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Clarkson Inequalities With Several Operators

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Abstract : We prove several inequalities for trace norms of sums of n operators with roots of unity coefficients. When $n = 2$ these reduce to the classical Clarkson inequalities and their non-commutative analogues.

1 Introduction

The classical inequalities of Clarkson [9] for the Lebesgue spaces L_p, and their non-commutative analogues for the Schatten trace ideals C_p play an important role in analysis, operator theory, and mathematical physics. They have been generalised in various directions. Among these are versions for more general symmetric norms [4] and for the Haagerup L_p-spaces [10], as well as refinements [2]. In this paper we obtain extensions of these (and related) inequalities in another direction, replacing pairs of operators by n-tuples. Let A be a linear operator on a complex separable Hilbert space. If A is compact, we denote by $\{s_j(A)\}$ the sequence of decreasingly ordered singular values of A. For $0 < p < \infty$, let

$$\|A\|_p = \left[\sum (s_j(A))^p\right]^{1/p}. \tag{1}$$

For $1 \leq p < \infty$, this defines a norm on the class C_p consisting of operators A for which $\|A\|_p$ is finite. This is called the Schatten p-norm. By convention $\|A\|_\infty = s_1(A)$ is the operator bound norm of A. These p-norms belong to a larger class of symmetric or unitarily invariant norms. Such a norm $|||.|||$ is characterized by the equality

$$|||A||| = |||UAV|||, \tag{2}$$

for all A and unitary U, V. When we use the symbol $\|A\|_p$ or $|||A|||$ it is implicit that the operator A belongs to the class of operators on which this norm is defined. See [3] for properties of these norms. For $1 \leq p \leq \infty$, we denote by q the conjugate index defined by the relation $1/p+1/q = 1$. The symbol $|A|$ stands for the positive operator $(A^\star A)^{1/2}$. We prove the following four theorems.

In each of the statements $A_0, A_1, \ldots, A_{n-1}$ are linear operators and $\omega_0, \omega_1, \ldots, \omega_{n-1}$ are the n roots of unity with $\omega_j = e^{2\pi i j/n}$, $0 \le j \le n-1$.

Theorem 1 *For $2 \le p \le \infty$, we have*

$$n^{\frac{2}{p}} \sum_{j=0}^{n-1} \|A_j\|_p^2 \le \sum_{k=0}^{n-1} \| \sum_{j=0}^{n-1} \omega_j^k A_j \|_p^2 \le n^{2-2/p} \sum_{j=0}^{n-1} \|A_j\|_p^2. \tag{3}$$

For $0 < p \le 2$ these two inequalities are reversed.

Theorem 2 *For $2 \le p < \infty$, we have*

$$n \sum_{j=0}^{n-1} \|A_j\|_p^p \le \sum_{k=0}^{n-1} \| \sum_{j=0}^{n-1} \omega_j^k A_j \|_p^p \le n^{p-1} \sum_{j=0}^{n-1} \|A_j\|_p^p. \tag{4}$$

For $0 < p \le 2$, these two inequalities are reversed.

Theorem 3 *For $2 \le p < \infty$, we have*

$$n||| \sum_{j=0}^{n-1} |A_j|^p ||| \le ||| \sum_{k=0}^{n-1} | \sum_{j=0}^{n-1} \omega_j^k A_j|^p ||| \le n^{p-1} ||| \sum_{j=0}^{n-1} |A_j|^p |||, \tag{5}$$

for every unitarily invariant norm $|||.|||$. For $0 < p \le 2$, these two inequalities are reversed.

Theorem 4 *For $2 \le p < \infty$, we have*

$$n \left(\sum_{j=0}^{n-1} \|A_j\|_p^p \right)^{q/p} \le \sum_{k=0}^{n-1} \| \sum_{j=0}^{n-1} \omega_j^k A_j \|_p^q. \tag{6}$$

For $1 < p \le 2$, this inequality is reversed.

