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On characterisation of Markov processes via Martingale problems

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Abstract

It is well-known that well-posedness of a martingale problem in the class of continuous (or r.c.l.l.) solutions implies measurability of the transition probability functions and hence Markovian property of the solution. We extend this result to the case when the martingale problem is well-posed in the class of solutions which are continuous in probability. This extension is used to improve on a criterion for a probability measure to be invariant for the semigroup associated with the Markov process. We also give an example of a martingale problem that is well posed in the class of solutions which are continuous in probability but for which no r.c.l.l. solution exists.

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1 Introduction

The seminal paper on Multi-dimensional diffusions by Stroock and Varadhan (1969) introduced Martinagle problems as a way of constructing and studying of Markov processes. Since then, this approach has been used succesfully in several contexts such as interacting particle systems, Markov processes associated with Boltzmann equation, nonlinear filtering theory, controlled markov processes, branching processes etc. An excellent account of the "Theory of Martingale problems" is given in the book by Ethier and Kurtz (1986). To construct a Markov process, the martingale problem approach allows one to construct the process for each initial condition separately and a general result gives the measurability of the associated transition probability function. To proceed, we give the basic definitions here. Given an operator A with domain $\mathcal{D}(A) \subseteq C_b(E)$ and range subset of $C_b(E)$, (where E is a complete separable metric space), a process X_t^x is said to be a solution to the (A, δ_x) martingale problem if for all $f \in \mathcal{D}(A)$

$$f(X_t^x) - \int_0^t Af(X_u^x)du \text{ is a martingale} \tag{1.1}$$

and

$$\mathbb{P}(X_0^x = x) = 1 \tag{1.2}$$

The martingale problem for A is said to be well posed in the class of r.c.l.l. solutions if for all x there exits a r.c.l.l. process (X_t^x) satisfying (1.1) and (1.2) and further for two such processes satisfying (1.1) and (1.2) (defined possibly on different probability spaces), the finite dimensional distributions are the same. Well posedness in the class of continuous solutions, solutions that are continuous in probability or measurable solutions is similarly defined. A well known result, which has its origins in the work of Stroock-Varadhan (1979) says that if the martingale problem for A is well posed in the class of r.c.l.l. solutions (or well posed in the class of continuous solutions), then (assuming that A, $\mathcal{D}(A)$ satisfy some mild conditions) it follows that $p_t(x, \cdot)$ defined by

$$p_t(x, A) = \mathbb{P}(X_t^x \in A) \tag{1.3}$$

is a transition probability function and any solution is a Markov process with p_t as its transition probability function. (See *e.g.* Theorem IV.4.2. of Ethier and Kurtz (1986)). This in turn gives us the associated semi-group (T_t) and its generator L. The generator L happens to be an extension of the operator A and thus A contains all the "relevant information" about L as well as about X. An important result on this theme is the following criterion for a probability measure to be invariant for (T_t): If $\mathcal{D}(A)$ is an algebra that separates points and if the martingale problem for A is well posed in the class of r.c.l.l. solutions, then for a probability measure μ on E

$$\int (Af)d\mu = 0 \; \forall f \in \mathcal{D}(A) \Rightarrow \mu \text{ is invariant for } (T_t). \tag{1.4}$$

This was proved by Echeverria (1982) assuming that E is locally compact. In Bhatt and Karandikar (1993) it was shown that the result continues to be true if the assumption of local compactness on E is replaced by the assumption that E is a complete separable metric space *provided we assume that the martingale problem is, in addition to being well posed in the class of r.c.l.l. solutions, also well posed in the class of progressively measurable solutions.* Thus, we need existence in the class of r.c.l.l. solutions and uniqueness in the class of progressively measurable solutions. This somewhat unusual assumption was required because one needed well posedness in r.c.l.l. solutions in order to give the measurability of the function $p_t(x, \cdot)$ defined by (1.3) and as a consequence, the associated semigroup and the Markov property of any solution. This approach required considering solutions to martingale problems without any reference to their path properties. The main result of this article is that if the Martingale problem is well posed in the class of solutions that are continuous in probability, then (under suitable conditions on $A, \mathcal{D}(A)$) the function p_t defined by (1.3) is measurable and that any solution satisfies the Markov property. This would allow us to drop the unusual assumption referred to above in the criterion for an invariant measure and replace it by *the martingale problem is well posed in the class of solutions that are continuous in probability.* The proof of measurability of $p_t(x, \cdot)$ defined by (1.3) when one assumes well posedness in the class of r.c.l.l. solutions is on the following lines. (See *e.g.* Theorem IV.4.6 of Ethier and Kurtz (1986)). One considers the set $\mathcal{P}$ of all probability measures on $D([0, \infty), E)$. It is a complete separable metric space under the topology of weak convergence. The set $\mathcal{M}$ of probability measures on $D([0, \infty), E)$ under which the coordinate process η is a solution to the martingale problem for some degenarate initial condition can be shown to be a Borel subset of $\mathcal{P}$. Well-posedness of the martingale problem in the class of r.c.l.l. solutions implies that for each $x \in E$, there exists a unique $P_x \in \mathcal{M}$ such that

$$P_x(\eta : \eta(0) = x) = 1.$$

Since the mapping $P \longrightarrow P \circ (\eta(0))^{-1}$ is Borel measurable on $\mathcal{P}$, its restriction to the Borel set $\mathcal{M}$ is also Borel measurable. But in view of the well-posedness assumption, the map restricted to $\mathcal{M}$ is the function $P_x \longrightarrow \delta_x$. (Here, $\delta_x \in \mathcal{P}(E)$ is the degenarate probability measure with unit mass at x). Since $\delta_x \longrightarrow x$ is measurable (it is indeed continuous), we get measurabilty of the mapping $P_x \longrightarrow x$. Using the fact that a one-to-one Borel measurable function between standard Borel spaces is bimeasurable, i.e. has a Borel measurable inverse we get that the mapping

$$x \longrightarrow P_x \text{ is Borel measurable.}$$

Once we have this, the proof of Markov property is easy. If one has well-posedness in the class of solutions with continuous paths, the proof is exactly the same- with $C([0,\infty),E)$ replacing $D([0,\infty),E)$. So in order to achieve our aim, we need to give a Borel structure on the set of distributions of processes that are continuous in probability. Once we have done this, we can deduce that well posedness in the class of solutions that are continuous in probability also gives measurability of the associated transition probability function and the Markov property. A crucial observation is that if the domain $\mathcal{D}(A)$ of A is a convergence determining class and X is any solution to the martingale problem, then the process X is continuous in probability. Thus, if $\mathcal{D}(A)$ is a convergence determining class, the phrase "solutions that are continuous in probability" in the discussion above can be replaced by "measurable solutions". In section 4, we give criterion for a measure to be invariant for the semigroup generated by a well-posed martingale problem. This is an improvement on the results mentioned above on this theme. In the last section, we give an example of a martingale problem that is well posed in the class of solutions that are continuous in probability but for which no r.c.l.l. solution exists.

