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Identification for linear stochastic Systems driven by fractional Brownian motion

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Identification for Linear Stochastic Systems Driven by Fractional Brownian Motion

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Abstract

We apply Grenander's method of sieves to the problem of identification or estimation of the "drift" function for linear stochastic systems driven by a fractional Brownian motion (fBm). We use an increasing sequence of finite dimensional subspaces of the parameter space as the natural sieves on which we maximise the likelihood function.

Keywords and phrases:Linear stochastic systems; Stochastic differential equations; fractional Ornstein-Uhlenbeck process; fractional Brownian motion; Identification; Nonparametric estimation; Consistency; Asymptotic normality; Method of sieves.

AMS Subject classification (2000): Primary 62M09, Secondary 60G15.

1 Introduction

Stochastic models for modeling long-range dependence has been the subject of investigation recently and it is interesting to study whether the theory developed for continuous time stochastic systems driven by a Brownian motion has an analogue for the systems driven by a fractional Brownian motion. Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999a). Inference for a general class of semimartingales is reviewed in Prakasa Rao (1999b). Since a fBm is not a semimartingale, there has been a recent interest to study similar problems for stochastic systems driven by a fractional Brownian motion. Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. In a recent paper, Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0. \tag{1. 1}$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$. Maximum likelihood estimation for a more general class of stochastic differential equations driven by a fBm were studied recently in Prakasa Rao (2003a,b). Sequential estimation of the drift for fractional Ornstein-Uhlenbeck type process was investigated in Prakasa Rao (2003c). We now discuss the problem of nonparametric estimation or identification of the "drift" function $\theta(t)$ for a class of stochastic processes satisfying a stochastic differential equation

(1. 2) $$dX_t = \theta(t)X_t dt + dW_t^H, X_0 = \tau, t \geq 0$$

where τ is a gaussian random variable and $\{W_t^H\}$ is a fBm . We use the method of sieves and study the asymptotic properties of the estimator. Identification of nonstationary diffusion models by the method of sieves is studied in Nguyen and Pham (1982).

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions and the processes discussed in the following are $(\mathcal{F}_T)$-adapted. Further the natural fitration of a process is understood as the P-completion of the filtration generated by this process. Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with Hurst parameter $H \in (0,1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

(2. 1) $$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, s \geq 0.$$

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

(2. 2) $$Y_t = \int_0^t C(s)ds + \int_0^t B(s)dW_s^H, t \geq 0$$

where $C = \{C(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process and $B(t)$ is a nonvanishing nonrandom function. For convenience we write the above integral equation in the form of a stochastic differential equation

(2. 3) $$dY_t = C(t)dt + B(t)dW_t^H, t \geq 0$$

driven by the fractional Brownian motion W^H. The integral

(2. 4) $$\int_0^t B(s)dW_s^H$$

is not a stochastic integral in the Ito sense but one can define the integral of a deterministic function with respect to the fBM in a natural sense(cf. Norros et al. (1999)). Even though the process Y is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration $(\mathcal{Z}_t)$ of the process Z coincides with the natural filtration $(\mathcal{Y}_t)$ of the process Y (Kleptsyna et al. (2000)). Define, for $0 < s < t$,

(2. 5) $$k_H = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(H + \frac{1}{2}\right),$$

(2. 6) $$k_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H},$$

(2. 7) $$\lambda_H = \frac{2H\ \Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)},$$

(2. 8) $$w_t^H = \lambda_H^{-1} t^{2-2H},$$

and

(2. 9) $$M_t^H = \int_0^t k_H(t,s)dW_s^H, t \geq 0.$$

The process M^H is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al. (1999)), and its quadratic variance $< M_t^H >= w_t^H$. Further more the natural filtration of the martingale M^H coincides with the natural fitration of the fBm W^H. In fact the stochastic integral

(2. 10) $$\int_0^t B(s)dW_s^H$$

can be represented in terms of the stochastic integral with respect to the martingale M^H. For a measurable function f on $[0,T]$, let

(2. 11) $$K_H^f(t,s) = -2H\frac{d}{ds}\int_s^t f(r)r^{H-\frac{1}{2}}(r-s)^{H-\frac{1}{2}}dr, 0 \leq s \leq t$$

where the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure(see Samko et al. (1993) for sufficient conditions). The following result is due to Kleptsyna et al. (2000).

Therorem 2.1: Let M^H be the fundamental martingale associated with the fBm W^H defined by (2.9). Then

(2. 12) $$\int_0^t f(s)dW_s^H = \int_0^t K_H^f(t,s)dM_s^H, t \in [0,T]$$

a.s $[P]$ whenever both sides are well defined.

Suppose the sample paths of the process $\{\frac{C(t)}{B(t)}, t \geq 0\}$ are smooth enough (see Samko et al. (1993)) so that

(2. 13) $$Q_H(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{C(s)}{B(s)}ds, t \in [0,T]$$

is welldefined where w^H and k_H are as defined in (2.8) and (2.6) respectively and the derivative is understood in the sense of absoulute continuity. The following theorem due to Kleptsyna et al. (2000) associates a *fundamental semimartingale* Z associated with the process Y such that the natural filtration $(\mathcal{Z}_t)$ coincides with the natural filtration $(\mathcal{Y}_t)$ of Y.

