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# Minimum $L_1$-norm Estimation for Fractional Ornstein-Uhlenbeck Type Process

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# Minimum $L_1$-norm Estimation for Fractional Ornstein-Uhlenbeck Type Process

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**Abstract**

We investigate the asymptotic properties of the minimum $L_1$-norm estimator of the drift parameter for fractional Ornstein-Uhlenbeck type process satisfying a linear stochastic differential equation driven by a fractional Brownian motion.

**Keywords and phrases**: Minimum $L_1$-norm estimation; fractional Ornstein-Uhlenbeck process; fractional Brownian motion.

**Mathematics Subject Classification**: Primary 62 M 09, Secondary 60 G 15.

## 1 Introduction

Long range dependence phenomenon is said to occur in a stationary time series $\{X_n, n \geq 0\}$ if the $Cov(X_0, X_n)$ of the time series tend to zero as $n \to \infty$ and yet it satisfies the condition

$$\sum_{n=0}^{\infty} |Cov(X_0, X_n)| = \infty.$$

In other words $Cov(X_0, X_n)$ tends to zero but so slowly that their sums diverge. This phenomenon was first observed by the hydrologist Hurst (1951) on projects involving the design of reservoirs along the Nile river (cf. Montanari (2003)) and by others in hydrological time series. It was recently observed that a similar phenomenon occurs in problems concerned with traffic patterns of packet flows in high-speed data net works such as the Internet (cf. Willinger et al. (2003), Norros (2003)). The long range dependence pattern is also observed in macroeconomics and finance (cf. Henry and Zafforoni (2003)). Long range dependence is also related to the concept of self-similarity for a stochastic process. A stochastic process $\{X(t), t \in R\}$ is said to be $H$-self-similar with index $H > 0$ if for evey $a > 0$, the processes $\{X(at), t \in R\}$ and the process $\{a^H X(t), t \in R\}$ have the same finite dimensional distributions. Supose a self-similar process has stationary increments. Then the increments form a stationary time series which exhibits long range dependence. A gaussian $H$-self-similar process with stationary increments with $0 < H < 1$ is called a fractional Brownian motion. A recent monograph by Doukhan et al.(2003) discusses theory and applications of long range dependence and properties of fractional brownian motion (Taqqu (2003)). If $H = \frac{1}{2}$, then the fractional Brownian motion reduces to the standard Brownian motion also called the Wiener process.

Diffusion processes and diffusion type processes satisfying stochastic differential equations driven by Wiener processes are used for stochastic modeling in wide variety of sciences such as population genetics, economic processes, signal processing as well as for modeling sunspot activity and more recently in mathematical finance. Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion to model processes involving long range dependence. Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. In a recent paper, Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in (1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = X_0 + \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0. \tag{1. 1}$$

They investigate the problem of estimation of the parameters $\theta$ and $\sigma^2$ based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

Parametric estimation for more general classes of stochastic processes satisfying the linear stochastic differential equations driven fractional Brownian motion, observed over a fixed period of time $T$, is studied in Prakasa Rao (2003a,b). It is well known that the sequential estimation methods might lead to equally efficient estimators from the process observed possibly over a shorter expected period of observation time. We have investigated the conditions for such a phenomenon for estimating the drift parameter of a fractional Ornstein-Uhlenbeck type process in Prakasa Rao (2003c). Novikov (1972) investigated the asymptotic properties of a sequential maximum likelihood estimator for the drift parameter in the Ornstein-Uhlenbeck process.

In spite of the fact that maximum likelihood estimators (MLE) are consistent and asymptotically normal and also asymptotically efficient in general, they have some short comings at the same time. Their calculation is often cumbersome as the expression for MLE involve stochastic integrals which need good approximations for computational purposes. Further more MLE are not robust in the sense that a slight perturbation in the noise component will change the properties of MLE substantially. In order to circumvent such problems, the minimum distance approach is proposed. Properties of the minimum distance estimators (MDE) were discussed in Millar (1984) in a general frame work.