When $n = 2$, Theorem 1 gives for any pair A, B the inequalities

$$2^{2/p} \left(\|A\|_p^2 + \|B\|_p^2 \right) \le \|A+B\|_p^2 + \|A-B\|_p^2 \le 2^{2-2/p} \left(\|A\|_p^2 + \|B\|_p^2 \right), \tag{7}$$

for $2 \le p \le \infty$, and the reverse inequalities for $0 < p \le 2$. Theorem 2 gives

$$2 \left(\|A\|_p^p + \|B\|_p^p \right) \le \|A+B\|_p^p + \|A-B\|_p^p \le 2^{p-1} \left(\|A\|_p^p + \|B\|_p^p \right), \tag{8}$$

for $2 \le p < \infty$, and the reverse inequalities for $0 < p \le 2$. For $p = 2$, (7) and (8) both reduce to the *parallelogram law*

$$\|A+B\|_2^2 + \|A-B\|_2^2 = 2 \left(\|A\|_2^2 + \|B\|_2^2 \right). \tag{9}$$

The special norm $||.||_2$ arises from an inner product $\langle A, B\rangle = \operatorname{tr} A^\star B$ and must satisfy this law. The generalisation given in Theorem 1 can be obtained easily in this case. The inequalities (8) are one half of the celebrated Clarkson inequalities. A recent generalisation due to Hirzallah and Kittaneh [11] says

$$2|||\, |A|^p + |B|^p \,||| \leq |||\, |A+B|^p + |A-B|^p \,||| \leq 2^{p-1}|||\, |A|^p + |B|^p \,|||, \tag{10}$$

for $2 \leq p < \infty$; and the two inequalities are reversed for $0 < p \leq 2$. The inequalities (8) follow from these by choosing for $|||.|||$ the special norm $||.||_1$. Theorem 3 includes the inequalities (10) as a special case. When $n = 2$, (6) reduces to the inequality

$$2\left(\|A\|_p^p + \|B\|_p^p\right)^{q/p} \leq \|A+B\|_p^q + \|A-B\|_p^q, \tag{11}$$

for $2 \leq p < \infty$, and the reverse inequality for $1 < p \leq 2$. These are the other half of the Clarkson inequalities. They are much harder to prove, and are stronger, than the inequalities (8). A simple proof and a generalisation of the inequalities (8) were given by Bhatia and Holbrook in [4]. Some of their ideas were developed further in our paper [5]. In Section 2 we give a proof of Theorems 1 and 2 using these results. In Section 3 we discuss some extensions of these results as in [4]. In section 4, we outline a proof of Theorem 3 and of some more general theorems. We follow the approach in [11]. This was based on results of Ando and Zhan [1], and we show how these can be generalised to n-tuples. The harder Clarkson inequalities (11) are usually proved by complex interpolation methods. In section 5, we show how one such proof as given by Fack and Kosaki [10] can be modified to give Theorem 4. Sharper versions of (7), (8), (11) have been proved by Ball, Carlen and Lieb [2] by deeper arguments. Our results go in a different direction.

2 Proofs of Theorems 1 and 2

Consider the $n \times n$ matrix

$$T = [T_{jk}]\,, \quad 0 \leq j, k \leq n-1 \tag{12}$$

where the entries T_{jk} are operators. In [5, Thm 1] we showed that

$$\|T\|_p^2 \leq \sum_{j,k} \|T_{jk}\|_p^2 \quad \text{for } 2 \leq p \leq \infty. \tag{13}$$

Now, given n operators $A_0, \ldots, A_{n-1}$ let T be the block circulant matrix

$$T = \operatorname{circ}(A_0, \ldots, A_{n-1})\,. \tag{14}$$

This is the $n \times n$ matrix whose first row has entries $A_0, \ldots, A_{n-1}$ and the other rows are obtained by successive cyclic permutations of these entries. Let

$$F_n = \frac{1}{\sqrt{n}} \begin{bmatrix} \omega_0^0 & \omega_1^0 & \ldots & \omega_{n-1}^0 \\ \omega_0^1 & \omega_1^1 & \ldots & \omega_{n-1}^1 \\ \ldots & \ldots & \ldots & \ldots \\ \omega_0^{n-1} & \omega_1^{n-1} & \ldots & \omega_{n-1}^{n-1} \end{bmatrix}$$

be the finite Fourier transform matrix of size n. Let $W = F_n \otimes I$. This is the block matrix whose jk entry is $\omega_k^j I$. It is easy to see that if T is the block circulant matrix in (14) then $X = W^\star T W$ is a block-diagonal matrix and the kth entry on its diagonal is the operator

$$X_{kk} = \sum_{j=0}^{n-1} \omega_j^k A_j. \tag{15}$$

Now note that

$$\|T\|_p = \|X\|_p = \left(\sum_{k=0}^{n-1} \|X_{kk}\|_p^p \right)^{1/p}. \tag{16}$$