Theorem 2.2: Suppose the sample paths of the process Q_H defined by (2.13) belong P-a.s to $L^2([0,T], dw^H)$ where w^H is as defined by (2.8). Let the process $Z = (Z_t, t \in [0,T])$ be defined by

(2. 14) $$Z_t = \int_0^t k_H(t,s)B^{-1}(s)dY_s$$

where the function $k_H(t,s)$ is as defined in (2.6). Then the following results hold: (i) The process Z is an $(\mathcal{F}_t)$ -semimartingale with the decomposition

$$Z_t = \int_0^t Q_H(s)dw_s^H + M_t^H \tag{2. 15}$$

where M^H is the fundamental martingale defined by (2.9), (ii) the process Y admits the representation

$$Y_t = \int_0^t K_H^B(t,s)dZ_s \tag{2. 16}$$

where the function K_H^B is as defined in (2.11), and (iii) the natural fitrations of $(\mathcal{Z}_t)$ and $(\mathcal{Y}_t)$ coincide.

Kleptsyna et al. (2000) derived the following Girsanov type formula as a consequence of the Theorem 2.2.

Theorem 2.3: Suppose the assumptions of Theorem 2.2 hold. Define

$$\Lambda_H(T) = \exp\{-\int_0^T Q_H(t)dM_t^H - \frac{1}{2}\int_0^t Q_H^2(t)dw_t^H\}. \tag{2. 17}$$

Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^* = \Lambda_H(T)P$ is a probability measure and the probability measure of the process Y under P^* is the same as that of the process V defined by

$$V_t = \int_0^t B(s)dW_s^H, 0 \le t \le T. \tag{2. 18}$$

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3 Estimation by the method of sieves

Let us consider the linear stochastic system

$$dX(t) = \theta(t)X(t)dt + dW_t^H, X(0) = \tau, \ \ 0 \le t \le T \tag{3. 1}$$

where $\theta(t) \in L^2([0,T], dt)$, $W = \{W_t^H, t \ge 0\}$ is a fractional Brownian motion with Hurst parameter$H$ and τ is a gaussian random variable indeopendent of the fBm W. In other words $X = \{X_t, t \ge 0\}$ is a stochastic process satisfying the stochastic integral equation

$$X(t) = \tau + \int_0^t \theta(s)X(s)ds + W_t^H, 0 \le t \le T. \tag{3. 2}$$

Let

$$C_\theta(t) = \theta(t)\ X(t), 0 \le t \le T \tag{3. 3}$$

and assume that the sample paths of the process $\{C_\theta(t), 0 \le t \le T\}$ are smooth enough so that the process

$$Q_{H,\theta}(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)C_\theta(s)ds, 0 \le t \le T \tag{3. 4}$$

is welldefined where w_t^H and $k_H(t,s)$ are as defined in (2.8) and (2.6) respectively. Suppose the sample paths of the process $\{Q_H(t), 0 \leq t \leq T\}$ belong almost surely to $L^2([0,T], dw_t^H)$. Define

$$Z_t = \int_0^t k_H(t,s) dX_s, 0 \leq t \leq T. \tag{3. 5}$$

Then the process $Z = \{Z_t, 0 \leq t \leq T\}$ is an $(\mathcal{F}_t)$-semimartingale with the decomposition

$$Z_t = \int_0^t Q_{H,\theta}(s) dw_s^H + M_t^H \tag{3. 6}$$

where M^H is the fundamental martingale defined by (2.9) and the process X admits the representation

$$X_t = X_0 + \int_0^t K_H(t,s) dZ_s \tag{3. 7}$$

where the function K_H is as defined by (2.11) with $f \equiv 1$.. Let P_θ^T be the measure induced by the process $\{X_t, 0 \leq t \leq T\}$ when $\theta(.)$ is the true "drift" function. Following Theorem 2.3, we get that the Radon-Nikodym derivative of P_θ^T with respect to P_0^T is given by

$$\frac{dP_\theta^T}{dP_0^T} = \exp[\int_0^T Q_{H,\theta}(s) dZ_s - \frac{1}{2} \int_0^T Q_{H,\theta}^2(s) dw_s^H]. \tag{3. 8}$$