Our aim in this paper is to obtain the minimum $L_1$-norm estimates of the drift parameter of a fractional Ornstein-Uhlenbeck type process and investigate the asymptotic properties of such estimators following the work of Kutoyants and Pilibossian (1994).

## 2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions and the processes discussed in the following are $(\mathcal{F}_t)$-adapted. Further the natural fitration of a process is understood as the $P$-completion of the filtration generated by this process.

Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, s \geq 0. \tag{2. 1}$$

Let us consider a stochastic process $\{X_t, t \geq 0\}$ defined by the stochastic integral equation

$$X_t = x_0 + \theta \int_0^t X(s)ds + \varepsilon W_t^H, 0 \leq t \leq T, \tag{2. 2}$$

where $\theta$ is an unknown drift parameters respectively. For convenience, we write the above integral equation in the form of a stochastic differential equation

$$dX_t == \theta X(t)dt + \varepsilon dW_t^H, X_0 = x_0, 0 \leq t \leq T, \tag{2. 3}$$

driven by the fractional Brownian motion $W^H$. Even though the process $X$ is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration $(\mathcal{Z}_t)$ of the process $Z$ coincides with the natural filtration $(\mathcal{X}_t)$ of the process $X$ (Kleptsyna et al. (2000)). Define, for $0 < s < t$,

$$k_H = 2H\Gamma\,(\frac{3}{2} - H)\Gamma(H + \frac{1}{2}), \tag{2. 4}$$

$$k_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}, \tag{2. 5}$$

$$\lambda_H = \frac{2H\,\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}, \tag{2. 6}$$

$$w_t^H = \lambda_H^{-1} t^{2-2H}, \tag{2. 7}$$

and

$$M_t^H = \int_0^t k_H(t,s)dW_s^H, t \geq 0. \tag{2. 8}$$

The process $M^H$ is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al. (1999)) and its quadratic variance $< M_t^H >= w_t^H$. Further more the natural filtration of the martingale $M^H$ coincides with the natural fitration of the fBM $W^H$. Let

$$K_H(t,s) = H(2H-1)\frac{d}{ds}\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr, 0 \leq s \leq t. \tag{2. 9}$$

The sample paths of the process $\{X_t, t \geq 0\}$ are smooth enough so that the process $Q$ defined by

(2. 10) $$Q(t) = \frac{d}{dw_t^H} \int_0^t k_H(t,s) X_s ds, t \in [0,T]$$

is welldefined where $w^H$ and $k_H$ are as defined in (2.7) and (2.5) respectively and the derivative is understood in the sense of absolute continuity with respect to the measure generated by $w^H$. More over the sample paths of the process $Q$ belong to $L^2([0,T], dw^H)$ a.s. [P]. The following theorem due to Kleptsyna et al. (2000) associates a *fundamental semimartingale* $Z$ associated with the process $X$ such that the natural filtration $(\mathcal{Z}_t)$ coincides with the natural filtration $(\mathcal{X}_t)$ of $X$.

**Theorem 2.1:** Let the process $Z = (Z_t, t \in [0,T])$ be defined by

(2. 11) $$Z_t = \int_0^t k_H(t,s) dX_s$$

where the function $k_H(t,s)$ is as defined in (2.5). Then the following results hold:
(i) The process $Z$ is an $(\mathcal{F}_t)$ -semimartingale with the decomposition

(2. 12) $$Z_t = \theta \int_0^t Q(s) dw_s^H + \sigma M_t^H$$

where $M^H$ is the gaussian martingale defined by (2.8),
(ii) the process $X$ admits the representation

(2. 13) $$X_t = \int_0^t K_H(t,s) dZ_s$$

where the function $K_H$ is as defined in (2.9), and
(iii) the natural fitrations of $(\mathcal{Z}_t)$ and $(\mathcal{X}_t)$ coincide.