2 Preliminaries

We will denote by (E,d) a complete, separable metric space. A will denote an operator with domain $D(A) \subset C_b(E)$, the space of real valued bounded continuous functions on E and with range contained in $M(E)$, the class of all real valued Borel measurable functions on E. Let $B(E)$ denote the class of all bounded Borel measurable functions. For $C \subset B(E)$, we define the *bp*-closure of C to be the smallest subset of $B(E)$ containing C which is closed under bounded pointwise convergence of sequences of functions. $\mathcal{B}(E)$ will denote the Borel σ-field on E, $\mathcal{P}(E)$ will denote the space of probability measures on E. For a random varibale Z taking values in E, $\mathcal{L}(Z)$ will denote the law of Z- *i.e.* the probability measure $\mathbb{P} \circ Z^{-1}$, if Z is defined on $(\Omega, \mathcal{F}, \mathbb{P})$. For a measurable Process (X_t) defined on $(\Omega, \mathcal{F}, \mathbb{P})$, let

$$^*\mathcal{F}_t^X = \sigma\left\{X_u,\ \int_0^u h(X_s)ds :\ u \le t\ ,\ h \in C_b(E)\right\}$$

Throughout this article, we will assume the following:

Assumption A1 *There exists a $[0,\infty)$ valued measurable function Φ on E such that*

$$(2.1) \qquad |Af(x)| \le C_f \Phi(x) \quad \forall x \in E,\ f \in \mathcal{D}(A).$$

Definition 2.1 : An E valued process $(X_t)_{0\le t\le T}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a solution to the martingale problem for (A, μ) if
(i) X is a measurable process with $\mathcal{L}(X_0) = \mu$,
(ii) $\mathbb{E}_{\mathbb{P}}[\int\limits_0^T \Phi(X_s)ds] < \infty$
and
(iiii) for every $f \in \mathcal{D}(A)$, $M_t^f = f(X_t) - \int\limits_0^t Af(X_s)ds$ is a $({}^*\mathcal{F}_t^X)$ -martingale. X will be called a solution to A martingale problem if it is a solution to (A, μ) martingale problem for some μ. Let $\mathcal{W}$ be a class of E valued processes. For example, we could consider $\mathcal{W}$ to be the class of E valued processes with r.c.l.l. paths or $\mathcal{W}$ can be the class of solutions that are continuous in probability. **Definition 2.2** : The martingale problem for A is said to well posed in the class $\mathcal{W}$ if for all $x \in E$, there exists a solution $X^x \in \mathcal{W}$ to the (A, δ_x) martingale problem and if $Y \in \mathcal{W}$ is any other solution to the (A, δ_x) martingale problem, then the finite dimensional distributions of X^x and Y are the same. We begin with some observations on solutions to the A -martingale problem.

LEMMA **2.1** *Let X be a solution to the A martingale problem (on $(\Omega, \mathcal{F}, \mathbb{P})$). Suppose that $\mathcal{D}(A)$ is a determining class and further that*

$$t \longrightarrow \mathcal{L}(X_t) \text{ is continuous.} \tag{2.2}$$

Then $t \longrightarrow X_t$ is continuous in probability.

Proof : Fix $f \in \mathcal{D}(A)$ and let $Z_t = f(X_t)$. The assumption (2.2) implies that $t \longrightarrow \mathbb{E}_{\mathbb{P}}(Z_t^2)$ is continuous (recall: $\mathcal{D}(A) \subset C_b(E)$). Further,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[Z_s(Z_t - Z_s)] &= \mathbb{E}_{\mathbb{P}}[Z_s \int_s^t (Af)(X_u)du] \\ &\le C_f ||f|| \mathbb{E}_{\mathbb{P}}[\int_s^t \Phi(X_u)du]. \end{aligned}$$

Hence, if $s_k \to v$ and $t_k \to v$ with $s_k < t_k$, we have

$$\mathbb{E}_{\mathbb{P}}[Z_{s_k} Z_{t_k}] \to \mathbb{E}_{\mathbb{P}}[Z_v^2]$$

and so

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[(Z_{t_k} - Z_{s_k})^2] &= \mathbb{E}_{\mathbb{P}}(Z_{t_k})^2 + \mathbb{E}_{\mathbb{P}}(Z_{s_k})^2 - 2\mathbb{E}_{\mathbb{P}}[Z_{s_k} Z_{t_k}] \\ &\to \mathbb{E}_{\mathbb{P}}[Z_v^2] + \mathbb{E}_{\mathbb{P}}[Z_v^2] - 2\mathbb{E}_{\mathbb{P}}[Z_v^2] \\ &= 0. \end{aligned}$$

Hence for all $f \in \mathcal{D}(A)$ the mapping $t \longrightarrow Z_t = f(X_t)$ is continuous in probability. As a consequence, for $f, g \in \mathcal{D}(A)$

$$(s,t) \longrightarrow \mathbb{E}_{\mathbb{P}}[f(X_s)g(X_t)] \text{ is continuous.} \tag{2.3}$$

Let $s_k \to s$. The assumption (2.2) implies that the family of distributions $\{\mathcal{L}(X_{s_k})\}$ is tight and so the family of distributions (on $E \times E$)

$$\{\mathcal{L}(X_{s_k}, X_s) : k \geq 1\} \text{ is tight.} \tag{2.4}$$

Since the class of functions $(x,y) \longrightarrow f(x)g(y),\ f, g \in \mathcal{D}(A)$ constitutes a determining class, (2.3) -(2.4) together imply that

$$\mathcal{L}(X_{s_k}, X_s) \to \mathcal{L}(X_s, X_s) \tag{2.5}$$

Now for any $\epsilon > 0$, $\mathbb{P}(d(X_s, X_s) \geq \epsilon) = 0$. Thus in view of (2.5)

$$\limsup_{k \to \infty} \mathbb{P}(d(X_{s_k}, X_s) \geq \epsilon) \leq 0$$

i.e.

$$\mathbb{P}(d(X_{s_k}, X_s) \geq \epsilon) \to 0.$$

This completes the proof. ■

Remark **2.2** The proof given above contains the proof of the following: if for a process Y, the mapping $(s,t) \longrightarrow \mathcal{L}(Y_s, Y_t)$ is continuous, then Y is continuous in probability.