Suppose the process X is observable on $[0,T]$ and $X_i, 1 \leq i \leq n$ is a random sample of n independent observations of the process X on $[0,T]$. Following the representation of the Radon-Nikodym derivative of P_θ^T with respect to P_0^T given above, it follows that the log-likelihood function corresponding to the observations $\{X_i, 1 \leq i \leq n\}$ is given by

$$\begin{aligned} L_n(X_1, \ldots, X_n; \theta) &\equiv L_n(\theta) \\ &= \sum_{i=1}^n \left(\int_0^T Q_{H,\theta}^{(i)}(s) dZ_i(s) - \frac{1}{2} \int_0^T [Q_{H,\theta}^{(i)}]^2(s) dw_s^H\right). \end{aligned} \tag{3. 9}$$

where the process $Q_{H,\theta}^{(i)}$ is as defined by the relation (3.4) for the process X_i. For convenience in notation, we write $Q_{i,\theta}(s)$ hereafter for $Q_{H,\theta}^{(i)}(s)$. Let $\{V_n, n \geq 1\}$ be an increasing sequence of subspaces of finite dimensions $\{d_n\}$ such that $\cup_{n \geq 1} V_n$ is dense in $L^2([0,T], dt)$. The method of sieves consists in maximizing $L_n(\theta)$ on the subspace V_n. Let $\{e_i\}$ be a set of linearly independent vectors in $L^2([0,T], dt)$ such that the set of vectors $\{e_1, \ldots, e_{d_n}\}$ is a basis for the subspace V_n for every $n \geq 1$. For $\theta \in V_n$, $\theta(.) = \sum_{j=1}^{d_n} \theta_j e_j(.)$, we have

$$\begin{aligned} Q_{i,\theta}(t) &= \frac{d}{dw_t^H} \int_0^t k_H(t,s) \theta(s) X_i(s) ds \\ &= \frac{d}{dw_t^H} \int_0^t k_H(t,s) [\sum_{j=1}^{d_n} \theta_j e_j(s)] X_i(s) ds \\ &= \sum_{j=1}^{d_n} \theta_j \frac{d}{dw_t^H} \int_0^t k_H(t,s) e_j(s) X_i(s) ds \\ &= \sum_{j=1}^{d_n} \theta_j \Gamma_{i,j}(t) \text{ (say).} \end{aligned} \tag{3. 10}$$

Furthermore

$$\int_0^T Q_{i,\theta}(t)dZ_i(t) = \int_0^T [\sum_{j=1}^{d_n} \theta_j \Gamma_{i,j}(t)]dZ_i(t) = \sum_{j=1}^{d_n} \theta_j \int_0^T \Gamma_{i,j}(t)dZ_i(t) = \sum_{j=1}^{d_n} \theta_j R_{i,j} \text{ (say)} \tag{3. 11}$$

and

$$\int_0^T Q_{i,\theta}^2(t)dw_t^H = \int_0^T [\sum_{j=1}^{d_n} \theta_j \Gamma_{i,j}(t)]^2 dw_t^H = \sum_{j=1}^{d_n}\sum_{k=1}^{d_n} \theta_j\theta_k \int_0^T \Gamma_{i,j}(t)\Gamma_{i,k}(t)dw_t^H = \sum_{j=1}^{d_n}\sum_{k=1}^{d_n} \theta_j\theta_k < R_{i,j}, R_{i,k} > \tag{3. 12}$$

where $< .,. >$ denotes the quadratic covariation. Therefore the log-likelihood function corresponding to the observations $\{X_i, 1 \leq i \leq n\}$ is given by

$$L_n(\theta) = \sum_{i=1}^n (\int_0^T Q_{i,\theta}(t)dZ_i(t) - \frac{1}{2}\int_0^T Q_{i,\theta}^2(t)dw_t^H) = \sum_{i=1}^n [\sum_{j=1}^{d_n} \theta_j R_{i,j} - \frac{1}{2}\sum_{j=1}^{d_n}\sum_{k=1}^{d_n} \theta_j\theta_k < R_{i,j}, R_{i,k} >] = n[\sum_{j=1}^{d_n} \theta_j B_j^{(n)} - \frac{1}{2}\sum_{j=1}^{d_n}\sum_{k=1}^{d_n} \theta_j\theta_k A_{j,k}^{(n)}] \tag{3. 13}$$

where

$$B_j^{(n)} = n^{-1}\sum_{i=1}^n R_{i,j}, \quad 1 \leq j \leq d_n \tag{3. 14}$$

and

$$A_{j,k}^{(n)} = n^{-1}\sum_{i=1}^n < R_{i,j}, R_{i,k} >, \quad 1 \leq j,k \leq d_n. \tag{3. 15}$$

Let $\theta^{(n)}, B^{(n)}$ and $A^{(n)}$ be the vectors and the matrix with elements $\theta_j, j = 1, \ldots, d_n$, $B_j^{(n)}, j = 1, \ldots, d_n$ and $A_{j,k}^{(n)}, j,k = 1, \ldots, d_n$ as defined above. Then the log-likelihood function can be written in the form

$$L_n(\theta) = n[B^{(n)}\theta^{(n)} - \frac{1}{2}\theta^{(n)\prime}A^{(n)}\theta^{(n)}]. \tag{3. 16}$$

Here α' denotes the transpose of the vector α. The restricted maximum likelihood estimator $\hat{\theta}^{(n)}(.)$ of $\theta(.)$ is given by

$$\hat{\theta}^{(n)}(.) = \sum_{j=1}^{d_n} \hat{\theta}_j^{(n)} e_j(.) \tag{3. 17}$$

where

(3. 18) $$\hat{\theta}^{(n)} = (\hat{\theta}_1^{(n)}, \ldots, \hat{\theta}_{d_n}^{(n)})$$

is the solution of the equation

(3. 19) $$A^{(n)}\hat{\theta}^{(n)} = B^{(n)}.$$

Assuming that $A^{(n)}$ is invertible, we get that

(3. 20) $$\hat{\theta}^{(n)} = (A^{(n)})^{-1}B^{(n)}.$$

We now construct an orthonormal basis for V_n with respect to a suitable inner product so that the matrix $A^{(n)}$ is transformed into an identity matrix as $n \to \infty$. Note that