Even though the fBm $\{W_t^H, t \geq 0\}$ is not a semi-martingale, it is still possible to define stochastic integration with respect to the fBm for deterministic integrands. For instance, for $f \in L_2(R_+) \cap L_1(R_+)$, one can define a stochastic integral of the form

$$\int_0^T f(s) dW_s^H$$

(cf. Grippenberg and Norris (1996), Norros et al. (1999)). Such a stochastic integral

$$\int_0^T f(s) dW_s^H$$

can be represented in terms of another stochastic integral with respect to the fundamental gaussian martingale $M^H$. The following result is due to Kleptsyna et al. (2000).

For any measurable function $f$ on $[0,T]$ and for $t \in [0,T]$, define

(2. 14) $$K_H^f(t,s) = -2H \frac{d}{ds} \int_s^t f(r) r^{H-\frac{1}{2}} (r-s)^{H-\frac{3}{2}} dr, 0 \leq s \leq t.$$

where the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure (cf. Samko et al.(1993)).

**Lemma 2.2:** Let $M^H$ be the fundamental martingale associated to the fBm $W^H$. Then the following equality holds a.s $[P]$:

$$\int_0^T f(s)dW_s^H = \int_0^T K_H^f(t,s)dM_s^H, t \in [0,T]. \tag{2. 15}$$

provided both the integrals on both sides are well defined.

The following lemma due to Grippenberg and Norris (1996) gives the covariance between the two stochastic integrals $\int_0^T f(s)dW_s^H$ and $\int_0^T g(s)dW_s^H$.

**Lemma 2.3:** For $f, g \in L_2(R_+) \cap L_1(R_+)$,

$$E(\int_0^\infty f(s)dW_s^H \int_0^\infty g(s)dW_s^H) = H(2H-1)\int_0^\infty \int_0^\infty f(s)g(t)|s-t|^{2H-2}dtds. \tag{2. 16}$$

## 3 Minimum $L_1$-norm Estimation

We now consider the problem of estimation of the parameter $\theta$ based on the observation of fractional Ornstein-Uhlenbeck type process $X = \{X_t, 0 \leq t \leq T\}$ satisfying the stochastic differential equation

$$dX_t = \theta X(t)dt + \varepsilon dW_t^H, X_0 = x_0, 0 \leq t \leq T \tag{3. 1}$$

for a fixed time T where $\theta \in \Theta \subset R$ and study its asymptotic properties as $\varepsilon \to 0$.

Let $x_t(\theta)$ be the solution of the above differential equation with $\varepsilon = 0$. It is obvious that

$$x_t(\theta) = x_0 e^{\theta t}, 0 \leq t \leq T. \tag{3. 2}$$

Let

$$S_T(\theta) = \int_0^T |X_t - x_t(\theta)|dt. \tag{3. 3}$$

We define $\theta_\varepsilon^*$ to be a *minimum $L_1$-norm estimator* if there exists a measurable selection $\theta_\varepsilon^*$ such that

$$S_T(\theta_\varepsilon^*) = \inf_{\theta \in \Theta} S_T(\theta). \tag{3. 4}$$

Conditions for the existence of a measurable selection are given in Lemma 3.1.2 in Prakasa Rao (1987). We assume that there exists a measurable selection $\theta_\varepsilon^*$ satisfying the above equation. An alternate way of defining the estimator $\theta_\varepsilon^*$ is by the relation

$$\theta_\varepsilon^* = \arg \inf_{\theta \in \Theta} \int_0^T |X_t - x_t(\theta)|dt. \tag{3. 5}$$

**Consistency:**

Let $W_T^{H*} = \sup_{0\leq t\leq T}|W_t^H|$. The self-similarity of the fractional Brownian motion $W_t^H$ implies that the random variables $W_{at}^H$ and $a^H W_t$ have the same probability distribution for any $a > 0$. Further more it follows from the self-similarity that the supremum process $W^{H*}$ has the property that the random variables $W_{at}^{H*}$ and $a^H W_t^{H*}$ have the same probability distribution for any $a > 0$. Hence we have the following observation due to Novikov and Valkeila (1999).

