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Berry-Esseen bound for MLE for linear stochastic differential equations driven by fractional Brownian motion

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Berry-Esseen Bound for MLE for Linear Stochastic Differential Equations Driven by Fractional Brownian Motion

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Abstract

We investigate the rate of convergence of the distribution of the maximum likelihood estimator (MLE) of an unknown parameter in the drift coefficient of a stochastic process described by a linear stochastic differential equation driven by a fractional Brownian Motion (fBM). As a special case, we obtain the rate of convergence for the case of the fractional Ornstein-Uhlenbeck type process studied recently by Kleptsyna and Le Breton (2002).

Keywords and phrases: Linear stochastic differential equations ; fractional Ornstein-Uhlenbeck type process; fractional Brownian motion; Maximum likelihood estimation; Berry-Esseen bound.

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1 Introduction

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999a). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion. Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. In a recent paper, Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck type process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0. \tag{1. 1}$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$. In a recent paper (cf. Prakasa Rao (2003)), We studied more general classes of stochastic processes satisfying linear stochastic differential equations driven by a fractional

Brownian motion and investigated the asymptotic properties of the maximum likelihood and the Bayes estimators for parameters involved in such processes. We now discuss rates of convergence of the distribution of the maximum likelihood estimator for such processes.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions and the processes discussed in the following are $(\mathcal{F}_t)$-adapted. Further the natural fitration of a process is understood as the P-completion of the filtration generated by this process. Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with Hurst parameter $H \in (0,1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, s \geq 0. \tag{2. 1}$$

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$Y_t = \int_0^t C(s)ds + \int_0^t B(s)dW_s^H, t \geq 0 \tag{2. 2}$$

where $C = \{C(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process and $B(t)$ is a nonvanishing nonrandom function. For convenience we write the above integral equation in the form of a stochastic differential equation

$$dY_t = C(t)dt + B(t)dW_t^H, t \geq 0; Y_0 = 0 \tag{2. 3}$$

driven by the fractional Brownian motion W^H. The integral

$$\int_0^t B(s)dW_s^H \tag{2. 4}$$

is not a stochastic integral in the Ito sense but one can define the integral of a deterministic function with respect to the fBM in a natural sense (cf. Norros et al. (1999), Alos et al. (2001)). Even though the process Y is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration $(\mathcal{Z}_t)$ of the process Z coincides with the natural filtration $(\mathcal{Y}_t)$ of the process Y (Kleptsyna et al. (2000)). Define, for $0 < s < t$,

$$k_H = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(H + \frac{1}{2}\right), \tag{2. 5}$$

$$k_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}, \tag{2. 6}$$

$$\lambda_H = \frac{2H\,\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}, \tag{2. 7}$$

$$w_t^H = \lambda_H^{-1} t^{2-2H}, \tag{2. 8}$$

and

$$M_t^H = \int_0^t k_H(t,s)dW_s^H, t \geq 0. \tag{2. 9}$$

The process M^H is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al. (1999)) and its quadratic variation $< M_t^H >= w_t^H$. Further more the natural filtration of the martingale M^H coincides with the natural fitration of the fBM W^H. In fact the stochastic integral

(2. 10) $$\int_0^t B(s)dW_s^H$$

can be represented in terms of the stochastic integral with respect to the martingale M^H. For a measurable function f on $[0,T]$, let

(2. 11) $$K_H^f(t,s) = -2H\frac{d}{ds}\int_s^t f(r)r^{H-\frac{1}{2}}(r-s)^{H-\frac{1}{2}}dr, 0 \le s \le t$$

when the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure(see Samko et al. (1993) for sufficient conditions). The following result is due to Kleptsyna et al. (2000).

Therorem 2.1: Let M^H be the fundamental martingale associated with the fBM W^H defined by (2.9). Then

(2. 12) $$\int_0^t f(s)dW_s^H = \int_0^t K_H^f(t,s)dM_s^H, t \in [0,T]$$

a.s $[P]$ whenever both sides are well defined.