(3. 21) $$A_{j,k}^{(n)} \to \int_0^T E[(\frac{d}{dw_t^H}\int_0^t k_H(t,s)e_j(s)X(s)ds)(\frac{d}{dw_t^H}\int_0^t k_H(t,s)e_k(s)X(s)ds)]dw_t^H$$

almost surely as $n \to \infty$ by the strong law of large numbers. We now consider a sequence $\psi_j, j \geq 1$ such that $\psi_j, 1 \leq j \leq d_n$ is an orthonormal basis of V_n in the sense of the inner product

(3. 22)
$$< h, g >= \int_0^T E[(\frac{d}{dw_t^H}\int_0^t k_H(t,s)h(s)X(s)ds)(\frac{d}{dw_t^H}\int_0^t k_H(t,s)g(s)X(s)ds)]dw_t^H.$$

Let $\hat{\eta}_1^{(n)}, \hat{\eta}_2^{(n)}, \ldots, \hat{\eta}_{d_n}^{(n)}$ be the coordinates of $\hat{\theta}^{(n)}(.)$ in the new basis $\psi_j, 1 \leq j \leq d_n$. Then the vector

(3. 23) $$\hat{\eta}^{(n)} = (\hat{\eta}_1^{(n)}, \hat{\eta}_2^{(n)}, \ldots, \hat{\eta}_{d_n}^{(n)})$$

is the solution of the equation

(3. 24) $$a^{(n)}\hat{\eta}^{(n)} = b^{(n)}$$

where $a^{(n)}$ and $b^{(n)}$ are the matrix and the vector with general elements

(3. 25)
$$a_{j,k}^{(n)} = n^{-1}\sum_{i=1}^{n}\int_0^T (\frac{d}{dw_t^H}[\int_0^t k_H(t,s)\psi_j(s)X_i(s)ds]\frac{d}{dw_t^H}[\int_0^t k_H(t,s)\psi_k(s)X_i(s)ds)])dw_t^H,$$

and

(3. 26) $$b_j^{(n)} = n^{-1}\sum_{i=1}^{n}\int_0^T \frac{d}{dw_t^H}[\int_0^t k_H(t,s)\psi_j(s)X_i(s)ds]dZ_i(t).$$

Let $\theta^{(n)}(.) = \sum_{k=1}^{d_n}\eta_i\psi_i(.)$ be the orthogonal projection of $\theta(.)$ onto V_n in the sense of the innerproduct $< .,. >$ defined above. Observe that

(3. 27) $$b_j^{(n)} - \sum_{k=1}^{d_n} a_{j,k}^{(n)}\eta_k$$
$$= n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)dZ_i(t) - \sum_{k=1}^{d_n} a_{j,k}^{(n)}\eta_k$$

$$= n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)[Q_{i,\theta}(t)dw_t^H + dM_t^H]$$

$$-\sum_{k=1}^{d_n} a_{j,k}^{(n)}\eta_k$$

$$= n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)Q_{i,\theta}(t)dw_t^H + n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)dM_t^H$$

$$-\sum_{k=1}^{d_n} a_{j,k}^{(n)}\eta_k$$

$$= n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)(\sum_{r=1}^{\infty}\eta_r Q_{i,\psi_r}(t))dw_t^H$$

$$+n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)dM_t^H$$

$$-\sum_{k=1}^{d_n} a_{j,k}^{(n)}\eta_k$$

$$= n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)(\sum_{r=1}^{d_n}\eta_r Q_{i,\psi_r}(t) + \sum_{r=d_n}^{\infty}\eta_r Q_{i,\psi_r}(t))dw_t^H$$

$$+n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)dM_t^H$$

$$-n^{-1}\sum_{k=1}^{d_n}\eta_k\int_0^T Q_{i,\psi_j}(t)Q_{i,\psi_k}(t)dw_t^H$$

$$= n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)Q_{i,\theta-\theta^{(n)}}(t)dw_t^H$$

$$+n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)dM_t^H$$

$$= n^{-1}\sum_{i=1}^{n}\int_0^T [Q_{i,\psi_j}(t)Q_{i,\theta-\theta^{(n)}}(t) - E(Q_{i,\psi_j}(t)Q_{i,\theta-\theta^{(n)}}(t))]dw_t^H$$

$$+n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)dM_t^H$$

since

(3. 28) $$< \theta - \theta^{(n)}, \psi_j >= 0$$

for $1 \leq j \leq d_n$ by the orthogonality of the basis $\{\psi_k, k \geq 1\}$ and the fact that