Using (13)-(16) we obtain

$$\left[\sum_{k=0}^{n-1} \| \sum_{j=0}^{n-1} \omega_j^k A_j \|_p^p \right]^{2/p} \leq n \sum_{j=0}^{n-1} \|A_j\|_p^2, \tag{17}$$

for $2 \leq p < \infty$. For these values of p the function $f(x) = x^{2/p}$ is concave on the positive half-line. Hence

$$n^{2/p-1} \left(x_0^{2/p} + \cdots + x_{n-1}^{2/p} \right) \leq (x_0 + \cdots + x_{n-1})^{2/p}. \tag{18}$$

Using this we get from (17) the inequality

$$n^{2/p-1} \sum_{k=0}^{n-1} \| \sum_{j=0}^{n-1} \omega_j^k A_j \|_p^2 \leq n \sum_{j=0}^{n-1} \|A_j\|_p^2, \tag{19}$$

for $2 \leq p \leq \infty$. This is the second inequality in (3). The first inequality in (3) can be obtained from this by a change of variables. Let

$$B_k = \sum_{j=0}^{n-1} \omega_j^k A_j \quad \text{for } 0 \leq k \leq n-1. \tag{20}$$

Replace the n-tuple $(A_0, \ldots, A_{n-1})$ in the inequality just proved by $(B_0, \ldots, B_{n-1})$. Note that the n-tuple whose kth entry is $\sum_j \omega_j^k B_j$ is the same as the n-tuple $(nA_0, nA_1, \ldots, nA_{n-1})$ up to a permutation. This leads to the first inequality in (3). When $1 \leq p \leq 2$, the inequality (13) is reversed [5, Thm 1]. So the inequality (17) is reversed. The function $f(x) = x^{2/p}$ is convex in this case, and the inequality (18) is reversed. As a result both inequalities in (3) are reversed. This completes the proof of Theorem 1 for $1 \le p \le \infty$. The case $0 < p < 1$ is discussed in Section 3. The proof of Theorem 2 runs parallel to that of Theorem 1. For T as in (12) we have from [5, Thm 2]

$$\sum_{j,k} \|T_{jk}\|_p^p \le \|T\|_p^p \quad \text{for } 2 \le p < \infty, \tag{21}$$

and the inequality is reversed for $0 < p \le 2$. Start with this instead of (13) and follow the steps of the proof of Theorem 1. One obtains Theorem 2 for $1 \le p < \infty$. The case $0 < p < 1$ is discussed in Section 3. The inequalities of Theorems 1 and 2 are sharp. For $0 \le j \le n-1$ let A_j be the diagonal matrix with its jj entry equal to 1 and all its other entries equal to 0. In this case the first inequality in (3) and in (4) is an equality. On the other hand if we choose $A_j = \left(\omega_0^j, \omega_1^j, \ldots, \omega_{n-1}^j\right)$ for $0 \le j \le n-1$, we see that the other two inequalities are equalities in this case. A simple consequences of the inequality (7) is the following result proved in [6]. Let T be any operator and let $T = A + iB$ be its Cartesian decomposition with A, B Hermitian. Then for $2 \le p \le \infty$

$$2^{2/p-1}\left(\|A\|_p^2 + \|B\|_p^2\right) \le \|T\|_p^2 \le 2^{1-2/p}\left(\|A\|_p^2 + \|B\|_p^2\right), \tag{22}$$

and the inequalities are reversed for $0 < p \le 2$. Note that in this case we have from (8)

$$\|A\|_p^p + \|B\|_p^p \le \|T\|_p^p \le 2^{p-2}\left(\|A\|_p^p + \|B\|_p^p\right), \tag{23}$$

for $2 \le p < \infty$, and the reverse inequalities for $0 < p \le 2$. The inequalities (22) can be derived from (23) by a simple convexity argument. More subtle norm inequalities for the Cartesian decomposition may be found in [7,8].

