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Rio-type inequality for the expectation of products of random variables

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Abstract: We develop an inequality for the expection of a product of n random variables generalizing the recent work of Dedecker and Doukhan (2003) and the earlier results in Rio (1993).

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let (X, Y) be a bivariate random vector defined on it. Suppose that $E(X^2) < \infty$ and $E(Y^2) < \infty$. Hoeffding proved that

$$Cov(X,Y) = \int_{R^2} [P(X \le x, Y \le y) - P(X \le x)P(Y \le y)]dxdy. \tag{1. 1}$$

Lehmann (1966) gave a simple proof of this identity and used it in his study of some concepts of dependence. This identity was generalised to functions $h(X)$ and $g(Y)$ with $E[h^2(X)] < \infty$ and $E[g^2(Y)] < \infty$ and with finite derivatives $h'(.)$ and $g'(.)$ by Newman (1980). Multidimensional versions of these results were proved by Block and Fang (1988), Yu(1993) and more recently in Prakasa Rao (1998). Related covariance identities for exponential and other distributions are given in Prakasa Rao (1999a, 2000).

Suppose that $\mathcal{M}$ is a sub σ-algebra of clf and Y is measurable with respect to $\mathcal{M}$. Let $\sigma(X)$ be the sub σ-algebra generated by the random variable X.. Define

$$\alpha(\mathcal{M}, X) = \sup\{|P(A \cap B) - P(A)P(B)|, A \in \mathcal{M}, B \in \sigma(X)\}.$$

Define

$$Q_X(u) = \inf\{x : P(|X| > x) \le u\},$$
$$G_X(s) = \inf\{z : \int_0^z Q_X(t)dt \ge s\},$$

and

$$H_{X,Y}(s) = \inf\{t : E(|X|I_{[|Y|>t]}) \le s\}.$$

Rio (1993) proved that

$$|Cov(X,Y)| \le 2\int_0^{\alpha(\mathcal{M},X)} Q_Y(u)Q_X(u)du \tag{1. 2}$$

Related results are given in Rio (2000), p.9. These results were generalised by Bradley (1996) for a strong-mixing process and by Prakasa Rao (1999b) for r-th order joint cumulant under r-th order strong mixing. In a recent work, Dedecker and Doukhan (2003) proved that

$$|E(XY)| \leq \int_0^{||E(X|\mathcal{M})||_1} H_{X,Y}(t)dt \leq \int_0^{||E(X|\mathcal{M})||_1} Q_Y o G_X(t)dt$$

and obtained an improved version of the above inequality. If $X_i, 1 \leq i \leq n$ are positive valued random variables, it is easy to see that

$$E(X_1X_2\ldots X_n) \leq \int_0^1 Q_{X_1}(u)Q_{X_2}(u)\ldots Q_{X_n}(u)du.$$

For a proof, see Lemma 2.1 in Rio (2000), p.35.

We now obtain an improved version of the above inequality following the techniques in Dedecker and Doukhan (2003) and Block and Fang (1988).

2 Main Result

Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of nonnegative random variables defined on a probability space $\{\Omega, \mathcal{F}, P)\}$. Then the random variable X_i can be represented in the form

$$X_i = \int_0^\infty I_{(x_i,\infty)}(X_i)dx_i \tag{2. 1}$$

where

$$\begin{aligned} I_{(x_i,\infty)}(X_i) &= 1 \text{ if } X_i > x_i \\ &= 0 \text{ if } X_i \leq x_i. \end{aligned}$$

Hence

$$\begin{aligned} E(X_1X_2\ldots X_n) &= E[X_1\Pi_{i=2}^n \int_0^\infty I_{(x_i,\infty)}(X_i)dx_i] \\ &= \int_{R_+^{n-1}} E[X_1\Pi_{i=2}^n I_{(x_i,\infty)}(X_i)]dx_2\ldots dx_n \\ &= \int_{R_+^{n-1}} E[X_1 I_{(X_i>x_i,2\leq i\leq n]}(X_2,\ldots X_n)]dx_2\ldots dx_n \end{aligned} \tag{2. 2}$$

by the Fubini's theorem where $R_+^{n-1} = \{(x_2,\ldots x_n) : x_i \geq 0, 2 \leq i \leq n\}$. Observe that

$$E(X_1 I_{[X_i>x_i,2\leq i\leq n]}(X_2,\ldots,X_n)) \leq \min(E[X_1], E(X_1 I_{[X_i>x_i,2\leq i\leq n]}(X_2,\ldots,X_n)))$$

and hence

$$E(X_1X_2\ldots X_n) \leq \int_{R_+^{n-1}} \{\int_0^{EX_1} \chi_{(E[X_1 I_{[X_i>x_i,2\leq i\leq n]}(X_2,\ldots,X_n)]>u)}(u)du\}dx_2\ldots dx_n. \tag{2. 3}$$