Theorem **2.3** *Let X be a solution to the A martingale problem (on $(\Omega, \mathcal{F}, \mathbb{P})$). Suppose that the domain $\mathcal{D}(A)$ of A is a convergence determining class on E. Then the process X is continuous in probability.*

Proof : Since $f(X_t) - \int_0^t Af(X_s)ds$ is a martingale for $f \in \mathcal{D}(A)$, it follows that the mapping $t \longrightarrow \mathbb{E}_{\mathbb{P}}[f(X_t)]$ is continuous. Since $\mathcal{D}(A)$ is a convergence determining class, this implies continuity of the mapping

$$t \longrightarrow \mathcal{L}(X_t).$$

Thus, by Lemma 2.1, $t \longrightarrow X_t$ is continuous in probability. ■

3 Main Result

We have seen in the previous section that under suitable conditions, all solutions to a Martingale problem are continous in probability. Thus we now construct a Borel structure on the class of distributions of such processes. For $m \geq 1$, E^m with the product topology is again a complete separable metric space. Let $\mathcal{P}(E^m)$ be equipped with the toplology of weak convergence. Let $\mathcal{C}_m = C([0,\infty)^m, \mathcal{P}(E^m))$ be equipped with the topology of uniform convergence on compact subsets. Then $\mathcal{C}_m$ is a complete separable metric space. Let $\mathcal{S}_m$ be the set of $\mu^m = \mu^m(t_1, t_2, \ldots t_m) \in \mathcal{C}_m$ satisfying

$$\mu^2(t,t)(D) = 1 \ \ \forall t \in [0,\infty) \tag{3.1}$$

where D is the diagonal in $E \times E$ and for $f \in C_b(E^m)$

$$\begin{aligned} &\int (\pi f)(x_1, x_2, \ldots, x_m)\mu^m(t_{\pi 1}, t_{\pi 2}, \ldots t_{\pi m})(dx_1, dx_2, \ldots dx_m) \\ &= \int f(x_1, x_2, \ldots, x_m)\mu^m(t_1, t_2, \ldots t_m)(dx_1, dx_2, \ldots dx_m) \end{aligned} \tag{3.2}$$

for all permutations π of $\{1, 2, \ldots, m\}$ where πf is defined by

$$\pi f(x_1, x_2, \ldots, x_m) = f(x_{\pi 1}, x_{\pi 2}, \ldots, x_{\pi m})$$

It is easy to see that $\mathcal{S}_m$ is a closed subset of $\mathcal{C}_m$ and hence $\mathcal{S}_m$ is a complete separable metric space. Let $\mathcal{S}_\infty = \Pi_{m=1}^\infty \mathcal{S}_m$. Under the product toplology, $\mathcal{S}_\infty$ is also a complete separable metric space. Elements of $\mathcal{S}_\infty$ will be denoted by $\mu = (\mu^1, \mu^2, \ldots)$ $(\mu^k \in \mathcal{S}_k)$. Let

$$\mathcal{S}^* = \{\mu \in \mathcal{S}_\infty : \mu^m(t_1, \ldots, t_m) \circ (h_m)^{-1} = \mu^{m-1}(t_1, \ldots, t_{m-1}), \forall m > 1\}$$

where $h_m : E^m \longrightarrow E^{m-1}$ is the projection map defined by

$$h_m(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_{m-1}).$$

Then clearly $\mathcal{S}^*$ is also a complete separable metric space since it is a closed subspace of $\mathcal{S}_\infty$. Every element of $\mathcal{S}^*$ is a *consistent* family of finite dimensional distributions and hence by the Kolmogorov consistency therorem, given $\mu = (\mu^1, \mu^2, \ldots) \in \mathcal{S}_\infty$, there exists a probability space $(\Omega^*, \mathcal{F}^*, P^*)$ and a stochastic process (X_t) on it such that for all $m \geq 1$,

$$\mathcal{L}(X_{t_1}, X_{t_2}, \ldots, X_{t_m}) = \mu^m(t_1, t_2, \ldots, t_m) \tag{3.3}$$

In view of Remark (2.2), the process X is continuous in probability. Conversely, given a E valued process X that is continuous in probability, μ^m defined by (3.3) belongs to $\mathcal{S}_m$ and clearly $\{\mu^1, \mu^2, \ldots\}$ is a consistent family and hence $\mu = (\mu^1, \mu^2, \ldots) \in \mathcal{S}^*$. Thus, $\mathcal{S}^*$ can be identified with the class of distributions of E valued processes that are continuous in probability. Having given a topological structure to the class of (distributions) of processes that are continuous in probability, we now identify the class of (distributions of) solutions to the martingale problem for A and show that it is a Borel set. As in the corresponding result on solutions with r.c.l.l. paths (see Ethier and Kurtz (1986)), we assume that $A, \mathcal{D}(A)$ satisfy the following:

Assumption A2 *There exists a countable set* $\{f_n : n \geq 1\} \subset \mathcal{D}(A)$ *such that*

$$bp - closure\{(f_n, \Phi^{-1}Af_n) : n \geq 1\} \supset \{(f, \Phi^{-1}Af) : f \in \mathcal{D}(A)\}.$$

Let X be a process that is continuous in probability (on some $(\Omega, \mathcal{F}, \mathbb{P})$). Since every such process admits a measurable modification (see Dellacherie and Meyer (1978)), we assume that X is measurable. Let $\mathcal{G}$ be a countable dense subset of $C_b(E)$ (in sup norm). Then X is a solution to A martingale problem if and only if

$$\int_0^N \Phi(X_u)du < \infty \ \ \forall N \geq 1$$

and

$$\mathbb{E}_{\mathbb{P}}\Big[g_1(X_{s_1}) \ldots g_k(X_{s_k})(f_m(X_t) - f_m(X_s) - \int_s^t (Af_m)(X_u)du)\Big] = 0$$

for all $s_1, s_2, \ldots s_k, s, t$ rationals with $s_i \leq s \leq t$, $g_i \in \mathcal{G}$, $1 \leq i \leq k$, $k \geq 1, m \geq 1$, where $\{f_j : j \geq 1\}$ are as in Assumption A2. Thus, a measurable process X is a solution to the A martingale problem if and only if its finite dimensional distributions $\mu = (\mu^1, \mu^2, \ldots)$ defined by (3.3) belong to $\mathcal{M} \subset \mathcal{S}^*$ defined as follows: $\mathcal{M}$ is the set of $\mu = (\mu^1, \mu^2, \ldots) \in \mathcal{S}^*$ satisfying

$$\int_0^N \langle \mu^1(s), \Phi \rangle ds < \infty, ; \forall N \geq 1 \tag{3.4}$$