**Lemma 3.1:** Let $T > 0$ and $\{W_t^H, 0 \leq t \leq T\}$ be a fBm with Hurst index $H$. Let $W_T^{H*} = \sup_{0\leq t\leq T} W_t^H$. Then

$$E(W_T^{H*})^p = K(p,H)T^{pH} \tag{3. 6}$$

for every $p > 0$, where $K(p,H) = E(W_1^{H*})^p$.

Let $\theta_0$ denote the true parameter, For any $\delta > 0$, define

$$g(\delta) = \inf_{|\theta-\theta_0|>\delta} \int_0^T |X_t(\theta) - x_t(\theta_0)|dt. \tag{3. 7}$$

Note that $g(\delta) > 0$ for any $\delta > 0$.

**Theorem 3.2:** For every $p > 0$, there exists a constant $K(p,H)$ such that for every $\delta > 0$,

$$\begin{aligned} P_{\theta_0}^{(\varepsilon)}\{|\theta_\varepsilon^* - \theta_0| > \delta\} &\leq 2^p T^{pH+p} K(p,H) e^{|\theta_0|Tp}(g(\delta))^{-p}\varepsilon^p \\ &= O((g(\delta))^{-p}\varepsilon^p). \end{aligned} \tag{3. 8}$$

**Proof:** Let $||.||$ denote the $L_1$-norm. Then

$$\begin{aligned} P_{\theta_0}^{(\varepsilon)}\{|\theta_\varepsilon^* - \theta_0| > \delta\} &= P_{\theta_0}^{(\varepsilon)}\{\inf_{|\theta-\theta_0|\leq\delta} ||X - x(\theta)|| > \inf_{|\theta-\theta_0|>\delta} ||X - x(\theta)||\} \\ &\leq P_{\theta_0}^{(\varepsilon)}\{\inf_{|\theta-\theta_0|\leq\delta}(||X - x(\theta_0)|| + ||x(\theta) - x(\theta_0)||) \\ &\qquad > \inf_{|\theta-\theta_0|>\delta}(||x(\theta) - x(\theta_0)|| - ||X - x(\theta_0)||)\} \\ &= P_{\theta_0}^{(\varepsilon)}\{2||X - x(\theta_0)|| > \inf_{|\theta-\theta_0|>\delta} ||x(\theta) - x(\theta_0)||\} \\ &= P_{\theta_0}^{(\varepsilon)}\{||X - x(\theta_0)|| > \frac{1}{2}g(\delta)\}. \end{aligned} \tag{3. 9}$$

Since the process $X_t$ satisfies the stochastic differential equation (3,2), it follows that

$$\begin{aligned} X_t - x_t(\theta_0) &= x_0 + \theta_0 \int_0^t X_s ds + \varepsilon W_t^H - x_t(\theta_0) \\ &= \theta_0 \int_0^t (X_s - x_s(\theta_0))ds + \varepsilon W_t^H \end{aligned} \tag{3. 10}$$

since $x_t(\theta) = x_0 e^{\theta t}$. Let $U_t = X_t - x_t(\theta_0)$. Then it follows from the above equation that

$$U_t = \theta_0 \int_0^t U_s \ ds + \varepsilon W_t^H. \tag{3. 11}$$

Let $V_t = |U_t| = |X_t - x_t(\theta_0)|$. The above relation implies that

$$V_t = |X_t - x_t(\theta_0)| \leq |\theta_0| \int_0^t V_s ds + \varepsilon |W_t^H|. \tag{3. 12}$$

Applying Gronwall-Bellman Lemma, we obtain that

$$\sup_{0 \leq t \leq T} |V_t| \leq \varepsilon e^{|\theta_0 T|} \sup_{0 \leq t \leq T} |W_t^H|. \tag{3. 13}$$