Here $\chi_A(.)$ denotes the indicator function of the set A. Let

$$g_{X_1}(x_2,\ldots,x_n) = E[X_1 I_{[X_i>x_i,2\leq i\leq n]}(X_2,\ldots,X_n)].$$

Then

$$(2.\ 4)\qquad \begin{aligned} E(X_1X_2\dots X_n) &\le \int_{R_+^{n-1}}\{\int_0^{EX_1}\chi_{[g_{X_1}(x_2,\dots,x_n)>u]}(u)du\}dx_2\dots dx_n \\ &\le \int_0^{E(X_1)}\{\int_{[(x_2,\dots,x_n):g_{X_1}(x_2,\dots,x_n)>u]} 1\ \ dx_2\dots dx_n\}du. \end{aligned}$$

Let

$$H_{X_1,X_2,\dots,X_n}(u)=\lambda[(x_2,\dots,x_n):g_{X_1}(x_2,\dots,x_n)>u]$$

where λ is the Lebesgue measure on the space R_+^{n-1}. Hence

$$(2.\ 5)\qquad E(X_1X_2\dots X_n)\le\int_0^{E(X_1)}H_{X_1,X_2,\dots,X_n}(u).$$

Observe that

$$(2.\ 6)\qquad \begin{aligned} g_{X_1}(x_2,\dots,x_n) &= E[X_1I_{[X_i>x_i,2\le i\le n]}(X_2,\dots,X_n)] \\ &\le \int_0^{E[I_{[X_i>x_i,2\le i\le n]}(X_2,\dots,X_n)]} Q_{X_1}(u)du \end{aligned}$$

from the Frechet's inequality (1957). Here $Q_{X_1}(.)$ is the generalized inverse of the function $T_{X_1}(x)=P(X_1>x)$ as defined earlier. Let

$$M_{X_1}(y)=\int_0^y Q_{X_1}(t)dt.$$

Observe that $M_{X_1}(.)$ is nondecreasing in y. Let $G_{X_1}(u)=\inf\{z:M_{X_1}(z)\ge u\}$ as defined earlier. Let

$$T_{X_2,\dots,X_n}(x_2,\dots,x_n)=P(X_i>x_i,2\le i\le n).$$

For any $0\le u\le 1$, define

$$Q^*_{X_2,\dots,X_n}(u)=\inf\Pi_{i=2}^n x_i$$

where the infimum is taken over $x_i\ge 0, 2\le i\le n$ such that $P(X_i>x_i,2\le i\le n)\le u$. If there exists a point $(x_{20},\dots,x_{n0})\in R_+^{n-1}$ such that

$$Q^*_{X_2,\dots,X_n}(u)=\Pi_{i=2}^n x_{i0},$$

then we define

$$Q_{X_2,\dots,X_n}(u)=(x_{20},\dots,x_{n0}).$$

If there are more than one such point, we choose any one of them. We will see later that this choice does not affect the final inequality. Note that

$$g_{X_1}(x_2,\dots,x_n)\le M_{X_1}(E(I_{[X_i>x_i,2\le i\le n]}(X_2,\dots,X_n)))$$

and

$$(2.\ 7)\qquad \begin{aligned} g_{X_1}(x_2,\dots,x_n)>u &\Leftrightarrow M_{X_1}(E(I_{[X_i>x_i,2\le i\le n]}(X_2,\dots,X_n)))>u \\ &\Leftrightarrow E(I_{[X_i>x_i,2\le i\le n]}(X_2,\dots,X_n))>G_{X_1}(u) \\ &\Leftrightarrow P[X_i>x_i,2\le i\le n]>G_{X_1}(u). \end{aligned}$$

Hence the set

$$[(x_2,\ldots,x_n)\in R_+^{n-1} : g_{X_1}(x_2,\ldots,x_n) > u]$$

is contained in the set

$$[(x_2,\ldots,x_n)\in R_+^{n-1} : Q_{X_2,\ldots,X_n}(G_{X_1}(u)) > (x_2,\ldots,x_n)]$$

with the interpretation that the inequality holds in the sense that $(x_2,\ldots,x_n)\le(y_2,\ldots,y_n)$ in R_+^{n-1} if and only if $\Pi_{i=2}^n x_i \le \Pi_{i=2}^n y_i$. In particular, it follows that the Lebesgue measure of the former set is less than or equal to that of the latter. Therefore

$$H_{X_1,X_2,\ldots,X_n}(u) \le Q^*_{X_2,\ldots,X_n}(G_{X_1}(u)) \tag{2. 8}$$

for all $0\le u\le 1$. Hence

$$E(X_1X_2\ldots X_n) \le \int_0^{E(X_1)} Q^*_{X_2,\ldots,X_n}(G_{X_1}(u))du. \tag{2. 9}$$

We have proved the following inequality.