3 Extensions and Remarks

We have proved Theorems 1 and 2 using results in [5]. There are other connections between [4,5] and the present paper. We point out some of them.

1. Let T be the block matrix (12) and let U_j be the block-diagonal operator

$$U_j = \operatorname{diag}\left(\omega_0^j I, \ldots, \omega_{n-1}^j I\right), \quad 0 \le j \le n-1.$$

Let $A_j = U_j^* T U_j$. The second inequality in (3) then gives

$$n^{4/p-2} \sum_{j,k} \|T_{jk}\|_p^2 \le \|T\|_p^2 \quad \text{for } 2 \le p \le \infty.$$

This is the inequality complementary to (13) proved in [5] by other arguments.

2. A unitarily invariant norm $|||.|||$ is called a *Q-norm* if there exists another unitarily invariant norm $|||.|||^{\widehat{}}$ such that $|||A|||^2 = |||A^\star A|||^{\widehat{}}$. The Schatten p-norms for $p \geq 2$ are Q-norms since $\|A\|_p^2 = \|A^\star A\|_{p/2}$. The crucial observation in [4] was a reinterpretation of the Clarkson inequalities (8) in such a way that a generalisation to Q-norms and their duals became possible. The next remarks concern similar generalisations of Theorems 1 and 2.

3. The following useful identity can be easily verified.

$$\frac{1}{n}\sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-1}\omega_j^k A_j\right)^\star\left(\sum_{j=0}^{n-1}\omega_j^k A_j\right) = \sum_{j=0}^{n-1} A_j^\star A_j. \tag{24}$$

For $n = 2$ this reduces to

$$\frac{(A+B)^\star(A+B)+(A-B)^\star(A-B)}{2} = A^\star A + B^\star B. \tag{25}$$

4. We use the notation $A_0 \oplus \cdots \oplus A_{n-1}$, or $\oplus A_j$, for the block-diagonal operator with operators A_j as its diagonal entries. For positive operators A_j, $0 \leq j \leq n-1$, we have the inequality

$$|||A_0 \oplus \cdots \oplus A_{n-1}||| \leq |||\left(\sum_{j=0}^{n-1} A_j\right) \oplus 0 \cdots \oplus 0|||, \tag{26}$$

for all unitarily invariant norms [5, Lemma 4]. For the p-norms this gives (for positive operators)

$$\sum_{j=0}^{n-1}\|A_j\|_p^p \leq \|\sum_{j=0}^{n-1} A_j\|_p^p \quad 1 \leq p < \infty. \tag{27}$$

For $n = 2$, this is a starting point of a proof of the Clarkson inequalities (8), and its generalisation as in (26) led to stronger versions in [4]. To bring out the relevance of Q-norms we give a different proof of Theorem 1 based on the identity (24) and the inequality (27). Let $A_0, \ldots, A_{n-1}$ be any operators and let B_k be the sum defined in (20). Then for $2 \leq p < \infty$

$$\begin{aligned}
\sum_{k=0}^{n-1}\|B_k\|_p^2 &= \sum_{k=0}^{n-1}\|B_k^\star B_k\|_{p/2} \\
&\geq \|\sum_{k=0}^{n-1} B_k^\star B_k\|_{p/2} \quad \text{(triangle inequality)} \\
&= n\|\sum_{j=0}^{n-1} A_j^\star A_j\|_{p/2} \quad \text{(using (24))} \\
&\geq n\left[\sum_{j=0}^{n-1}\|A_j^\star A_j\|_{p/2}^{p/2}\right]^{2/p} \quad \text{(using (27))} \\
&= n\left[\sum_{j=0}^{n-1}\left(\|A_j\|_p^2\right)^{p/2}\right]^{2/p}
\end{aligned}$$

$$
\begin{aligned}
&\geq n\left[n^{1-p/2}\left(\sum_{j=0}^{n-1}\|A_j\|_p^2\right)^{p/2}\right]^{2/p} \qquad \text{(using (18))}\\
&= n^{2/p}\sum_{j=0}^{n-1}\|A_j\|_p^2.
\end{aligned}
$$