(3. 29) $$< \theta - \theta^{(n)}, \psi_j >= E[\int_0^T Q_{i,\psi_j}(t)Q_{i,\theta-\theta^{(n)}}(t)dw_t^H].$$

Hence

(3. 30) $$a^{(n)}(\hat{\eta}^{(n)} - \eta^{(n)}) = c^{(n)}$$

where $\eta^{(n)}$ and $c^{(n)}$ are vectors with components $\eta_j, 1 \leq j \leq d_n$ and

$$c_j^{(n)} = n^{-1}\sum_{i=1}^{n}\int_0^T [Q_{i,\psi_j}(t)Q_{i,\theta-\theta^{(n)}}(t) - E(Q_{i,\psi_j}(t)Q_{i,\theta-\theta^{(n)}}(t))]dw_t^H + n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)dM_t^H. \quad (3.\ 31)$$

Let $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ if $j = k$. In view of the orthonormality of the basis $\{\psi_j, j \geq 1\}$, it follows that

$$(3.\ 32) \quad a_{j,k}^{(n)} - \delta_{j,k} = n^{-1}\sum_{i=1}^{n}\int_0^T (Q_{i,\psi_j}(t)Q_{i,\psi_k}(t) - E[Q_{i,\psi_j}(t)Q_{i,\psi_k}(t)])dw_t^H = n^{-1}\zeta_{ijk} \text{ (say)}$$

and

$$(3.\ 33) \quad c_j^{(n)} = n^{-1}\sum_{i=1}^{n}\int_0^T [Q_{i,\psi_j}(t)Q_{i,\theta-\theta^{(n)}}(t) - E(Q_{i,\psi_j}(t)Q_{i,\theta-\theta^{(n)}}(t))]dw_t^H + n^{-1}\sum_{i=1}^{n}\int_0^T Q_{i,\psi_j}(t)dM_t^H = n^{-1}\sum_{i=1}^{n}\zeta_{ij}^{(n)} + n^{-1}\sum_{i=1}^{n}\tilde{\zeta}_{ij} \text{ (say).}$$

Note that $E[a_{j,k}^{(n)}] = \delta_{jk}$ and $E(\zeta_{ijk}) = 0$. Hence

$$\begin{aligned}(3.\ 34)\ E[a_{j,k}^{(n)} - \delta_{jk}]^2 &= Var(a_{j,k}^{(n)}) \\ &= n^{-1}Var(\zeta_{1jk}) \text{ (since } X_i, 1 \leq i \leq n \text{ are i.i.d.)} \\ &= n^{-1}E(\zeta_{1jk}^2) \\ &= n^{-1}E[\int_0^T (Q_{i,\psi_j}(t)Q_{i,\psi_k}(t) - E[Q_{i,\psi_j}(t)Q_{i,\psi_k}(t)])dw_t^H]^2 \\ &\leq n^{-1}E[\int_0^T (Q_{i,\psi_j}(t)Q_{i,\psi_k}(t) - E[Q_{i,\psi_j}(t)Q_{i,\psi_k}(t)])^2dw_t^H\ \ w_T^H] \\ &\quad \text{(by the Cauchy-Schwarz inequality)} \\ &= n^{-1}(\int_0^T E[Q_{i,\psi_j}(t)Q_{i,\psi_k}(t) - E[Q_{i,\psi_j}(t)Q_{i,\psi_k}(t)]^2]dw_t^H)\ \ w_T^H \\ &\leq n^{-1}w_T^H\int_0^T E[Q_{i,\psi_j}(t)Q_{i,\psi_k}(t)]^2dw_t^H.\end{aligned}$$

Note that the process $\{Q_{H,\theta}(t), t \geq 0\}$ defined by the equation (3.4) is a gaussian process and the fundamental martingale M^H is a gaussian martingale. This follows from the remarks made in the equation (19) in Kleptsyna et al. (2000) and the representation given in the equation (15) of Kleptsyna et al.(2000). We now prove a Lemma to get an upper bound for the expression on the right side of the equation (3.34).

Lemma 3.1: Let $f_i, i = 1, 2$ be gaussian random variables. Then

$$(3.\ 35) \qquad E[f_1^2 f_2^2] \leq 32E(f_1^2)E(f_2^2).$$

Proof: Observe that

$$(E[f_1^2 f_2^2])^2 \leq E(f_1^4)E(f_2^4) \tag{3. 36}$$

by the Cauchy-Schwartz inequality. But

$$\begin{aligned} E(f_i^4) &\leq 8[E|f_i - Ef_i|^4 + |Ef_i|^4] \\ &\quad \text{(by the } C_r\text{-inequality)} \\ &\leq 8[E(f_i - Ef_i)^4 + (E|f_i|)^4] \\ &\leq 8[3(Var(f_i))^2 + (E|f_i|)^4] \\ &\quad \text{(since } f_i \text{ is Gaussian)} \\ &\leq 8[3(Ef_i^2)^2 + ((Ef_i^2)^{1/2})^4] \\ &= (32)(E(f_i^2))^2. \end{aligned} \tag{3. 37}$$