Suppose the sample paths of the process $\{\frac{C(t)}{B(t)}, t \ge 0\}$ are smooth enough (see Samko et al. (1993)) so that

(2. 13) $$Q_H(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{C(s)}{B(s)}ds, t \in [0,T]$$

is well-defined where w^H and k_H are as defined in (2.8) and (2.6) respectively and the derivative is understood in the sense of absoulute continuity. The following theorem due to Kleptsyna et al. (2000) associates a *fundamental semimartingale* Z associated with the process Y such that the natural filtration $(\mathcal{Z}_t)$ of Z coincides with the natural filtration $(\mathcal{Y}_t)$ of Y.

Theorem 2.2: Suppose the sample paths of the process Q_H defined by (2.13) belong P-a.s to $L^2([0,T], dw^H)$ where w^H is as defined by (2.8). Let the process $Z = (Z_t, t \in [0,T])$ be defined by

(2. 14) $$Z_t = \int_0^t k_H(t,s)B^{-1}(s)dY_s$$

where the function $k_H(t,s)$ is as defined in (2.6). Then the following results hold:
(i) The process Z is an $(\mathcal{F}_t)$ -semimartingale with the decomposition

(2. 15) $$Z_t = \int_0^t Q_H(s)dw_s^H + M_t^H$$

where M^H is the fundamental martingale defined by (2.9), (ii) the process Y admits the representation

(2. 16) $$Y_t = \int_0^t K_H^B(t,s)dZ_s$$

where the function K_H^B is as defined in (2.11), and (iii) the natural fitrations $(\mathcal{Z}_t)$ and $(\mathcal{Y}_t)$ coincide.

Kleptsyna et al. (2000) derived the following Girsanov-type formula as a consequence of the Theorem 2.2.

Theorem 2.3: Suppose the assumptions of Theorem 2.2 hold. Define

$$\Lambda_H(T) = \exp\{-\int_0^T Q_H(t)dM_t^H - \frac{1}{2}\int_0^t Q_H^2(t)dw_t^H\}. \tag{2. 17}$$

Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^* = \Lambda_H(T)P$ is a probability measure and the probability measure of the process Y under P^* is the same as that of the process V defined by

$$V_t = \int_0^t B(s)dW_s^H, 0 \leq t \leq T \tag{2. 18}$$

under the probability measure P.

3 Maximum likelihood estimation

Let us consider the stochastic differential equation

$$dX(t) = [a(t, X(t)) + \theta\, b(t, X(t))]dt + \sigma(t)dW_t^H, t \geq 0 \tag{3. 1}$$

where $\theta \in \Theta \subset R, W = \{W_t^H, t \geq 0\}$ is a fractional Brownian motion with known Hurst parameter $H \in (0, 1)$ and $\sigma(t)$ is a positive nonvanishing function on $[0, \infty)$. In other words $X = \{X_t, t \geq 0\}$ is a stochastic process satisfying the stochastic integral equation

$$X(t) = X(0) + \int_0^t [a(s, X(s)) + \theta\, b(s, X(s))]ds + \int_0^t \sigma(s)dW_s^H, t \geq 0. \tag{3. 2}$$

Let

$$C(\theta, t) = a(t, X(t)) + \theta\, b(t, X(t)), t \geq 0 \tag{3. 3}$$

and assume that the sample paths of the process $\{\frac{C(\theta,t)}{\sigma(t)}, t \geq 0\}$ are smooth enough so that the the process

$$Q_{H,\theta}(t) = \frac{d}{dw_t^H}\int_0^t k_H(t, s)\frac{C(\theta, s)}{\sigma(s)}ds, t \geq 0 \tag{3. 4}$$

is well-defined where w_t^H and $k_H(t, s)$ are as defined in (2.8) and (2.6) respectively. Suppose the sample paths of the process $\{Q_{H,\theta}, 0 \leq t \leq T\}$ belong almost surely to $L^2([0, T], dw_t^H)$. Define

$$Z_t = \int_0^t \frac{k_H(t, s)}{\sigma(s)}dX_s, t \geq 0. \tag{3. 5}$$

Then the process $Z = \{Z_t, t \geq 0\}$ is an $(\mathcal{F}_t)$-semimartingale with the decomposition

$$Z_t = \int_0^t Q_{H,\theta}(s)dw_s^H + M_t^H \tag{3. 6}$$

where M^H is the fundamental martingale defined by (2.9) and the process X admits the representation