(here, $\langle F, \Gamma \rangle$ denotes $\int F d\Gamma$) and

$$\begin{aligned} &\langle \mu^{k+1}(s_1, s_2, \ldots, s_k, t), G \otimes f_m \rangle - \langle \mu^{k+1}(s_1, s_2, \ldots, s_k, s), G \otimes f_m \rangle \\ &\qquad = \int_s^t \langle \mu^{k+1}(s_1, s_2, \ldots, s_k, u), G \otimes Af_m \rangle du \end{aligned} \tag{3.5}$$

for all $s_1, s_2, \ldots s_k, s, t$ rationals with $s_i \leq s \leq t$, $g_i \in \mathcal{G}$, $1 \leq i \leq k$, $k \geq 1, m \geq 1$, where $\{f_j : j \geq 1\}$ are as in Assumption A2 and

$$G \otimes f_m(x_1, x_2, \ldots, x_k, z) = g_1(x_1)g_2(x_2) \ldots g_k(x_k)f_m(z).$$

Since $\mathcal{M}$ is defined via countably many conditions with each condition in turn involving measurable functions of $\mu = (\mu^1, \mu^2, \ldots)$, it follows that $\mathcal{M}$ is a Borel subset of $\mathcal{S}^*$. Moreover, given $\mu = (\mu^1, \mu^2, \ldots) \in \mathcal{M}$, as noted above there exists a process X such that its finite dimensional distributions are those given by $\mu = (\mu^1, \mu^2, \ldots)$. This process is continuous in probability and can be assumed to be measurable. It follows that X is a solution to the A martingale problem. We have thus proved

Theorem **3.1** *Suppose that $A, \mathcal{D}(A)$ satisfy A1 and A2. Then $\mu = (\mu^1, \mu^2, \ldots) \in \mathcal{M}$ if and only if there exsists a process X that is (i) continuous in probability, (ii) the finite dimensional distributions of X are given by $\mu = (\mu^1, \mu^2, \ldots)$ and (iii) X is a solution to the martingale problem for A.*

We are now ready to prove the measurability of p_t when the martingale problem for A is well posed. We introduce the following

Assumption A3 *The martingale problem for (A, δ_x) is well posed in the class of solutions that are continuous in probability for each $x \in E$.*

Theorem **3.2** *Suppose that $A, \mathcal{D}(A)$ satisfy A1, A2 and A3. Let X^x denote a solution that is continuous in probability to the (A, δ_x) martingale problem. Let $p_t(x, B)$, $t \in [0, \infty)$, $x \in E$, $B \in \mathcal{B}(E)$ be defined by*

$$p_t(x, B) = P(X_t^x \in B). \tag{3.6}$$

Then for all $t \in [0, \infty)$, $B \in \mathcal{B}(E)$, $x \longrightarrow p_t(x, B)$ is Borel measurable.

Proof : Note that $F = \{\delta_x : x \in E\}$ is a Borel measurable subset of $\mathcal{P}(E)$ (indeed it is a closed subset) and the function $\theta(\delta_x) = x$ is a Borel measurable function on it (again this is a continuous function). Let $\psi_t : \mathcal{M} \longrightarrow \mathcal{P}(E)$ for $0 \le t < \infty$ be defined by

$$\psi_t(\mu) = \mu^1(t), \qquad \mu = (\mu^1, \mu^2, \ldots) \in \mathcal{M}.$$

The functions ψ_t are continuous and hence measurable. Let $\mathcal{M}_0 = (\psi_0)^{-1}(F)$. It follows that $\mathcal{M}_0$ is a Borel subset of $\mathcal{S}^*$. Also, $\Psi = \theta(\psi)$ is a measurable function from $\mathcal{M}_0$ into E. In view of the Assumption A3, for a given $x \in E$, $\mathcal{M}$ has exactly one element $\mu = (\mu^1, \mu^2, \ldots)$ such that

$$\mu^1(0) = \delta_x$$

and hence the function Ψ is one-to-one. Hence by Kuratowski's theorem (See *e.g.* Corollary I.3.3 of Parthasarathy (1967)) the function is bimeasurable, or it has a measurable inverse. Let us note that $\Psi^{-1}(x)$ denotes the finite dimensional distributions of X^x- the (unique in law) solution to (A, δ_x) martingale problem which is continuous in probability. The required conclusion follows by noting that

$$p_t(x, B) = \psi_t(\Psi^{-1}(x))(B).$$

■

Remark **3.3** Let us note that

Assumption A4 *$\mathcal{D}(A)$ is convergence determining*

and

Assumption A5 *The (A, δ_x) martingale problem is well posed in the class of measurable processes for all $x \in E$.*

imply Assumtion A3. This is because assumption A4 implies that every solution to the A martingale problem is continuous in probability. Thus the conclusion of the above theorem remains valid with the same proof if instead we assume that $A, \mathcal{D}(A)$ satisfy A1, A2 , A4 and A5.

Remark **3.4** Assume that A1, A2 and A3 are true. Denote by

$$\mu_x = (\mu_x^1, \mu_x^2, \ldots)$$

the finite dimensional distributions of the (unique in law) solution to the (A, δ_x) martingale problem that is continuous in probability. We have seen in the proof above that

$$x \longrightarrow \mu_x (= \Psi^{-1}(x))$$

is Borel measurable and hence for all $t_1, t_2, \ldots, t_m$, $m \geq 1$

(3.7) $$x \longrightarrow \mu_x^m(t_1, t_2, \ldots, t_m) \text{ is Borel measurable}$$

The next step is to prove that $\{T_t : t \geq 0\}$ defined by

(3.8) $$T_t f(x) = \int f(y) p_t(x, dy) = \int f(y) \mu_x^1(t)(dy)$$

is a semigroup on the class of bounded Borel measurable functions f on E. For this we need to consider the martingale problem with non-degenarate initial distributions. When the martingale problem is well posed in the class of solutions with r.c.l.l. paths for all degenarate initials and Af is bounded for every $f \in \mathcal{D}(A)$, uniqueness follows for all initial distributions. However, since we are dealing with the case where Af can be unbounded, we can no longer assert well-posedness for all initial distributions. To proceed further, let us introduce the following notation:

$$\Phi_N^*(x) = \int_0^N \langle \mu_x^1(s), \Phi \rangle ds \tag{3.9}$$

Then in view of Remark 3.4, it follows that Φ_N^* is measurable $[0,\infty)$ valued function. The next Lemma shows that existence holds for large class of initials. Let $\mathcal{P}_\Phi$ be the set of all measures $\lambda \in \mathcal{P}(E)$ such that

$$\langle \Phi_N^*, \lambda \rangle < \infty \quad \forall N \geq 1. \tag{3.10}$$

LEMMA **3.5** *Suppose that $A, \mathcal{D}(A)$ satisfy assumptions A1, A2 and A3. Let $\lambda \in \mathcal{P}_\Phi$. Then $\nu = (\nu^1, \nu^2, \ldots)$ defined by*

$$\langle \nu^m(t_1, t_2, \ldots, t_m), g \rangle = \int \langle \mu_x^m(t_1, t_2, \ldots, t_m), g \rangle d\lambda(x) \tag{3.11}$$

belongs to $\mathcal{M}$ with $\nu^1(0) = \lambda$. Hence there exists a solution to the martingale problem for (A, λ) (whose finite dimensional distributions are $\{\nu^m\}$).

