \tag{2. 4}$$

Note that for any positive integer $k \geq 1$,

$$\begin{aligned} E(D_n) &= \int_0^\infty P(D_n > x)dx \qquad (2.\ 5) \\ &\leq 1 + \int_1^\infty P(D_n > x)dx \\ &\leq 1 + \int_1^\infty C_k x^{-2k} dx \\ &= 1 + C_k(\frac{1}{2k-1}) \end{aligned}$$

and, in general, it is easy to see that for $r \geq 1$ and positive integer $k \geq \frac{r}{2}$,

$$E(D_n) \leq \{1 + C_k(\frac{r}{2k-r})\}^{1/r}. \tag{2. 6}$$

3 Application to Density Estimation

Suppose a stationary ϕ-mixing process $\{X_i, i \geq 1\}$ is observed up to time n with the ϕ-mixing sequence satisfying the condition (2.2) and with the one-dimensional probability density function f. The problem is to estimate the density fuction f based on the observations $\{X_i, 1 \leq i \leq n\}$.

Let $J(.)$ be a bounded symmetric probability density function with mean zero and finite variance σ_J^2. Further suppose that it is of bounded variation with total variation V_J. Let h_n be a sequence of positive numbers such that

$$h_n \to 0 \ \text{ and } \ nh_n \to \infty \ \text{ as } \ n \to \infty. \tag{3. 1}$$

Define, for any x,

$$\hat{f}_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} J(\frac{x - X_i}{h_n}). \tag{3. 2}$$

The estimator $\hat{f}_n(x)$ is a kernel type estimator of the probability density function $f(x)$. Properties of such estimators are discussed in Prakasa Rao (1983).

Suppose that the probability density function f has continuous and bounded second derivative with $\sup_x |f''| = C_f < \infty$. Let

$$W_n = \sup_{-\infty<x<\infty} |\hat{f}_n(x) - f(x)|. \tag{3. 3}$$

Observe that

$$\begin{aligned}
|\hat{f}_n(x) - f(x)| &\leq |\frac{1}{h_n}\int_{-\infty}^{\infty} J(\frac{x-y}{h_n})dF_n(y) - \frac{1}{h_n}\int_{-\infty}^{\infty} J(\frac{x-y}{h_n})dF(y)| \qquad (3.\ 4)\\
&\quad +|\frac{1}{h_n}\int_{-\infty}^{\infty} J(\frac{x-y}{h_n})dF(y) - f(x)| \\
&\leq \frac{1}{h_n}\sup_{-\infty<y<\infty} |F_n(y) - F(y)| \, |\int_{-\infty}^{\infty} dJ(\frac{x-y}{h_n})| + \frac{h_n^2}{2}\sigma_J^2 C_f \\
&= \frac{1}{h_n}\sup_{-\infty<y<\infty} |F_n(y) - F(y)| V_J + \frac{h_n^2}{2}\sigma_J^2 C_f \\
&\leq \frac{1}{h_n\sqrt{n}} D_n V_J + \frac{h_n^2}{2}\sigma_J^2 C_f.
\end{aligned}$$

Hence

$$E(W_n) \leq \frac{1}{h_n\sqrt{n}} E(D_n) V_J + \frac{h_n^2}{2}\sigma_J^2 C_f. \tag{3. 5}$$

Applying the bound on $E(D_n)$ derived in the equation (2.5), we have

$$E(W_n) \leq \frac{1}{h_n\sqrt{n}}(1 + C_k(\frac{1}{2k-1}))V_J + \frac{h_n^2}{2}\sigma_J^2 C_f \tag{3. 6}$$

for any positive integer $k \geq 1$. Choosing h_n such that $\frac{1}{h_n\sqrt{n}} = h_n^2$, that is, $h_n = n^{-1/6}$, one can get an optimum bound on $E(W_n)$ as far as the rate of convergence is concerned and

$$E(W_n) \leq n^{-1/3}[(1 + C_k(\frac{1}{2k-1}))V_J + \frac{1}{2}\sigma_J^2 C_f]. \tag{3. 7}$$