Hence

$$\begin{aligned} P_{\theta_0}^{(\varepsilon)}\{||X - x(\theta_0)|| > \frac{1}{2} g(\delta)\} &\leq P\{\sup_{0 \leq t \leq T} |W_t^H| > \frac{e^{-|\theta_0 T|} g(\delta)}{2\varepsilon T}\} \\ &= P\{W_T^{H*} > \frac{e^{-|\theta_0 T|} g(\delta)}{2\varepsilon T}\}. \end{aligned} \tag{3. 14}$$

Applying the Lemma 3.1 to the estimate obtained above, we get that

$$\begin{aligned} P_{\theta_0}^{(\varepsilon)}\{|\theta_\varepsilon^* - \theta_0| > \delta\} &\leq 2^p T^{pH+p} K(p, H) e^{|\theta_0 T| p} (g(\delta))^{-p} \varepsilon^p \\ &= O((g(\delta))^{-p} \varepsilon^p). \end{aligned} \tag{3. 15}$$

**Remarks:** As a consequence of the above theorem, we obtain that $\theta_\varepsilon^*$ converges in probability to $\theta_0$ under $P_{\theta_0}^{(\varepsilon)}$-measure as $\varepsilon \to 0$. Further more the rate of convergence is of the order $(O(\varepsilon^p))$ for every $p > 0$.

**Asymptotic distribution**

We will now study the asymptotic distribution if any of the estimator $\theta_\varepsilon^*$ after suitable scaling. It can be checked that

$$X_t = e^{\theta_0 t}\{x_0 + \int_0^t e^{-\theta_0 s} \varepsilon dW_s^H\} \tag{3. 16}$$

or equivalently

$$X_t - x_t(\theta_0) = \varepsilon e^{\theta_0 t} \int_0^t e^{-\theta_0 s} dW_s^H. \tag{3. 17}$$

Let

$$Y_t = e^{\theta_0 t} \int_0^t e^{-\theta_0 s} dW_s^H. \tag{3. 18}$$

Note that $\{Y_t, 0 \leq t \leq T\}$ is a gaussian process and can be interpreted as the "derivative" of the process $\{X_t, 0 \leq t \leq T\}$ with respect to $\varepsilon$. Applying Lemma 2.2, we obtain that, $P$-a.s.,

$$Y_t e^{-\theta_0 t} = \int_0^t e^{-\theta_0 s} dW_s^H = \int_0^t K_H^f(t, s) dM_s^H, t \in [0, T] \tag{3. 19}$$

where $f(s) = e^{-\theta_0 s}, s \in [0, T]$ and $M^H$ is the fundamental gaussian martingale associoated with the fBm $W^H$. In particular it follows that the random variable $Y_t e^{-\theta_0 t}$ and hence $Y_t$ has normal distribution with mean zero and further more, for any $h \geq 0$,

$$(3.\ 20) \qquad \begin{aligned} Cov(Y_t, Y_{t+h}) &= e^{2\theta_0 t+\theta_0 h} E[\int_0^t e^{-\theta_0 u} dW_u^H \int_0^{t+h} e^{-\theta_0 v} dW_v^H] \\ &= e^{2\theta_0 t+\theta_0 h} H(2H-1) \int_0^t \int_0^t e^{-\theta_0(u+v)} |u-v|^{2H-2} dudv \\ &= e^{2\theta_0 t+\theta_0 h} \gamma_H(t) \ \text{(say)}. \end{aligned}$$

In particular

$$(3.\ 21) \qquad Var(Y_t) = e^{2\theta_0 t} \gamma_H(t).$$

Hence $\{Y_t, 0 \leq t \leq T\}$ is a zero mean gaussian process with $Cov(Y_t, Y_s) = e^{\theta_0(t+s)} \gamma_H(t)$ for $s \geq t$.

Let

$$(3.\ 22) \qquad \zeta = \arg \inf_{-\infty<u<\infty} \int_0^T |Y_t - utx_0 e^{\theta_0 t}| dt.$$

**Theorem 3.3:** The random variable converges in probability to a random variable whose probability distribution is the same as that of $\zeta$ under $P_{\theta_0}$.