Proof : It is easy to see that $\{\nu^m\}$ satisfy (3.5) since each $\{\mu_x^m\}$ satisfies the same. Further, condition (3.10) on λ along with the definition of Φ_N^* implies that ν^1 satisfies (3.4) and hence $\{\nu^m\}$ belongs to $\mathcal{M}$. Thus the corresponding process Y is a solution to the the martingale problem for (A, λ). ■We need one more observation on Martingale problems before we can state our result on Markov property of solutions.

LEMMA **3.6** *Let a process X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ be a solution to the (A, λ) martingale problem and let g be a $[0, M]$ valued measurable function on E (where $M < \infty$) such that $\langle \lambda, g \rangle = 1$. Let γ be defined by $\frac{d\gamma}{d\lambda} = g$. Let $\mathbb{Q}$ be defined by*

$$\frac{d\,\mathbb{Q}}{d\mathbb{P}} = g(X_0).$$

Then, considered as a process on $(\Omega, \mathcal{F}, \mathbb{Q})$, X is a solution to the (A, γ) martingale problem.

Proof : Since g is bounded it follows that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\Big[\int_0^N \Phi(X_u) du\Big] &\leq M \mathbb{E}_{\mathbb{P}}\Big[\int_0^N \Phi(X_u) du\Big] \\ &< \infty \end{aligned}$$

Moreover, since $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is $\sigma(X_0)$ measurable, it follows that $f(X_t) - \int_0^t Af(X_u)du$ is a martingale on $(\Omega, \mathcal{F}, \mathbb{Q})$ (as it is a martingale on $(\Omega, \mathcal{F}, \mathbb{P})$). The result follows upon noting that $\mathbb{Q} \circ (X_0)^{-1} = \gamma$. ■

In addition to the assumption A3, we need to asuume the following in order to get Markov property of any solution and the semigroup property of the $\{T_t\}$:

Assumption A6 *There exists a sequence $\{h_n : n \geq 1\}$ of $[0,\infty)$ valued Borel measurable functions on E such that for every $\lambda \in \mathcal{P}(E)$ satisfying*

$$\langle h_n, \lambda \rangle < \infty \quad \forall n \geq 1, \tag{3.12}$$

any two solutions to the (A,λ) martingale problem that are continuous in probability have the same finite dimensional distributions.

Thus, in order to verify that A6 holds in a given example, we can show that the uniqueness holds under a (or even countably many) integrability condition(s).

Theorem **3.7** *Suppose that $A, \mathcal{D}(A)$ satisfy assumptions A1, A2, A3 and A6.*

(i) *Let $\lambda \in \mathcal{P}_\Phi$. Then the Martingale problem for (A,λ) is well-posed. Further, the finite dimensional law of any solution Y that is continuous in probability are gievn by (3.11)*

(ii) *Let $\lambda \in \mathcal{P}(E)$ be such that the martingale problem for (A,λ) admits a solution that is continuous in probability. Then $\lambda \in \mathcal{P}_\Phi$.*

(iii) *Let $\lambda \in \mathcal{P}_\Phi$. Let X be a solution to the (A,λ) martingale problem (defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$). Further let X be continuous in probability. Then for any $s < t$*

$$\mathbb{E}_{\mathbb{P}}[g(X_{s+t})|\sigma(X_u : u \leq s)] = T_t g(X_s) \tag{3.13}$$

where $\{T_t : t \geq 0\}$ is defined by (3.8) and $g \in C_b(E)$. As a consequence, X is a Markov process and $\{T_t : t \geq 0\}$ is the semigroup associated with X.

Proof : (i) Let $\lambda \in \mathcal{P}_\Phi$. We have seen in Lemma 3.5 that the (A,λ) martingale problem admits a solution X whose finite dimensional distributions are given by (3.11). Let X be defined on $(\Omega, \mathcal{F}, \mathbb{P})$. This process X is continuous in probability. Let Y be another solution to the (A,λ) martingale problem defined on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ such that Y is continuous in probability. Define g on E by

$$g(x) = C \sum_{n=1}^{\infty} 2^{-n} \frac{1}{1 + h_n(x)}$$

where C is a constant that is chosen so that $\langle \lambda, g\rangle = 1$. Define probability measures γ, $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ by

$$\frac{d\gamma}{d\lambda} = g, \quad \frac{d\,\mathbb{Q}}{d\mathbb{P}} = g(X_0) \text{ and } \frac{d\,\widetilde{\mathbb{Q}}}{d\widetilde{\mathbb{P}}} = g(Y_0).$$

By Lemma 3.6, X on $(\Omega, \mathcal{F}, \mathbb{Q})$ and Y on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{Q}})$ are solutions to the (A, γ) martingale problem. Further, these processes are continuous in probability. By construction, γ satisfies (3.12) and hence by assumption A6, the finite dimensional distributions of X on $(\Omega, \mathcal{F}, \mathbb{Q})$ are the same as those of Y on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{Q}})$. This in turn implies that the finite dimensional distributions of X on $(\Omega, \mathcal{F}, \mathbb{P})$ are the same as those of Y on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. This proves part (i). For (ii), let X be a solution that is continuous in probability to the (A, λ) martingale problem. This time define

$$g(x) = C \sum_{n=1}^{\infty} 2^{-n} \frac{1}{1 + \Phi_n^*(x)}$$

where C is a constant that is chosen so that $\langle \lambda, g\rangle = 1$. Define probability measures γ and $\mathbb{Q}$ by

$$\frac{d\gamma}{d\lambda} = g, \text{ and } \frac{d\,\mathbb{Q}}{d\mathbb{P}} = g(X_0)$$

By Lemma 3.6, X is a solution to the the (A, γ) martingale problem under $\mathbb{Q}$ and X is continuous in $\mathbb{Q}$ probability. By part (i), we have that the regular conditional probability distribution of $(X_{t_1}, X_{t_2}, \ldots, X_{t_m})$ given $\sigma(X_0)$ is $\mu^m_{X_0}(t_1, t_2, \ldots, t_m)$. As a consequence

$$\mathbb{E}_{\mathbb{Q}}\Big[\int_0^N \Phi(X_s)ds|\sigma(X_0)\Big] = \Phi_N^*(X_0) \tag{3.14}$$