Theorem 2.1: Let $X_i, 1\le i\le n$ be nonnegative random variables defined on a probability space $(\Omega,\mathcal{F},P)$. Then

$$E(X_1X_2\ldots X_n) \le \int_0^{E(X_1)} H_{X_1,X_2,\ldots,X_n}(u)du \le \int_0^{E(X_1)} Q^*_{X_2,\ldots,X_n} oG_{X_1}(u)du$$

where the functions H, Q and G are as defined earlier.

3 Applications

Let us now suppose that the random variables $\{X_i, 1\le i\le n\}$ are arbitrary but with

$$E|X_1X_2\ldots X_n| < \infty.$$

Define

$$g_{X_1}(x_2,\ldots,x_n) = E(|X_1|I_{[|X_i|>x_i, 2\le i\le n]}(X_2,\ldots,X_n)),$$

$$H_{X_1,X_2,\ldots,X_n}(u) = \lambda[(x_2,\ldots,x_n) : g_{X_1}(x_2,\ldots,x_n)\le u],$$

$$T_{X_2,\ldots,X_n}(x_2,\ldots,x_n) = P(|X_i|>x_i, 2\le i\le n),$$

and define $M_{X_1}(.), Q_{X_1}(.), Q^*_{X_2,\ldots,X_n}$ and G_{X_1} accordingly. The following theorem follows by arguments analogous to those given in the Section 2.

Theorem 3.1: Let $X_i, 1\le i] \le n$ be arbitrary random variables defined on a probability space $(\Omega,\mathcal{F},P)$.. Then

$$E(|X_1X_2\ldots X_n|) \le \int_0^{E(|X_1|)} H_{X_1,X_2,\ldots,X_n}(u)du \le \int_0^{E(|X_1|)} Q^*_{X_2,\ldots,X_n} oG_{X_1}(u)du$$

where the functions H, Q^* and G are as defined above.

In particular, for $n = 2$, we have

$$E(|X_1X_2|) \leq \int_0^{E(|X_1|)} H_{X_1,X_2}(u)du \leq \int_0^{E(|X_1|)} Q_{X_2}oG_{X_1}(u)du$$

since $Q_X^* = Q_X$ for any univariate random variable X. Further more

$$G_{X_1-E(X_1)}(u) \geq G_{X_1}(u/2), 0 \leq u \leq 1$$

(cf. Dedecker and Doukhan (2003)). Hence

$$E[|X_1X_2|] \leq \int_0^{G_{X_1}^{-1}(\frac{E(|X_1|)}{2})} Q_{X_2}(u)Q_{X_1}(u)du. \tag{3. 1}$$

Therefore, for any two fuctions $f_i(.), i = 1, 2$ with $f_i(0) = 0$ such that $E|f_1(X_1)f_2(X_2)| < \infty$, we obtain that

$$E[|f_1(X_1)f_2(X_2)|] \leq \int_0^{G_{f_1(X_1)}^{-1}(\frac{E(|f_1(X_1)|)}{2})} Q_{f_2(X_2)}(u)Q_{f_1(X_1)}(u)du. \tag{3. 2}$$

Applying the Theorem 3.1 for the random variables $X_1 - E(X_1), X_2, \ldots, X_n$, we get that

$$E[|(X_1 - E(X_1))X_2 \ldots X_n|] \leq \int_0^{E(|X_1-E(X_1)|)} Q^*_{X_2,\ldots,X_n}oG_{X_1-E(X_1)}(u)du.$$

But

$$G_{X_1-E(X_1)}(u) \geq G_{X_1}(u/2), u \geq 0$$

(cf. Dedecker and Doukhan (2003)). Hence

$$E[|(X_1 - E(X_1))X_2 \ldots X_n|] \leq \int_0^{\frac{E(|X_1-E(X_1)|)}{2}} Q^*_{X_2,\ldots,X_n}oG_{X_1}(u)du.$$

Observing that $G_{X_1}(.)$ is the inverse of the function $M_{X_1}(y) = \int_0^y Q_{X_1}(t)dt$, it follows that

$$E[|(X_1 - E(X_1))X_2 \ldots X_n|] \leq \int_0^{G_{X_1}^{-1}(\frac{E(|X_1-E(X_1)|)}{2})} Q^*_{X_2,\ldots,X_n}(u)Q_{X_1}(u)du.$$

Hence we have the following result.