Let us now consider the problem of estimation of

$$I(f) = \int_{-\infty}^{\infty} f^2(x)dx. \tag{3. 8}$$

An estimator of $I(f)$ is $I(\hat{f}_n)$. Note that

$$\begin{aligned}
|I(\hat{f}_n) - I(f)| &= |\int_{-\infty}^{\infty} (\hat{f}_n(x) - f(x))(\hat{f}_n(x) + f(x))dx| \\
&\leq \int_{-\infty}^{\infty} |(\hat{f}_n(x) - f(x))||(\hat{f}_n(x) + f(x))|dx \\
&\leq 2 \sup_{-\infty<x<\infty} |\hat{f}_n(x) - f(x)| \\
&= 2W_n.
\end{aligned}$$

Hence

$$E|I(\hat{f}_n) - I(f)| \leq 2[\frac{1}{h_n\sqrt{n}}(1 + C_k(\frac{1}{2k-1}))V_J + \frac{h_n^2}{2}\sigma_J^2 C_f]. \quad (3.\ 9)$$

If $h_n = n^{-1/6}$, then the above bound reduces to

$$E|I(\hat{f}_n) - I(f)| \leq 2n^{-1/3}[(1 + C_k(\frac{1}{2k-1}))V_J + \frac{1}{2}\sigma_J^2 C_f]. \quad (3.\ 10)$$

4 Application to Estimation of Distribution Function

Suppose a stationary ϕ-mixing process $\{X_i, i \geq 1\}$ is observed up to time n with the ϕ-mixing sequence satisfying the condition (2.2) and with the one-dimensional probability distribution function F. The problem is to estimate the distribution function F based on the observations $\{X_i, 1 \leq i \leq n\}$.

Let $R_n(x)$ be a sequence of distribution functions converging weakly to the distribution function $R(x)$ degenerate at zero such that

$$\sup_{-\infty<x<\infty} |R_n(x) - R(x)| = o(\delta_n) \quad (4.\ 1)$$

where $\delta_n \to 0$ as $n \to \infty$. Define

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} R_n(x - X_i). \quad (4.\ 2)$$

Let

$$\begin{aligned} Z_n &= \sup_{-\infty<x<\infty} |\hat{F}_n(x) - F(x)| \quad (4.\ 3) \\ &\leq \sup_{-\infty<x<\infty} |\hat{F}_n(x) - E\hat{F}_n(x)| + \sup_{-\infty<x<\infty} |E\hat{F}_n(x) - F(x)|. \end{aligned}$$

But

$$(4.\ 4)$$

$$\begin{aligned} \sup_{-\infty<x<\infty} |\hat{F}_n(x) - E\hat{F}_n(x)| &= \sup_{-\infty<x<\infty} |\int_{-\infty}^{\infty} R_n(x-y)dF_n(y) \\ &\quad - \int_{-\infty}^{\infty} R_n(x-y)dF(y)| \\ &= \sup_{-\infty<x<\infty} |\int_{-\infty}^{\infty} (F_n(y) - F(y))dR_n(x-y)| \\ &\leq \frac{D_n}{\sqrt{n}}. \end{aligned}$$

Therefore

$$Z_n \leq \frac{D_n}{\sqrt{n}} + \sup_{-\infty<x<\infty} |E\hat{F}_n(x) - F(x)|. \quad (4.\ 5)$$

It can be checked that

$$
\begin{aligned}
\sup_{-\infty<x<\infty} |E\hat{F}_n(x) - F(x)| &\leq \sup_{-\infty<x<\infty} |\int_{-\infty}^{\infty} |R_n(x-y) - R(x-y)| f(y)dy \qquad (4.\ 6) \\
&\leq \sup_{-\infty<x<\infty} |R_n(x) - R(x)| \int_{-\infty}^{\infty} f(y)dy \\
&\leq \delta_n.
\end{aligned}
$$

Hence

$$
E(Z_n) \leq \frac{E(D_n)}{\sqrt{n}} + \delta_n. \qquad (4.\ 7)
$$

Applying the inequality (2.5), we get that

$$
E(Z_n) \leq \frac{1 + C_k(\frac{1}{2k-1})}{\sqrt{n}} + \delta_n. \qquad (4.\ 8)
$$

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