$$X_t = \int_0^t K_H^\sigma(t,s)dZ_s \tag{3. 7}$$

where the function K_H^σ is as defined by (2.11). Let P_θ^T be the measure induced by the process $\{X_t, 0 \le t \le T\}$ when θ is the true parameter. Following Theorem 2.3, we get that the Radon-Nikodym derivative of P_θ^T with respect to P_0^T is given by

$$\frac{dP_\theta^T}{dP_0^T} = \exp[\int_0^T Q_{H,\theta}(s)dZ_s - \frac{1}{2}\int_0^T Q_{H,\theta}^2(s)dw_s^H]. \tag{3. 8}$$

We now consider the problem of maximum likelihood estimation of the parameter θ based on the observation of the process $X = \{X_t, 0 \le t \le T\}$ and study the rate of convergence of the distribution of a suitably normalized maximum likelihood estimator as $T \to \infty$.

Let $L_T(\theta)$ denote the Radon-Nikodym derivative $\frac{dP_\theta^T}{dP_0^T}$. The maximum likelihood estimator (MLE) is defined by the relation

$$L_T(\hat{\theta}_T) = \sup_{\theta\in\Theta} L_T(\theta). \tag{3. 9}$$

We assume that there exists a measurable maximum likelihood estimator. Sufficient conditions can be given for the existence of such an estimator (cf. Lemma 3.1.2, Prakasa Rao (1987)). Note that

$$\begin{aligned} Q_{H,\theta}(t) &= \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{C(\theta,s)}{\sigma(s)}ds \\ &= \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{a(s,X(s))}{\sigma(s)}ds + \theta\frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{b(s,X(s))}{\sigma(s)}ds \\ &= J_1(t) + \theta J_2(t).(say) \end{aligned} \tag{3. 10}$$

Then

$$\log L_T(\theta) = \int_0^T (J_1(t) + \theta J_2(t))dZ_t - \frac{1}{2}\int_0^T (J_1(t)+\theta J_2(t))^2 dw_t^H \tag{3. 11}$$

and the likelihood equation is given by

$$\int_0^T J_2(t)dZ_t - \int_0^T (J_1(t)+\theta J_2(t))J_2(t)dw_t^H = 0. \tag{3. 12}$$

Hence the MLE $\hat{\theta}_T$ of θ is given by

$$\hat{\theta}_T = \frac{\int_0^T J_2(t)dZ_t - \int_0^T J_1(t)J_2(t)dw_t^H}{\int_0^T J_2^2(t)dw_t^H}. \tag{3. 13}$$

Let θ_0 be the true parameter. Using the fact that

$$dZ_t = (J_1(t)+\theta_0 J_2(t))dw_t^H + dM_t^H \tag{3. 14}$$

when θ_0 is the true parameter, it can be shown that

$$\frac{dP_\theta^T}{dP_{\theta_0}^T} = \exp[(\theta - \theta_0)\int_0^T J_2(t)dM_t^H - \frac{1}{2}(\theta - \theta_0)^2 \int_0^T J_2^2(t)dw_t^H]. \tag{3. 15}$$

Following this representation of the Radon-Nikodym derivative, we obtain that

$$\hat{\theta}_T - \theta_0 = \frac{\int_0^T J_2(t)dM_t^H}{\int_0^T J_2^2(t)dw_t^H}. \tag{3. 16}$$

Note that the quadratic variation $< Z >$ of the process Z is the same as the quadratic variation $< M^H >$ of the martingale M^H which in turn is equal to w^H. This follows from the relations (2.15) and (2.9). Hence we obtain that

$$[w_T^H]^{-1} \lim_n \Sigma[Z_{t_{i+1}^{(n)}} - Z_{t_i^{(n)}}]^2 = 1 \text{ a.s } [P_{\theta_0}]$$

where $(t_i^{(n)})$ is a partition of the interval $[0, T]$ such that $\sup_i |t_{i+1}^{(n)} - t_i^{(n)}|$ tends to zero as $n \to \infty$. If the function $\sigma(t)$ is an unknown constant σ, the above property can be used to obtain a strongly consistent estimator of σ^2 based on the continuous observation of the process X over the interval $[0, T]$. Here after we assume that the nonrandom function $\sigma(t)$ is known.