**Proof:** Let $x_t'(\theta) = x_0 t e^{\theta t}$ and let

$$(3.\ 23) \qquad Z_\varepsilon(u) = ||Y - \varepsilon^{-1}(x(\theta_0 + \varepsilon u) - x(\theta_0))||$$

and

$$(3.\ 24) \qquad Z_0(u) = ||Y - ux'(\theta_0)||.$$

Further more, let

$$(3.\ 25) \qquad A_\varepsilon = \{\omega : |\theta_\varepsilon^* - \theta_0| < \delta_\varepsilon\}, \delta_\varepsilon = \varepsilon^\tau, \tau \in (\frac{1}{2}, 1), L_\varepsilon = \varepsilon^{\tau-1}.$$

Observe that the random variable $u_\varepsilon^* = \varepsilon^{-1}(\theta_\varepsilon^* - \theta_0)$ satisfies the equation

$$(3.\ 26) \qquad Z_\varepsilon(u_\varepsilon^*) = \inf_{|u|<L_\varepsilon} Z_\varepsilon(u), \omega \in A_\varepsilon.$$

Define

$$(3.\ 27) \qquad \zeta_\varepsilon = \arg \inf_{|u|<L_\varepsilon} Z_0(u).$$

Observe that, with probability one,

$$(3.\ 28) \qquad \begin{aligned} \sup_{|u|<L_\varepsilon} |Z_\varepsilon(u) - Z_0(u)| &= |||Y - ux'(\theta_0) - \frac{1}{2}\varepsilon u^2 x''(\tilde{\theta})|| - ||Y - ux'(\theta_0)||| \\ &\leq \frac{\varepsilon}{2} L_\varepsilon^2 \sup_{|\theta-\theta_0<\delta_\varepsilon} \int_0^T |x''(\theta)| dt \\ &\leq C\varepsilon^{2\tau-1}. \end{aligned}$$

Here $\tilde{\theta} = \theta_0 + \alpha(\theta - \theta_0)$ for some $\alpha \in (0, 1)$. Note that the last term in the above inequality tends to zero as $\varepsilon \to 0$. Further more the process $\{Z_0(u), -\infty < u < \infty\}$ has a unique minimum $u^*$ with probability one. This follows from the arguments given in Theorem 2 of Kutoyants and Pilibossian (1994). In addition, we can choose the interval $[-L, L]$ such that

$$P_{\theta_0}^{(\varepsilon)}\{u_\varepsilon^* \in (-L, L)\} \geq 1 - \beta g(L)^{-p} \tag{3. 29}$$

and

$$P\{u^* \in (-L, L)\} \geq 1 - \beta g(L)^{-p} \tag{3. 30}$$

where $\beta > 0$. Note that $g(L)$ increases as $L$ increases. The processes $Z_\varepsilon(u), u \in [-L, L]$ and $Z_0(u), u \in [-L, L]$ satisfy the Lipschitz conditions and $Z_\varepsilon(u)$ converges uniformly to $Z_0(u)$ over $u \in [-L, L]$. Hence the minimizer of $Z_\varepsilon(.)$ converges to the minimizer of $Z_0(u)$. This completes the proof.

**Remarks :** We have seen earlier that the process $\{Y_t, 0 \leq t \leq T\}$ is a zero mean gaussian process with the covariance function

$$Cov(Y_t, Y_s) = e^{\theta_0(t+s)}\gamma_H(t)$$

for $s \geq t$. Recall that

$$\zeta = \arg \inf_{-\infty < u < \infty} \int_0^T |Y_t - utx_0e^{\theta_0 t}|dt. \tag{3. 31}$$

It is not clear what the distribution of $\zeta$ is. Observe that for every $u$, the integrand in the above integral is the absolute value of a gaussian process $\{J_t, 0 \leq t \leq T\}$ with the mean function $E(J_t) = -utx_0e^{\theta_0 t}$ and the covariance function

$$Cov(J_t, J_s) = e^{\theta_0(t+s)}\gamma_H(t)$$

for $s \geq t$.

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