This is the first inequality in (3). In this chain of reasoning inequalities entered at three stages. All get reversed for $0 < p \leq 2$. It has been noted [6, Lemma 1] that for positive operators A_j and $0 < p \leq 1$

$$\sum \|A_j\|_p \leq \|\sum A_j\|_p,$$

and also that the inequality (27) is reversed in this case [6, p.111] or [12, p.20]. The inequality (18) is reversed too in this case. So the statement of Theorem 1 for $1 \leq p \leq 2$ is, in fact, true when $0 < p \leq 2$.

5. Let us now recast Theorem 2 in the mould of [4]. Taking pth roots, the first inequality in (4) can be rewritten as

$$n^{1/p}\| \oplus_{j=0}^{n-1} A_j\|_p \leq \| \oplus_{k=0}^{n-1} B_k\|_p, \quad 2 \leq p < \infty,$$

where B_k is as in (20), and then as

$$\| \oplus_{n \text{ copies}} \left[\oplus_{j=0}^{n-1} A_j\right] \|_p \leq \| \oplus_{k=0}^{n-1} B_k\|_p, \quad 2 \leq p < \infty. \tag{28}$$

In the same way, the second inequality in (4) can be rewritten as

$$n^{1/p}\| \oplus_{k=0}^{n-1} B_k\|_p \leq n\| \oplus_{j=0}^{n-1} A_j\|_p, \quad 2 \leq p < \infty,$$

and then as

$$\| \oplus_{n \text{ copies}} \left[\oplus_{k=0}^{n-1} B_k\right] \|_p \leq n\| \oplus_{j=0}^{n-1} A_j\|_p, \quad 2 \leq p < \infty. \tag{29}$$

In this form the inequalities (28) and (29) shed some of their dependence on p compared to the (equivalent) inequalities (4). What is left of p can be removed too. The inequalities (28) and (29) are true for all Q-norms. For the duals of Q-norms they are reversed. This can be proved using the ideas in [4] and this paper. We do not give the details here.

6. The case $0 < p < 1$ of Theorem 2 is proved on the same lines as in Remark 4 above.

7. It is tempting to attempt a generalisation of Theorem 1 on the same lines as for Theorem 2 in Remark 5. Let us start with the special case $n = 2$. The first inequality in (7) can be rewritten as

$$\|A \oplus A\|_p^2 + \|B \oplus B\|_p^2 \leq \|A + B\|_p^2 + \|A - B\|_p^2 \quad \text{for } 2 \leq p \leq \infty. \tag{30}$$

This is the same as saying

$$\|A^\star A \oplus A^\star A\|_p + \|B^\star B \oplus B^\star B\|_p \leq \|(A+B)^\star (A+B)\|_p + \|(A-B)^\star (A-B)\|_p \quad \text{for} 1 \leq p \leq \infty. \tag{31}$$

To ask whether the inequality (30) might be true for all Q-norms is to ask whether (31) might be true for all unitarily invariant norms; i.e., whether we have

$$|||A^\star A \oplus A^\star A||| + |||B^\star B + B^\star B||| \leq |||(A+B)^\star (A+B) \oplus 0||| + |||(A-B)^\star (A-B) \oplus 0||| \tag{32}$$

for all unitarily invariant norms. The answer is no. On 8×8 matrices consider the norm

$$|||A||| = \left[(s_1(A) + s_2(A))^2 + (s_3(A) + s_4(A))^2 \right]^{1/2}.$$

Let $A = \operatorname{diag}(1,1,0,0)$, $B = \operatorname{diag}(0,0,2^{1/4},0)$. The inequality (32) breaks down for this choice.