The following results on the strong consistency and the asympotic distribution of the MLE $\hat{\theta}_T$ as $T \to \infty$ were proved in Prakasa Rao (2003).

Strong consistency: Theorem 3.1: The maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent, that is,

$$\hat{\theta}_T \to \theta_0 \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty \tag{3. 17}$$

provided

$$\int_0^T J_2^2(t)dw_t^H \to \infty \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty. \tag{3. 18}$$

Limiting distribution: Let

$$R_t = \int_0^t J_2(s)dM_s^H, t \geq 0. \tag{3. 19}$$

Note that $\{R_t, t \geq 0\}$ is a square integrable local continuous martingale.

We now discuss the limiting distribution of the MLE $\hat{\theta}_T$ as $T \to \infty$.

Theorem 3.2: Assume that the functions $b(t, s)$ and $\sigma(t)$ are such that the process $\{R_t, t \geq 0\}$ is a local continuous martingale and that there exists a norming function $I_t, t \geq 0$ such that

$$I_T^2 < R_T >= I_T^2 \int_0^T J_2^2(t)dw_t^H \to \eta^2 \text{ in probability as } T \to \infty \tag{3. 20}$$

where $I_T \to 0$ as $T \to \infty$ and η is a random variable such that $P(\eta > 0) = 1$. Then

$$(I_T R_T, I_T^2 < R_T >) \to (\eta Z, \eta^2) \text{ in law as } T \to \infty \tag{3. 21}$$

where the random variable Z has the standard normal distribution and the random variables Z and η are independent.

Observe that

$$I_T^{-1}(\hat{\theta}_T - \theta_0) = \frac{I_T R_T}{I_T^2 < R_T >} \tag{3. 22}$$

Applying the Theorem 3.2, we obtain the following result.

Theorem 3.3: Suppose the conditions stated in the Theorem 3.2 hold. Then

$$I_T^{-1}(\hat{\theta}_T - \theta_0) \to \frac{Z}{\eta} \text{ in law as } t \to \infty \tag{3. 23}$$

where the random variable Z has the standard normal distribution and the random variables Z and η are independent.

Remarks: If the random variable η is a constant with probability one, then the limiting distribution of the maximum likelihood estimator is normal with mean 0 and variance η^{-2}. Otherwise it is a mixture of the normal distributions with mean zero and variance η^{-2} with the mixing distribution as that of η.

4 Berry -Esseen type bound

Hereafter we assume that the random variable η in (3.20) is a positive constant with probability one. Hence

$$I_T^{-1}(\hat{\theta}_T - \theta_0) \to N(0, \eta^{-2}) \text{ in law as } t \to \infty \tag{4. 1}$$

where $N(0, \eta^{-2})$ denotes the Gaussian distribution with mean zero and variance η^{-2}.

We will now study the rate of convergence of the asymptotic distribution of the maximum likelihood estimator in (4.1). Suppose there exist nonrandom positive functions α_T decreasing to zero and ε_T decreasing to zero such that

$$\alpha_T^{-1}\varepsilon^2(T) \to \infty \text{ as } T \to \infty, \tag{4. 2}$$

and

$$\sup_{\theta \in \Theta} P_\theta^T[|\alpha_T < R_T > -1| \geq \varepsilon_T] = O(\varepsilon_T^{1/2}) \tag{4. 3}$$

where the process $\{R_t, t \geq 0\}$ is as defined in (3.19). Note that the process $\{R_t, t \geq 0\}$ is a locally square integrable continuous martingale. From the results on the representation of locally square integrable continuous martingales (cf. Ikeda and Watanabe (1981), Chapter II, Thoorem 7.2), it follows that there exists a standard Wiener process $\{\tilde{W}(t), t \geq 0\}$ adapted to $(\mathcal{F}_t)$ such that $R_t = \tilde{W}(< R_t >), t \geq 0$. In particular

$$R_T \alpha_T^{1/2} = \tilde{W}(< R_T > \alpha_T) \text{ a.s. } [P] \tag{4. 4}$$

for all $T \geq 0$.