Hence

$$\begin{aligned} (E[f_1^2 f_2^2])^2 &\leq E(f_1^4)E(f_2^4) \\ &\leq (32)^2(E(f_1^2))^2(E(f_2^2))^2 \end{aligned} \tag{3. 38}$$

which proves that

$$E[f_1^2 f_2^2]) \leq (32)E(f_1^2)E(f_2^2). \tag{3. 39}$$

Aplying the Lemma 3.1 on the right side of equation (3.34), we get that

$$\begin{aligned} E[a_{j,k}^{(n)} - \delta_{jk}]^2 &\leq n^{-1}w_T^H \int_0^T E[Q_{i,\psi_j}(t)Q_{i,\psi_k}(t)]^2 dw_t^H \\ &\leq (32)n^{-1}w_T^H \int_0^T E[Q_{i,\psi_j}(t)^2]E[Q_{i,\psi_k}(t)]^2] dw_t^H \\ &= (32)n^{-1}w_T^H \sup_{0\leq t\leq T} E[Q_{i,\psi_j}(t)^2] \int_0^T E[Q_{i,\psi_k}(t)]^2 dw_t^H \\ &= (32)n^{-1}w_T^H \sup_{0\leq t\leq T} E[Q_{i,\psi_j}(t)^2] \end{aligned} \tag{3. 40}$$

since $\int_0^T E(Q_{i,\psi_k}(t))^2 dw_t^H = 1$ by the choice of the orthonormal basis $\psi_j, j \geq 1$.

Observe that $E(\tilde{\zeta}_{ij}) = 0$ and $E(\zeta_{ij}^{(n)}) = 0$. Furthermore

$$\begin{aligned} E(\tilde{\zeta}_{ij}^2) &= E[\int_0^T Q_{i,\psi_j}(t)dM_t^H]^2 \\ &= \int_0^T E[Q_{i,\psi_j}^2(t)]dw_t^H \\ &= 1 \end{aligned} \tag{3. 41}$$

and it follows by the arguments given earlier and Lemma 3.1 that

$$E((\zeta_{ij}^{(n)})^2) \leq (32)w_T^H \sup_{0\leq t\leq T} E[Q_{i,\psi_j}(t)^2]||\theta - \theta^{(n)}||^2. \tag{3. 42}$$

We shall now estimate $E(c_j^{(n)})^2$. Note that $E(c_j^{(n)}) = 0$. Hence

$$
\begin{aligned}
E(c_j^{(n)})^2 &= Var(c_j^{(n)}) \\
&= n^{-1} Var(\zeta_{1j}^{(n)} + \tilde{\zeta}_{1j}) \\
&\leq n^{-1} E(\zeta_{1j}^{(n)} + \tilde{\zeta}_{1j})^2 \\
&\leq 2n^{-1}[E(\zeta_{1j}^{(n)})^2 + E(\tilde{\zeta}_{1j})^2] \\
&\leq 2n^{-1}[1 + (32) w_T^H \sup_{0 \leq t \leq T} E[Q_{1,\psi_j}(t)^2] ||\theta - \theta^{(n)}||^2]
\end{aligned}
\tag{3. 43}
$$

Lemma 3.2: Let $||M|| = \sup\{||Mx||, ||x|| \leq 1\}$ be the operator norm of a matrix M. Then $||M||^2 \leq \sum M_{jk}^2$ and

$$||M^{-1}|| \leq (1 + [\sum_{j,k}(M_{jk} - \delta_{jk})^2]^{-1/2})^{-1} \tag{3. 44}$$

provided that

$$\sum_{j,k}(M_{jk} - \delta_{jk})^2 < 1.$$

Proof: See Lemma 3 of Nguyen and Pham (1982).

We now have the following result.

Theorem 3.3: Suppose V_n is an increasing sequence of subspaces of $L^2([0,T], dt)$ of dimension d_n such that $d_n \to \infty$ and $\frac{d_n^2 \gamma_n}{n} \to 0$ as $n \to \infty$ where

$$\gamma_n = \sup_{0 \leq t \leq T} \sup_{f \in V_n} E[\frac{d}{dw_t^H} \int_0^t k_H(t,s) f(s) X(s) ds]^2. \tag{3. 45}$$

Then

$$||\hat{\eta}^{(n)} - \eta^{(n)}|| \to 0 \tag{3. 46}$$

in probability as $n \to \infty$.

Proof:Observe that

$$\hat{\eta}^{(n)} - \eta^{(n)} = a^{(n)^{-1}} c^{(n)} \tag{3. 47}$$

from equation (3.30). Applying Lemma 3.2, we get that

$$||\hat{\eta}^{(n)} - \eta^{(n)}|| \leq [1 - \{\sum_{j=1}^{d_n} \sum_{k=1}^{d_n} (a_{j,k}^{(n)} - \delta_{jk})^2\}^{1/2}]^{-1} ||c^{(n)}||. \tag{3. 48}$$

Applying the estimates obtained in (3.42) and (3.43), we get that there exists a constant $C_{T,H}$ depending only on T and H such that

$$E\{\sum_{j=1}^{d_n} \sum_{k=1}^{d_n} (a_{j,k}^{(n)} - \delta_{jk})^2\} \leq C_{T,H} n^{-1} d_n^2 \gamma_n \tag{3. 49}$$

and the last term tends to zero as $n \to \infty$. Similarly

$$E||c^{(n)}||^2 \le C_{T,H}[n^{-1}d_n + n^{-1}d_n\gamma_n||\theta - \theta^{(n)}||^2] \tag{3. 50}$$

the last term tends to zero as $n \to \infty$. Hence

$$||\hat{\eta}^{(n)} - \eta^{(n)}|| \to 0 \tag{3. 51}$$

in probability as $n \to \infty$.