8. Ball, Carlen and Lieb [2] have proved the following inequalities for $1 \leq p \leq 2$:

$$\begin{aligned} \|A\|_p^2 + (p-1)\|B\|_p^2 &\leq \frac{1}{2}\left(\|A+B\|_p^2 + \|A-B\|_p^2\right), \text{ and} \\ \|A\|_p^2 + (p-1)\|B\|_p^2 &\leq \frac{1}{2^{2/p}}\left(\|A+B\|_p^p + \|A-B\|_p^p\right)^{2/p}. \end{aligned}$$

Compare the first of these with one of the inequalities in (7)

$$2^{1-2/p}\left(\|A\|_p^2 + \|B\|_p^2\right) \leq \frac{1}{2}\left(\|A+B\|_p^2 + \|A-B\|_p^2\right),$$

and compare the second with the inequality obtained by following some of the steps of Remark 4 :

$$\|A\|_p^2 + \|B\|_p^2 \leq \frac{1}{2}\left(\|A+B\|_p^p + \|A-B\|_p^p\right)^{2/p}.$$

4 Proof of Theorem 3 and Generalisations

This part has to be read along with the papers of Ando-Zhan [1] and Hirzallah-Kittaneh [11]. We indicate how results obtained there for $n = 2$ can be proved for $n > 2$. Recall that a non-negative function f on $[0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ whenever A, B are positive operators with $A \geq B$. The function $f(t) = t^p$ is operator monotone for $0 < p \leq 1$. Thus for $1 \leq p < \infty$ the inverse function of $f(t) = t^p$ is operator monotone. See [3, Chapter V].

Theorem 5 *(Generalised Ando-Zhan Theorem) Let $A_0, \ldots, A_{n-1}$ be positive operators. Then for every unitarily invariant norm*

(i)

$$|||\sum_{j=0}^{n-1} f(A_j)||| \geq |||f\left(\sum_{j=0}^{n-1} A_j\right)||| \tag{33}$$

for every non-negative operator monotone function f on $[0, \infty)$; and

(ii) this inequality is reversed if f is a non-negative increasing function on $[0, \infty)$ such that $f(0) = 0$, $f(\infty) = \infty$, and the inverse function of f is operator monotone.

Ando and Zhan [1] have proved this for $n = 2$. An analysis of their proof shows that all their arguments can be suitably modified when $n > 2$. In particular, in their crucial Lemma 1 we can replace the sum $A + B$ by $\sum_j A_j$, and check that the same proof works. Using this we can prove the following.

Theorem 6 *Let $A_0, \ldots, A_{n-1}$ be any operators. Then for every unitarily invariant norm we have*

(i)

$$n|||\sum_{j=0}^{n-1} f\left(|A_j|\right)||| \leq |||\sum_{k=0}^{n-1} f\left(|\sum_{j=0}^{n-1} \omega_j^k A_j|\right)||| \leq \frac{1}{n}|||\sum_{j=0}^{n-1} f\left(n|A_j|\right)|||, \tag{34}$$

for every increasing function f on $[0, \infty)$ such that $f(0) = 0$, $f(\infty) = \infty$, and the inverse function of $g(t) = f\left(\sqrt{t}\right)$ is operator monotone;

(ii) the two inequalities in (34) are reversed for every nonnegative function f on $[0, \infty)$ such that $h(t) = f(\sqrt{t})$ is operator monotone.

The $n = 2$ case of Theorem 6 has been proved by Hirzallah and Kittaneh [11]. Their arguments can be modified replacing the Ando-Zhan theorem by its generalisation pointed out above. Their Lemma 1 needs no change. At one stage we need the identity

$$\frac{1}{n}\sum_{k=0}^{n-1}\left|\sum_{j=0}^{n-1} \omega_j^k A_j\right|^2 = \sum_{j=0}^{n-1} |A_j|^2. \tag{35}$$

This is just the identity (24). This substitutes for its $n = 2$ version used in [11] (p. 366 line 6). We leave the rest of the details to the reader. The two parts of Theorem 3 follow from the corresponding parts of Theorem 6 upon choosing $f(t) = t^p$ with $p \geq 2$ and $0 < p \leq 2$, respectively. We remark that Corollaries 1-3 of [1] and Corollaries 2,3 of [11] too can be generalised to n-tuples of operators in this manner.