We use the following lemmas in the sequel. **Lemma 4.1:** Let $(\Omega, \mathcal{F}, P)$ be a probability space and f and g be $\mathcal{F}$-measurable functions. Then, for any $\varepsilon > 0$,

$$\begin{aligned}(4.5)\qquad & \sup_x |P(\omega : \frac{f(\omega)}{g(\omega)} \le x) - \Phi(x)| \\ & \le \sup_y |P(\omega : f(\omega) \le y) - \Phi(x)| + P(\omega : |g(\omega) - 1| > \varepsilon) + \varepsilon\end{aligned}$$

where $\Phi(x)$ is the distribution function of the standard Gaussian distribution. Proof: See Michel and Pfanzagl (1971).

Lemma 4.2: Let $\{W(t), t \ge 0\}$ be a standard Wiener process and V be a nonnegative random variable. Then, for every $x \in R$ and $\varepsilon > 0$,

$$(4.6)\qquad |P(W(V) \le x) - \Phi(x)| \le (2\varepsilon)^{1/2} + P(|V - 1| > \varepsilon).$$

Proof: See Hall and Heyde (1980), p.85.

Let us fix $\theta \in \Theta$. It is clear from the earlier remarks that

$$(4.7)\qquad R_T =< R_T > (\hat{\theta}_T - \theta)$$

under P_θ^T measure. Then it follows, from the Lemmas 4.1 and 4.2, that

$$\begin{aligned}(4.8)\qquad & P_\theta^T[\alpha_T^{-1/2}(\hat{\theta}_T - \theta) \le x] - \Phi(x)| \\ & = |P_\theta^T[\frac{R_T}{< R_T >}\alpha_T^{-1/2} \le x] - \Phi(x)| \\ & = |P_\theta^T[\frac{R_T/\alpha_T^{-1/2}}{< R_T > /\alpha_T^{-1}} \le x] - \Phi(x)| \\ & \le \sup_x |P_\theta^T[R_T \alpha_T^{1/2} \le x] - \Phi(x)| \\ & \quad + P_\theta^T[|\alpha_T < R_T > -1| \ge \varepsilon_T] + \varepsilon_T \\ & = \sup_y |P(\tilde{W}(< R_T > \alpha_T) \le y) - \Phi(y)| + P_\theta^T[|\alpha_T < R_T > -1| \ge \varepsilon_T] + \varepsilon_T \\ & \le (2\varepsilon_T)^{1/2} + 2P_\theta^T[|\alpha_T < R_T > -1| \ge \varepsilon_T] + \varepsilon_T.\end{aligned}$$

It is clear that the bound obtained above is of the order $O(\varepsilon_T^{1/2})$ under the condition (4.3) and it is uniform in $\theta \in \Theta$. Hence we have the following result.

Theorem 4.3: Under the conditions (4.2) and (4.3),

$$\begin{aligned}(4.9)\qquad & \sup_{\theta \in \Theta} \sup_{x \in R} |P_\theta^T[\alpha_T^{-1/2}(\hat{\theta}_T - \theta) \le x] - \Phi(x)| \\ & \le (2\varepsilon_T)^{1/2} + 2P_\theta^T[|\alpha_T < R_T > -1| \ge \varepsilon_T] + \varepsilon_T = O(\varepsilon_T^{1/2}).\end{aligned}$$

As a consequence of this result, we have the following theorem giving the rate of convergence of the MLE $\hat{\theta}_T$.

Theorem 4.4: Suppose the conditions (4.2) and (4.3) hold. Then there exists a constant $c > 0$ such that for every $d > 0$,

$$\sup_{\theta\in\Theta} P_\theta^T[|\hat{\theta}_T - \theta| \geq d] \leq c\varepsilon_T^{1/2} + 2P_\theta^T[|\alpha_T < R_T > -1| \geq \varepsilon_T] = O(\varepsilon_T^{1/2}). \tag{4. 10}$$