As a consequence of the above theorem, we obtain the following corollary from the definition of the inner product defined in (3.22).

Corollary 3.4: Under the conditions stated in Theorem 3.3,

$$\lim_{n\to\infty} \frac{d}{dw_t^H}\int_0^t k_H(t,s)(\hat{\theta}^{(n)}(s) - \theta^{(n)}(s))X(s)ds = 0 \tag{3. 52}$$

in probability.

Proof: Observe that

$$||\hat{\theta}^{(n)} - \theta^{(n)}||^2 = \int_0^T E[\frac{d}{dw_t^H}\int_0^t k_H(t,s)(\hat{\theta}^{(n)}(s) - \theta^{(n)}(s))X(s)ds]^2 dw_t^H. \tag{3. 53}$$

which can also be written in the form

$$\sum_{j=1}^{d_n} |\hat{\eta}_j^{(n)} - \eta_j|^2 + \sum_{j=d_n+1}^{\infty} \eta_j^2.$$

The first term in the above sum tends to zero by Theorem 3.3. Since the set $\cup_{n\ge1}V_n$ is dense in $L^2([0,T],dt)$, it is also dense in the metric generated by the norm corresponding to the inner product $< .,>$.

Lemma 3.5: Let $\lambda^{(n)} = (\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_{d_n}^{(n)})$ be such that

$$\sum_{j=1}^{d_n} (\lambda_j^{(n)})^2 \to \lambda^2 \text{ as } n \to \infty. \tag{3. 54}$$

Then the random variable $\sqrt{n}\sum_{j=1}^{d_n} \lambda_j^{(n)} c_j^{(n)}$ is asymptotically normal with mean zero and variance λ^2.

Proof: In view of (3.33), it follows that

$$\sqrt{n}\sum_{j=1}^{d_n} \lambda_j^{(n)} c_j^{(n)} = n^{-1/2}\sum_{i=1}^{n}[\sum_{j=1}^{d_n} \lambda_j^{(n)}\zeta_{ij}^{(n)} + \sum_{j=1}^{d_n} \lambda_j^{(n)}\tilde{\zeta}_{ij}]. \tag{3. 55}$$

As in the derivation of the inequality (3.33), it can be checked that

$$E[\sum_{i=1}^{n}[\sum_{j=1}^{d_n} \lambda_j^{(n)}\zeta_{ij}^{(n)}]^2 \le (32)w_T^H\gamma_n\sum_{j=1}^{d_n}(\lambda_j^{(n)}]^2||\theta - \theta^{(n)}||^2. \tag{3. 56}$$

Note that $E(\zeta_{ij}^{(n)}) = 0$ and

$$(3.\ 57)\quad E(\sqrt{n}\sum_{j=1}^{d_n}\lambda_j^{(n)}c_j^{(n)})^2$$

$$= Var(\sqrt{n}\sum_{j=1}^{d_n}\lambda_j^{(n)}c_j^{(n)})$$

$$= n^{-1}\sum_{i=1}^{n}Var[\sum_{j=1}^{d_n}\lambda_j^{(n)}\zeta_{ij}^{(n)}]$$

$$= n^{-1}Var[\sum_{j=1}^{d_n}\lambda_j^{(n)}\zeta_{1j}^{(n)}] \text{ (since } X_i, 1 \le i \le n \text{ are i.i.d.)}$$

$$= n^{-1}E[[\sum_{j=1}^{d_n}\lambda_j^{(n)}\zeta_{1j}^{(n)}]^2$$

$$\le n^{-1}(32)w_T^H\gamma_n\sum_{j=1}^{d_n}(\lambda_j^{(n)})^2||\theta - \theta^{(n)}||^2.$$

The last term tends to zero since $\frac{\gamma_n}{n} \le \frac{\gamma_n d_n^2}{n} \to 0, ||\theta - \theta^{(n)}|| \to 0$ and

$$\sum_{j=1}^{d_n}(\lambda_j^{(n)})^2 \to \lambda^2$$

as $n \to \infty$. Hence

$$(3.\ 58)\qquad \sum_{i=1}^{n}[\sum_{j=1}^{d_n}\lambda_j^{(n)}\zeta_{ij}^{(n)}]^2 = o_p(1).$$

Furthermore

$$(3.\ 59)\qquad Var(\sum_{j=1}^{d_n}\lambda_j^{(n)}\tilde{\zeta}_{ij}) = \sum_{j=1}^{d_n}(\lambda_j^{(n)})^2$$

by (3.41) and the last term tends to λ^2 as $n \to \infty$. We now obtain the asymptotic normality from central limit theorems for triangular arrays.