5 Proof of Theorem 4

Imitating the standard complex interpolation proof of the $n = 2$ case, we give a proof of Theorem 4 for $1 < p \leq 2$. The ideas are the same as in [10]. At a crucial stage we need a generalisation of the parallelogram law provided by Theorem 1. **Lemma.** *Let $A_0, \ldots, A_{n-1}$ be operators in the Schatten p-class C_p for some $1 < p \leq 2$. Let B_k be the sum defined in (20) and let Y_k, $0 \leq k \leq n-1$ be operators in the dual class C_q. Then*

$$\left| \mathrm{tr} \sum_{k=0}^{n-1} Y_k B_k \right| \leq n^{1/q} \left(\sum_{j=0}^{n-1} \|A_j\|_p^p \right)^{1/p} \left(\sum_{k=0}^{n-1} \|Y_k\|_q^p \right)^{1/p}. \tag{36}$$

Proof. Let $A_j = |A_j|W_j$ and $Y_k = V_k|Y_k|$ be right and left polar decompositions of A_j and Y_k, respectively. Here W_j and Y_k are partial isometries. We have $\frac{1}{2} \leq \frac{1}{p} < 1$. For the complex variable $z = x + iy$ with $\frac{1}{2} \leq x \leq 1$ let

$$\begin{aligned} A_j(z) &= |A_j|^{pz} W_j \\ Y_k(z) &= \|Y_k\|_q^{pz - q(1-z)} V_k |Y_k|^{q(1-z)}. \end{aligned}$$

Note that $A_j(1/p) = A_j$ and $Y_k(1/p) = Y_k$. Let

$$f(z) = \mathrm{tr} \sum_{k=0}^{n-1} Y_k(z) B_k(z).$$

The left hand side of (36) is $|f(1/p)|$. We can estimate this if we have bounds for $|f(z)|$ at $x = \frac{1}{2}$ and $x = 1$. If $x = 1$, we have

$$|\mathrm{tr}\, Y_k(z) A_j(z)| = \|Y_k\|_q^p \left| \mathrm{tr}\, V_k |Y_k|^{-iqy} |A_j|^{p(1+iy)} W_j \right|$$

Using the facts that for any operator T, $|\mathrm{tr}\, T| \leq \|T\|_1$ and $|||XTZ||| \leq ||X||\ |||T|||\ ||Z||$ for any three operator X, T, Z and unitarily invariant norm $|||.|||$, we get from this

$$|\mathrm{tr}\, Y_k(z) A_j(z)| \leq \|Y_k\|_q^p \|A_j\|_p^p,$$

for all $0 \leq j,\ k \leq n-1$. Hence

$$|f(z)| = \left| \mathrm{tr} \sum_{k=0}^{n-1} Y_k(z) B_k(z) \right| \leq \left(\sum_{k=0}^{n-1} \|Y_k\|_q^p \right) \left(\sum_{j=0}^{n-1} \|A_j\|_p^p \right), \tag{37}$$

when $x = 1$. When $x = 1/2$, the operators $A_j(z)$ and $Y_k(z)$ are in C_2 and

$$|f(z)| \leq \sum_{k=0}^{n-1} |\mathrm{tr}\, Y_k(z) B_k(z)|$$

$$\begin{aligned}
&\leq \sum_{k=0}^{n-1} \|Y_k(z)\|_2 \|B_k(z)\|_2 \\
&\leq \left(\sum_{k=0}^{n-1} \|Y_k(z)\|_2^2\right)^{1/2} \left(\sum_{k=0}^{n-1} \|B_k(z)\|_2^2\right)^{1/2} \\
&= n^{1/2} \left(\sum_{k=0}^{n-1} \|Y_k(z)\|_2^2\right)^{1/2} \left(\sum_{j=0}^{n-1} \|A_j(z)\|_2^2\right)^{1/2}.
\end{aligned}$$