Since $\frac{d\,\mathbb{Q}}{d\mathbb{P}}$ is $\sigma(X_0)$ measurable, (3.14) implies that

$$\mathbb{E}_{\mathbb{P}}\Big[\int_0^N \Phi(X_s)ds|\sigma(X_0)\Big] = \Phi_N^*(X_0)$$

and hence that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}\Big[\int_0^N \Phi(X_s)ds\Big] &= \mathbb{E}_{\mathbb{P}}\Big[\Phi_N^*(X_0)\Big] \\ &= \langle \Phi_N^*, \lambda\rangle. \end{aligned}$$

Since X is a solution to the (A, λ) martingale problem, the LHS above is finite for all N and hence $\lambda \in \mathcal{P}_\Phi$. For (iii), let X be a solution to (A, λ) martingale problem that is continuous in probability (defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$). Fix $m \geq 1$ and $0 \leq u_1 < u_2 < \ldots u_m \leq s$ and $h_1, h_2 \ldots h_m$ bounded positive continuous functions, define a probabiltiy measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\frac{d\,\mathbb{Q}}{d\mathbb{P}} = C h_1(X_{u_1}) h_2(X_{u_2}) \ldots h_m(X_{u_m})$$

where the constant C is chosen such that $\mathbb{Q}$ is a probability measure. Define Y by

$$Y_t = X_{s+t}, \quad t \geq 0.$$

Then using that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is bounded (say by M) we get

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\Big[\int_0^T \Phi(Y_u)du\Big] &= \mathbb{E}_{\mathbb{Q}}\Big[\int_s^{T+s} \Phi(X_u)du\Big] \\ &\leq M\mathbb{E}_{\mathbb{P}}\Big[\int_s^{T+s} \Phi(X_u)du\Big] \\ &< \infty \end{aligned} \tag{3.15}$$

Further, noting that for $g_1, \ldots g_k \in C_b(E)$ and $0 \leq s_1 \ldots < s_k < v < t$,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\Big[g_1(Y_{s_1}) \ldots g_k(Y_{s_k})(f(Y_t) - f(Y_v) - \int_v^t (Af)(Y_u)du)\Big] \\ &= \mathbb{E}_{\mathbb{Q}}\Big[g_1(X_{s+s_1}) \ldots g_k(X_{s+s_k})(f(X_{s+t}) - f(X_{s+v}) - \int_v^t (Af)(X_{s+u})du)\Big] \\ &= \mathbb{E}_{\mathbb{P}}\Big[Ch_1(X_{u_1})h_2(X_{u_2}) \ldots h_m(X_{u_m} g_1(X_{s+s_1}) \ldots g_k(X_{s+s_k}) \times \\ &\qquad (f(X_{s+t}) - f(X_{s+v}) - \int_v^t (Af)(X_{s+u})du)\Big] \\ &= 0 \end{aligned}$$

we conclude that Y is a solution to the (A, γ) martingale problem where $\gamma = \mathbb{Q} \circ [Y(0)]^{-1}$. Of course, Y is continuous in probability. Hence, by parts (ii) above $\gamma \in \mathcal{P}_\Phi$ and then by part (i), the finite dimensional distributions are given by (3.11) (with λ replaced by γ). Thus, for $g_1, \ldots g_k \in C_b(E)$ and $0 \leq s_1 \ldots < s_k$,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\Big[g_1(Y_{s_1}) \ldots g_k(Y_{s_k})\Big] &= \int \langle \mu_x^k(s_1, \ldots, s_k), g_1 \otimes \ldots \otimes g_k \rangle d\gamma(x) \\ &= \mathbb{E}_{\mathbb{Q}}\Big[\langle \mu_{Y_0}^k(s_1, \ldots, s_k), g_1 \otimes \ldots \otimes g_k \rangle\Big] \end{aligned}$$

and so (using with $k = 1$, $s_1 = t$ and $g_1 = g$) we can conclude that

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}}\Big[Ch_1(X_{u_1})h_2(X_{u_2}) \ldots h_m(X_{u_m})g(X_{s+t})\Big] \\ &= \mathbb{E}_{\mathbb{Q}}\Big[g(Y_t)\Big] \\ &= \mathbb{E}_{\mathbb{Q}}\Big[\langle \mu_{Y_0}^1(t), g \rangle\Big] \\ &= \mathbb{E}_{\mathbb{P}}\Big[Ch_1(X_{u_1})h_2(X_{u_2}) \ldots h_m(X_{u_m})\langle \mu_{X_s}^1(t), g \rangle\Big] \end{aligned}$$

for all $0 \leq u_1 < u_2 < \ldots u_m \leq s$ and $h_1, h_2 \ldots h_m$ bounded positive continuous functions, $m \geq 1$. As a consequence,

$$\mathbb{E}_{\mathbb{P}}\Big[g(X_{s+t})\big|\sigma(X_u : 0 \leq u \leq s)\Big] = \langle \mu^1_{X_s}(t), g\rangle = (T_t g)(X_s).$$

This completes the proof. ■

4 Criterion for an invariant measure

In the light of the result proved above, we can improve the criterion for a measure to be invariant for the semigroup (T_t) arising from a well-posed martingale problem obtained in Bhatt and Karandikar (1993, 1995, 1999), Bhatt and Borkar (1996), Kurtz and Stockbridge (1998). In these papers, it was assumed that the martingale problem for (A, δ_x) is well-posed in the class of r.c.l.l. solutions and further that any measurable solution admits a r.c.l.l. modification. In other words, existence in the class of r.c.l.l. solutions and uniqueness in the class of measurable solutions. The existence of r.c.l.l. solutions was required to conclude that (i) The operator A satisfies the positive maximum principle and (ii) to deduce the measurability of the transition probability so as to give rise to a semigroup. We have already seen how to get around (ii). The following lemma takes care of (i) as well. We introduce another condition on A and Φ (appearing in assumption A1).