As a consequence of the above lemma, the following theorem can be proved.

Theorem 3.6: Let $\lambda^{(n)}$ be as in the Lemma 3.5. Suppose that the conditions stated in the Theorem 3.3 hold. In addition suppose that $\frac{d_n^3\gamma_n^2}{n} \to 0$ as $n \to \infty$. Then

$$(3.\ 60)\qquad \sqrt{n}\sum_{j=1}^{d_n}\lambda_j^{(n)}(\hat{\eta}_i^{(n)} - \eta_i)$$

is asymptotically normal with mean zero and variance λ^2.

Proof:Observe that

$$(3.\ 61)\qquad a^{(n)}(\hat{\eta}^{(n)} - \eta^{(n)}) = c^{(n)}$$

and hence

$$\hat{\eta}^{(n)} - \eta^{(n)} = (a^{(n)})^{-1}c^{(n)} \tag{3. 62}$$

or equivalently

$$\hat{\eta}^{(n)} - \eta^{(n)} - c^{(n)} = (a^{(n)})^{-1}(I - a^{(n)})c^{(n)}. \tag{3. 63}$$

Denoting the operator norm and the Euclidean norm by the same symbol $||.||$, we get that

$$|\lambda^{(n)'}(\hat{\eta}^{(n)} - \eta^{(n)} - c^{(n)})| \leq ||\lambda^{(n)}||\ \ ||a^{(n)})^{-1}||\ \ ||a^{(n)} - I||\ \ ||c^{(n)}||. \tag{3. 64}$$

Relations (3.48) and (3.49) prove that

$$\begin{aligned} E||a^{(n)} - I||^2 &\leq E\{\sum_{j=1}^{d_n}\sum_{k=1}^{d_n}(a_{j,k}^{(n)} - \delta_{jk})^2\} \\ &\leq C_{T,H}n^{-1}d_n^2\gamma_n \end{aligned} \tag{3. 65}$$

and

$$nE||c^{(n)}||^2 \leq C_{T,H}[d_n + d_n\gamma_n||\theta - \theta^{(n)}||^2]. \tag{3. 66}$$

Therefore

$$\begin{aligned} (E[||\sqrt{n}||a^{(n)} - I||||c^{(n)}||])^2 &\leq nE||c^{(n)}||^2E||a^{(n)} - I||^2 \\ &\leq C_T([d_n + d_n\gamma_n||\theta - \theta^{(n)}||^2])(n^{-1}d_n^2\gamma_n) \end{aligned} \tag{3. 67}$$

and the last term tends to zero provided $\frac{d_n^3\gamma_n^2}{n} \to 0$ as $n \to \infty$. Therefore

$$||\sqrt{n}||a^{(n)} - I||||c^{(n)}|| \to 0 \tag{3. 68}$$

in probability as $n \to \infty$. We have observed earlier that

$$||a^{(n)}|| \to 1 \tag{3. 69}$$

in probability as $n \to \infty$. Hence

$$\sqrt{n}\lambda^{(n)'}(\hat{\eta}^{(n)} - \eta^{(n)} - c^{(n)}) \to 0 \tag{3. 70}$$

in probabilty as $n \to \infty$. but

$$\sqrt{n}\lambda^{(n)'}c^{(n)}$$

is asymptotically normal with mean zero and variance λ^2 by the Lemma 3.5. This proves the result.

As an application of the previous theorem, we get the following result.

Corollary 3.7: Let $h(.)$ be a function such that $||h|| < \infty$ in the sense of the inner product defined by (3.22). Suppose that the conditions stated in Theorem 3.5 hold. Then

$$\sqrt{n} < h, \hat{\theta}^{(n)} - \theta^{(n)} > \tag{3. 71}$$

is asymptotically normal with mean zero and variance $< h, h >$.

Proof: Suppose that $h(t) = \sum_{j=1}^{\infty} h_j \psi_j(t)$. Note that

$$\hat{\theta}^{(n)} - \theta^{(n)} = \sum_{j=1}^{d_n} (\hat{\eta}_j^{(n)} - \eta_j)\psi_j \tag{3. 72}$$

and hence

$$< h, \hat{\theta}^{(n)} - \theta^{(n)} = \sum_{j=1}^{d_n} h_j(\hat{\eta}_j^{(n)} - \eta_j). \tag{3. 73}$$

Since

$$\sum_{j=1}^{d_n} h_j^2 \to < h, h >= ||h||^2 \tag{3. 74}$$

by the Parseval's theorem, the result follows from Theorem 3.5.

Remarks: If in addition to the conditions stated in Corollary 3.7, we have

$$\sqrt{n}(h, \theta^{(n)} - \theta^{(n)} > \to 0 \text{ as } n \to \infty, \tag{3. 75}$$

then

$$\sqrt{n} < h, \hat{\theta}^{(n)} - \theta > \tag{3. 76}$$

is asymptotically normal with mean zero and variance $< h, h >$.

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