The equality at the last step is a consequence of Theorem 1 specialised to the case $p = 2$. Note that when $x = 1/2$ we have $\|A_j(z)\|_2^2 = \|A_j\|_p^p$, and $\|Y_k(z)\|_2^2 = \|Y_k\|_q^p$. Hence

$$|f(z)| \leq n^{1/2} \left(\sum_{k=0}^{n-1} \|Y_k\|_q^p\right)^{1/2} \left(\sum_{j=0}^{n-1} \|A_j\|_p^p\right)^{1/2}, \tag{38}$$

when $x = 1/2$. If M_1 is the right hand side of (37) and M_2 that of (38), then by the three line theorem, we have for $\frac{1}{2} \leq \frac{1}{p} < 1$

$$|f(1/p)| \leq M_1^{2(1/p-1/2)} M_2^{2(1-1/p)}.$$

This gives (36). ■

Now to prove Theorem 4 let $B_k = U_k|B_k|$ be a polar decomposition and let

$$Y_k = \|B_k\|_p^{q-p} |B_k|^{p-1} U_k^\star.$$

It is easy to see that

$$\operatorname{tr} Y_k B_k = \|B_k\|_p^q = \|Y_k\|_q^p.$$

So we get from (36)

$$\sum_{k=0}^{n-1} \|B_k\|_p^q \leq n^{1/q} \left(\sum_{j=0}^{n-1} \|A_j\|_p^p\right)^{1/p} \left(\sum_{k=0}^{n-1} \|B_k\|_p^q\right)^{1/p}.$$

This is the same as saying

$$\sum_{k=0}^{n-1} \|B_k\|_p^q \leq n \left(\sum_{j=0}^{n-1} \|A_j\|_p^p\right)^{q/p}, \quad 1 < p \leq 2.$$

This proves Theorem 4 for $1 < p \leq 2$. The reverse inequality for $2 \leq p < \infty$ can be obtained from this by a duality argument. ■By a change of variables a pair of complementary inequalities can be obtained as in Theorems 1-3. As pointed out earlier [2,4] the inequalities of Theorem 2 follow from those of Theorem 4 by simple convexity arguments. Theorem 1 too can be derived from Theorem 4 by such arguments. For example, for $2 \leq p < \infty$ we have from (6)

$$\left(\sum_{j=0}^{n-1} \|A_j\|_p^p\right)^{1/p} \leq \left(\frac{1}{n}\sum_{k=0}^{n-1} \|\sum_{j=0}^{n-1} \omega_j^k A_j\|_p^q\right)^{1/q}. \tag{39}$$

On the positive half-line the function $f(x) = x^{2/q}$ is convex and the function $g(x) = x^{2/p}$ concave. Using this we can get the first inequality in (3) from the inequality (39). The proof given in Section 2 is based on easier ideas.

References :

1. T. Ando and X. Zhan, *Norm inequalities related to operator monotone functions,* Math. Ann., 315 (1999) 771-780.

2. K. Ball, E. A. Carlen and E. H. Lieb, *Sharp uniform convexity and smoothness inequalities for trace norms,* Invent. Math., 115 (1994) 463-482.

3. R. Bhatia, *Matrix Analysis,* Springer, New York, 1997.

4. R. Bhatia and J.A.R. Holbrook, *On the Clarkson-McCarthy inequalities,* Math. Ann., 281 (1988) 7-12.

5. R. Bhatia and F. Kittaneh, *Norm inequalities for partitioned operators and an application,* Math. Ann., 287 (1990) 719-726.

6. R. Bhatia and F. Kittaneh, *Cartesian decompositions and Schatten norms,* Linear Algebra Appl., 318 (2000) 109-116.

7. R. Bhatia and X. Zhan, *Compact operators whose real and imaginary parts are positive,* Proc. Amer. Math. Soc., 129 (2001) 2277-2281.

8. R. Bhatia and X. Zhan, *Norm inequalities for operators with positive real part,* J. Operator Theory, to appear.

9. J. A. Clarkson, *Uniformly convex spaces,* Trans. Amer. Math. Soc., 40 (1936) 396-414.

10. T. Fack and H. Kosaki, *Generalised s-numbers of τ-measurable operators,* Pacific J. Math., 123 (1986) 269-300.

11. O. Hirzallah and F. Kittaneh, *Non-commutative Clarkson inequalities for unitarily invariant norms,* Pacific J. Math., 202 (2002) 363-369.

12. B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, 1979.