Assumption A7 *Φ and Af, for every $f \in \mathcal{D}(A)$, are continuous*

LEMMA **4.1** *Suppose that $A, \mathcal{D}(A)$ satisfy assumptions A1, A2, A3 and A7. Then A satisfies the positive maximum principle, i.e. if $f \in \mathcal{D}(A)$ and $z \in E$ are such that $f(z) \geq 0$ and $f(z) \geq f(x)$ for all $x \in E$, then*

$$Af(z) \leq 0.$$

Proof : Let X be a solution to (A, δ_z) martingale problem defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that is continuous in probability. Let $\mathcal{F}_t = {}^*\mathcal{F}^X_t$ and

$$M_t = f(X_t) - \int_0^t Af(X_u)du.$$

Then $(M_t, \mathcal{F}_t)$ is a martingale. Let σ_t, $0 \leq t < \infty$ be the increasing family of $(\mathcal{F}_t)$ stopping times defined by

$$\sigma_t = \inf\Big\{s \geq 0 : \int_0^s (1 + \Phi(X_u))du \geq t\Big\}.$$

Since $\mathbb{E}_{\mathbb{P}}[\int_0^s \Phi(X_u)du] < \infty$, it follows that $\sigma_t < \infty$ for all t. Let $N_t = M_{\sigma_t}$, $Y_t = X_{\sigma_t}$ and $\mathcal{G}_t = \mathcal{F}_{\sigma_t}$. Then, it follows that $(N_t, \mathcal{G}_t)$ is a local martingale. Moreover, $t \longrightarrow \sigma_t$ is continuous and hence Y is also continuous in probability. Using change of variable, it is easy to see that

$$N_t = f(Y_t) - \int_0^t \frac{Af(Y_r)}{1+\Phi(Y_r)}dr.$$

Since $Af(x) \leq C_f\Phi(x)$, it follows that N is bounded and hence is a martingale. Since f has a maximum at z and

$$\mathbb{E}_{\mathbb{P}}[f(Y_t) - f(z) - \int_0^t \frac{Af(Y_r)}{1+\Phi(Y_r)}dr] = 0$$

it follows that (using Fubini's theorem)

$$\int_0^t \mathbb{E}_{\mathbb{P}}\big[\frac{Af(Y_r)}{1+\Phi(Y_r)}\big]dr \leq 0 \ \ \forall t > 0. \tag{4.1}$$

Since Y is continuous in probability and $Af(x) \leq C_f\Phi(x)$, it follows that

$$r \longrightarrow \mathbb{E}_{\mathbb{P}}\big[\frac{Af(Y_r)}{1+\Phi(Y_r)}\big]$$

is continuous. Now dividing the LHS in (4.1) by t and taking limit as $t \to 0$ we get

$$\frac{Af(z)}{1+\Phi(z)} \leq 0.$$

Since $\Phi(z) \geq 0$ this completes the proof. ■Here is yet another assumption on $A, \mathcal{D}(A)$.

Assumption A8 *$\mathcal{D}(A)$ is an algebra that contains constants and separates points in E.*

THEOREM **4.2** *Suppose that $A, \mathcal{D}(A)$ satisfy assumptions A1, A2, A3, A6, A7 and A8. Let (T_t) be the semigroup associated with $(A, \mathcal{D}(A))$ by Theorem 3.7. If $\lambda \in \mathcal{P}(E)$ is such that $\int \Phi d\lambda < \infty$ and*

$$\int (Af)(x)d\lambda(x) = 0 \ \ \forall f \in \mathcal{D}(A)$$

then λ is an invariant measure for the semigroup (T_t) and the solution to the (A, λ) martingale problem that is continuous in probability is a stationary process.

Proof : In view of Lemma 4.1 and the assumptions made in the statement of this theorem, the proof of Theorem 3.1 in Bhatt and Karandikar (1995) gives the existence of a stationary solution to the (A, λ) martingale problem. Since the solution (say X) is stationary, the mapping $t \longrightarrow \mathcal{L}(X_t)$ is continuous (it is a constant) and hence by Lemma 2.1, X is continuous in probability. Now, Theorem 3.7 implies that λ is an invariant measure for (T_t). ■

The criterion for invariant measure given above is true even if assumption A7 (continuity of Φ and of Af) does not hold. We instead need the following assumptions.

Assumption A9 *A satisfies the positive maximum principle*

Assumption A10 *There exists a complete separable metric space* U*, an operator* $\hat{A}: \mathcal{D}(A) \to C(E \times U)$ *and a transition function* η *from* $(E, \mathcal{B}(E))$ *into* $(U, \mathcal{B}(U))$ *such that*

$$(Af)(x) = \int_U \hat{A}f(x,u)\eta(x,du). \tag{4.2}$$

Assumption A11 *There exists* $\hat{\Phi} \in C(E \times U)$ *such that for all* $f \in \mathcal{D}(A)$*, there exists* $C_f < \infty$ *satisfying*

$$|\hat{A}f(x,u)| \leq C_f \hat{\Phi}(x,u) \qquad \forall x, u \in E \times U \tag{4.3}$$

$$\Phi(x) = \int_U \hat{\Phi}(x,u)\eta(x,du) < \infty. \tag{4.4}$$

Remark **4.3** *Note that assumptions A10 and A11 together imply assumption A1. Thus the conclusions of Theorem 3.7 remain valid under assumptions A2, A3, A6, A10 and A11. In particular, an operator* A *with domain* $\mathcal{D}(A)$ *satisfying A2, A3, A6, A10 and A11 determines a semigroup* $\{T_t : t \geq 0\}$ *defined by* (3.8).

We now have the following theorem.

Theorem **4.4** *Suppose that* $A, \mathcal{D}(A)$ *satisfy assumptions A2, A3, A6, A8, A9, A10 and A11. Let* (T_t) *be the semigroup associated with* $(A, \mathcal{D}(A))$ *as in Remark 4.3. If* $\lambda \in \mathcal{P}(E)$ *is such that* $\int \Phi d\lambda < \infty$ *and*

$$\int (Af)(x) d\lambda(x) = 0 \ \ \forall f \in \mathcal{D}(A)$$

then λ *is an invariant measure for the semigroup* (T_t) *and the solution to the* (A, λ) *martingale problem that is continuous in probability is a stationary process.*

Proof : Under conditions A2, A8, A9, A10 and A11, existence of a stationary solution to the (A, λ) martingale problem was proven in Bhatt and Karandikar (1999). Rest of the argument is as in the proof of Theorem 4.2. ∎

5 Example

Let $E = [0,1)$. Let $C^2(E)$ be the class of functions f that are restriction of some $g \in C_b^2(\mathbb{R})$ and

$$\mathcal{D}(A) = C^2(E) \cap \{f : \lim_{x\to 1} f(x) = f(0)\}.$$

For $f \in \mathcal{D}(A)$ define Af by $Af = \frac{1}{2}f''$. Fix $x \in E$. To construct a solution of the martingale problem for (A, δ_x) we proceed as follows. Let $(\beta_t^x)_{t\geq 0}$ be a Standard Brownian motion (on $\mathbb{R}$) starting at x. Let

$$X_t^x = \beta_t^x - [\beta_t^x] \qquad t \geq 0,$$

where $[z]$ denotes the integer part of a real number z. Clearly $P(X_0^x = x) = 1$. We will show that X^x is a solution of the martingale problem for A. First note that X^x is neither a r.c.l.l. nor a l.c.r.l. process. However

$$\lim_{s\uparrow t} X_s^x \text{ and } \lim_{s\downarrow t} X_s^x \text{ exist a.s. } \forall t.$$