Proof: Observe that

$$\begin{aligned}
&\sup_{\theta\in\Theta} P_\theta^T[|\hat{\theta}_T - \theta| \geq d] \\
&\quad\leq \sup_{\theta\in\Theta} |P_\theta^T[\alpha_T^{-1/2}(\hat{\theta}_T - \theta) \geq d\alpha_T^{-1/2}] - 2(1 - \Phi(d\alpha_T^{-1/2}))| \\
&\qquad +2(1 - \Phi(d\alpha_T^{-1/2})) \\
&\quad\leq (2\varepsilon_T)^{1/2} + 2\sup_{\theta\in\Theta} P_\theta^T[|\alpha_T < R_T > -1| \geq \varepsilon_T] + \varepsilon_T \\
&\qquad +2d^{-1/2}\alpha_T^{1/2}(2\pi)^{-1/2}\exp[-\frac{1}{2}\alpha_T^{-1}d^2]
\end{aligned} \tag{4. 11}$$

by Theorem 4.3 and the inequality

$$1 - \Phi(x) < \frac{1}{x\sqrt{2\pi}}\exp[-\frac{1}{2}x^2] \tag{4. 12}$$

for all $x > 0$ (cf. Feller (1968), p.175). Since

$$\alpha_T^{-1}\varepsilon^2(T) \to \infty \text{ as } T \to \infty,$$

by the condition (4.2), it follows that

$$\sup_{\theta\in\Theta} P_\theta^T[|\hat{\theta}_T - \theta| \geq d] \leq c\varepsilon_T^{1/2} + 2\sup_{\theta\in\Theta} P_\theta^T[|\alpha_T < R_T > -1| \geq \varepsilon_T] \tag{4. 13}$$

for some constant $c > 0$ and the last term is of the order $O(\varepsilon_T^{1/2})$ by the condition (4.3). This proves Theorem 4.4.

5 fractional Ornstein-Uhlenbeck type process

Following the notation introduced by Kleptsyna and Le Breton (2002), we now discuss the rate of convergence for the maximum likelihood estimator for fractional Ornstein-Uhlenbeck type process studied by them. Consider a stochastic process $X = \{X_t, t \geq 0\}$ satisfying the stochastic integral equation

$$X_t = \theta\int_0^t X_s ds + W_t^H, t \geq 0 \tag{5. 1}$$

where $\theta \in \Theta \subset R$ is a drift parameter and $\{W_t^H, t \geq 0$ is the fractional Brownian Motion with known Hurst coefficient H. We now assume that $H \in [\frac{1}{2}, 1)$. The equation (5.1) can be written formally in the form of a stochastic differential equation

$$dX_t = \theta X_t dt + dW_t^H, X_0 = 0, t \geq 0. \tag{5. 2}$$

Suppose the process X is observed over the interval $[0,T]$. Let $X^T = \{X_t, 0 \leq t \leq T\}$. For k_H and w^H defined earlier by (2.5) and (2.8), the sample paths of the process are smooth enough so that the process Q is well-defined by the integral

$$Q(t) = \frac{d}{dw_t^H} \int_0^t k_H(t,s)X(s)ds, t \geq 0 \tag{5. 3}$$

where the derivative is understood in the sense of absolute continuity with respect to the measure generated by w^H. Further more the sample paths of the process $\{Q(t), 0 \leq t \leq T\}$ belong P-a.s to $L^2([0,T], dw^H)$. For $H \in (\frac{1}{2}, 1)$, define

$$K_H(t,s) = H(2H-1)\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr, 0 \leq s \leq t \tag{5. 4}$$

and we define $K_{1/2} \equiv 1$. Let

$$M_t^H = \int_0^t k_H(t,s)dW_s^H, t \geq 0 \tag{5. 5}$$

and

$$Z_t = \int_0^t k_H(t,s)dX_s, t \geq 0.(19) \tag{5. 6}$$

The following result is due to Kleptsyna and Le Breton (2002).

Theorem 5.1: Let the process Z be defined by the relation (5.6). Then (i) The process Z is an $(\mathcal{F}_t)$ -semimartingale with the decomposition

$$Z_t = \theta \int_0^t Q(s)dw_s^H + M_t^H \tag{5. 7}$$

where M^H is the Gaussian martingale defined by (5.5),and (ii) the process X admits the representation

$$X_t = \int_0^t K_H(t,s)dZ_s \tag{5. 8}$$

where the function K_H^B is as defined in (5.4) and the natural filtrations $(\mathcal{Z}_t)$ and $(\mathcal{X}_t)$ of Z and X respectively coincide.

Let P_θ^T be the probability measure induced by the process X^T when θ is the true parameter. Let P_0^T be the probability measure induced by the fBM. Then it follows, from the Girasnov type formula given in Kleptsyna et al.(2000), that

$$\frac{dP_\theta^T}{dP_0^T} = \exp[\theta \int_0^T Q(t)dZ_t - \frac{1}{2}\theta^2 \int_0^T Q^2(t)dw_t^H]. \tag{5. 9}$$

Hence the maximum likelihood estimator $\hat{\theta}_T$ based on the observation of the process X on $[0,T]$ is given by

$$\hat{\theta}_T = \frac{\int_0^T Q(t)dZ_t}{\int_0^T Q^2(t)dw_t^H}. \tag{5. 10}$$

If θ is the true parameter, then

$$\hat{\theta}_T - \theta = \frac{\int_0^T Q(t)dM_t^H}{\int_0^T Q^2(t)dw_t^H}. \tag{5. 11}$$

Note that the quadratic variation process $< M >$ of the martingale M and the quadratic variation of the semimartingale Z are both equal to w^H almost surely. Kleptsyna and Le Breton (2002) obtained expressions for the bias and the mean square error of the estimator $\hat{\theta}_T$ in terms of the function

$$\psi_T^H(\theta; a) = E_\theta[\exp(-a\int_0^T Q^2(s)dw_s^H)], a > 0. \tag{5. 12}$$

where E_θ denotes the expectation when θ is the true parameter. They have obtained a closed form expression for this function involving modified Bessel functions of the first kind (cf. Watson (1995)) and analyzed the asymptotic behaviour as $T \to \infty$ for different values of θ. It follows that

$$E_\theta(\int_0^T Q^2(s)dw_s^H) = -\lim_{a\to 0+} \frac{d\psi_T^H(\theta; a)}{da} \tag{5. 13}$$

from (5.12). Let

$$\mathcal{L}_T^H(\theta; \rho) = E_\theta[\exp(-\rho\int_0^T Q(s)dZ_s)], \rho > 0. \tag{5. 14}$$

Kleptsyna and Le Breton (2002) have also obtained explicit expression for the function $\mathcal{L}_T^H(\theta; \rho)$ again in terms of the modified Bessel functions of the first kind and one can show that

$$E_\theta(\int_0^T Q(s)dZ_s) = -\lim_{\rho\to 0+} \frac{d\mathcal{L}_T^H(\theta; \rho)}{d\rho}. \tag{5. 15}$$

It seems to be difficult to obtain an explicit functional form for the expectations defined in (5.13) and (5.15). Suppose there exists functions α_T decreasing to zero as $T \to \infty$ and ε_T decreasing to zero as $T \to \infty$ such that

$$\sup_{\theta\in\Theta} P_\theta^T[|\alpha_T \int_0^T Q^2(t)dw_t^H - 1| \geq \varepsilon_T] = O(\varepsilon_T^{1/2}). \tag{5. 16}$$

Then it follows that

$$\sup_{\theta\in\Theta}\sup_x |P_\theta^T[|\alpha_T^{-1/2}(\hat{\theta}_T - \theta] \leq x] - \Phi(x)| = O((\varepsilon_T^{1/2}). \tag{5. 17}$$

and

$$\sup_{\theta\in\Theta}\sup_d P_\theta^T[|\alpha_T^{-1/2}(\hat{\theta}_T - \theta] \geq d] = O((\varepsilon_T^{1/2}) \tag{5. 18}$$

from the Theorems 4.3 and 4.4. **Remarks:** One can approach the above problem by computing the joint characteristic function of the vector

$$(\int_0^T Q(s)dZ_s, \int_0^T Q^2(s)dw_s^H)$$

explicitly by using the results in Kleptsyna and Le Breton (2002) and then following the technique in Bose (1986) using the Esseen's lemma. However this approach does not seem to be helpful in view of the complex nature of the above characteristic function involving the modified Bessel functions of first kind.

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