Let $(\Delta X)_t = \lim_{s\uparrow t} X_s^x - \lim_{s\downarrow t} X_s^x$. Let $\tau_0 \equiv 0$. For $i \geq 1$ let τ_i be defined by

$$\tau_i = \inf\{t > \tau_{i-1} : (\Delta X)_t \neq 0\}.$$

Then for all i, τ_i are $(\mathcal{F}_{t+}^X)$ stopping times. Note that for every $i \geq 1$, $(\Delta X)_{\tau_i} = \pm 1$ and $(\Delta X)_t = 0$ for $t \neq \tau_i$. Moreover,

$$\beta_{\tau_i}^x \equiv 0 \pmod 1 \qquad \forall i \geq 1, \tag{5.1}$$

which in turn implies $X_{\tau_i} = 0$ for all $i \geq 1$. As a consequence of (5.1), we also get

$$P(\tau_i < \infty) = 1 \quad \forall i \geq 1 \text{ and } P(\lim_{i\to\infty} \tau_i = \infty) = 1.$$

Also, it is easy to see that for every $i \geq 1$, $(\Delta X)_{\tau_i} = 1$ if and only if there exists $\varepsilon > 0$ such that $\beta_s^x > \beta_{\tau_i}^x$ for all $s \in (\tau_i - \varepsilon, \tau_i)$ and $\beta_s^x < \beta_{\tau_i}^x$ for all $s \in (\tau_i, \tau_i + \varepsilon)$. Similarly, $\{(\Delta X)_{\tau_i} = -1\}$ is the set $\{\exists \varepsilon > 0 : \beta_s^x < \beta_{\tau_i}^x \forall s \in (\tau_i - \varepsilon, \tau_i) \text{ and } \beta_s^x > \beta_{\tau_i}^x \forall s \in (\tau_i, \tau_i + \varepsilon)\}$. Hence we can reconstruct β^x from X^x as follows by noting that

$$\beta_s^x = X_s^x \text{ for } 0 \leq s < \tau_1, \tag{5.2}$$

and for $i \geq 1$

$$\beta_s^x = \begin{cases} \beta_{\tau_{i-1}}^x + X_s^x & \text{if } \tau_{i-1} \leq s < \tau_i, (\Delta X)_{\tau_{i-1}} = -1, \\ \beta_{\tau_{i-1}}^x + X_s^x - 1 & \text{if } \tau_{i-1} \leq s < \tau_i, (\Delta X)_{\tau_{i-1}} = 1. \end{cases} \tag{5.3}$$

This, in particular, implies for every t, $(\mathcal{F}^X_{t+}) = (\mathcal{F}^\beta_{t+}) = (\mathcal{F}^\beta_t) = (\mathcal{F}^X_t)$. Thus τ_i is a $(\mathcal{F}^X_t)$-stop-time for every i. Also, rewriting (5.2) and (5.3)) we get for $\tau_{i-1} \le t < \tau_i$

$$X^x_t = \begin{cases} \beta^x_t - \beta^x_{\tau_{i-1}} & \text{if } (\Delta X)_{\tau_{i-1}} = -1, \\ \beta^x_t - \beta^x_{\tau_{i-1}} + 1 & \text{if } (\Delta X)_{\tau_{i-1}} = 1. \end{cases}$$

The Strong Markov Property of Brownian motion now implies that

$$f(X^x_{t\wedge\tau_i}) - f(X^x_{t\wedge\tau_{i-1}}) - \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} Af(X^x_s)ds$$

is a martingale for all $i \ge 1, f \in \mathcal{D}(A)$. Now fix $f \in \mathcal{D}(A)$. Let $M_t = f(X^x_t) - \int_0^t Af(X^x_s)ds$. Let σ be a bounded $(\mathcal{F}^X_t)$ stop-time. Then using the observation made in the previous paragraph we get

$$\begin{aligned} E[M_\sigma] &= E\sum_{i=1}^\infty (M_{\sigma\wedge\tau_i} - M_{\sigma\wedge\tau_{i-1}}) \\ &= \sum_{i=1}^\infty E(M_{\sigma\wedge\tau_i} - M_{\sigma\wedge\tau_{i-1}}) \\ &= \sum_{i=1}^\infty E\left(f(X^x_{\sigma\wedge\tau_i}) - f(X^x_{\sigma\wedge\tau_{i-1}}) - \int_{\sigma\wedge\tau_{i-1}}^{\sigma\wedge\tau_i} Af(X^x_s)ds \right) \\ &= 0. \end{aligned} \tag{5.4}$$

This implies that X^x is a solution of the martingale problem for (A, δ_x). Uniqueness of solution is proved similarly. We give a brief sketch of the proof to avoid repetition. Let (Y_t) be any other solution of the martingale problem for (A, δ_x). Then for any $s > 0$ with $Y_s \in (0, 1)$, $t \longrightarrow Y_{s+t}$ is a Brownian motion till the next jump. See *e.g.* Section IV.6 of Ethier and Kurtz (1986). In particular the left hand and right limits of Y_s exist at all time points t a.s. and all the jumps are of size ±1. Let $\sigma_0 \equiv 0$ and for $i \ge 1, \sigma_i$ denote the successive jump times of Y. Thus following (5.2) and (5.3) we define a $\mathbb{R}$ valued process B by

$$B_s = Y_s \text{ for } 0 \le s < \sigma_1,$$

and then having defined (recursively) B_s for all $s < \sigma_{i-1}$ we define

$$B_s = \begin{cases} B_{\sigma_{i-1}} + Y_s & \text{if } \sigma_{i-1} \le s < \sigma_i, (\Delta X)_{\sigma_{i-1}} = -1, \\ B_{\sigma_{i-1}} + Y_s - 1 & \text{if } \sigma_{i-1} \le s < \sigma_i, (\Delta X)_{\sigma_{i-1}} = 1. \end{cases}$$

It is now easy to check, arguing as in (5.4), and using the fact that Y is a solution of the martingale problem for A, that for every $g \in C_b^2(\mathbb{R})$,

$$g(B_t) - \int_0^t \frac{1}{2} g''(B_s) ds$$

is a martingale. Thus B has to be a Brownian motion. This in turn gives the required uniqueness of solution of the martingale problem for A. As seen above, the unique solution X^x is neither a r.c.l.l. nor a l.c.r.l. process. However, (5.1) implies that the set of jumps of X^x is contained in the set

$$\{t : \beta_t^x \text{ is an integer.}\}$$

Hence, X^x is continuous in probability.

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