Theorem 3.2: Let $X_i, 1 \leq i \leq n$ be arbitrary random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ with $E|X_1| < \infty$ and $E|X_1X_2 \ldots X_n| < \infty$.. Then

$$E[|(X_1 - E(X_1))X_2 \ldots X_n|] \leq \int_0^{G_{X_1}^{-1}(\frac{E(|X_1-E(X_1)|)}{2})} Q^*_{X_2,\ldots,X_n}(u)Q_{X_1}(u)du.$$

Observe that $Q_X^* = Q_X$ for any univariate random variable X. Let $n = 2$ in Theorem 3.2. Then $Q_{X_2}^* = Q_{X_2}$ and the above result reduces to

$$E[|(X_1 - E(X_1))X_2|] \leq \int_0^{G_{X_1}^{-1}(\frac{E(|X_1 - E(X_1)|)}{2})} Q_{X_2}(u)Q_{X_1}(u)du.$$

As a further consequence, we get that

$$E[|(X_1 - E(X_1))(X_2 - E(X_2))|] \leq \int_0^{G_{X_1}^{-1}(\frac{E(|X_1 - E(X_1)|)}{2})} Q_{X_2 - E(X_2)}(u)Q_{X_1}(u)du.$$

Since

$$Q_{X_2 - E(X_2)} \leq Q_{X_2} + E|X_2|,$$

we obtain that

$$\begin{aligned} E[|(X_1 - E(X_1))(X_2 - E(X_2))|] &\leq \int_0^{G_{X_1}^{-1}(\frac{E(|X_1 - E(X_1)|)}{2})} Q_{X_2}(u)Q_{X_1}(u)du \\ &\quad + E|X_2| \int_0^{G_{X_1}^{-1}(\frac{E(|X_1 - E(X_1)|)}{2})} Q_{X_1}(u)du. \end{aligned}$$

Let

$$\alpha(X_1, X_2) = \max\{G_{X_1}^{-1}(\frac{E(|X_1 - E(X_1)|)}{2}), G_{X_2}^{-1}(\frac{E(|X_2 - E(X_2)|)}{2})\}. \tag{3. 3}$$

Then it follows that

$$\begin{aligned} E[|(X_1 - E(X_1))(X_2 - E(X_2))|] &\leq \int_0^{\alpha(X_1,X_2)} Q_{X_1}(u)Q_{X_2}(u)du \\ &\quad + \frac{1}{2}(E|X_1| \int_0^{\alpha(X_1,X_2)} Q_{X_1}(u)du + E|X_2| \int_0^{\alpha(X_1,X_2)} Q_{X_2}(u)du). \end{aligned} \tag{3. 4}$$

This inequality is different from the inequality in Rio (2000),p.9.

Let f_1 and f_2 be differentiable functions on R_+ with $f_i(0) = 0$. Let $X_i, i = 1, 2$ be nonnegative random variables. Supose that $E[f_i^2(X_i)] < \infty, i = 1, 2$. It is easy to that

$$f_i(X_i) = \int_0^\infty f_i'(X_i)I_{(x_i,\infty)}(X_i)dx_i.$$

Then

$$\begin{aligned} E(f_1(X_1)f_2(X_2)) &= E[f_1(X_1) \int_0^\infty f_2'(X_2)I_{(x_2,\infty)}(X_2)dx_2] \\ &= \int_{R_+} E[f_1(X_1)f_2'(X_2)I_{(x_2,\infty)}(X_2)]dx_2 \end{aligned} \tag{3. 5}$$

by the Fubini's theorem. Observe that

$$E(|f_1(X_1)f_2'(X_2)|I_{[X_2 > x_2]}(X_2)) \leq \min(E[|f_1(X_1)f_2'(X_2)|], E(|f_1(X_1)f_2'(X_2)|I_{[X_2 > x_2]}(X_2))$$

and hence

(3. 6)

$$|E(f_1(X_1)f_2(X_2)|)| \leq \int_{R+}\{\int_0^{E[|f_1(X_1)f_2'(X_2)|]} \chi_{(E[|f_1(X_1)f_2'(X_2)|I_{[X_2>x_2]}(X_2)]>u)}(u)du\}dx_2.$$

Here $\chi_A(.)$ denotes the indicator function of the set A. Let

$$g_{f_1(X_1),f_2'(X_2)}(x_2) = E[|f_1(X_1)f_2'(X_2)|I_{[X_2>x_2],}(X_2)].$$

Then

$$\begin{aligned} (3.\ 7) \quad E(|f_1(X_1)f_2(X_2)|) &\leq \int_{R+}\{\int_0^{E[|f_1(X_1)f_2'(X_2)|]} \chi_{([g_{f_1(X_1),f_2'(X_2)}(x_2)]>u)}(u)du\}dx_2 \\ &\leq \int_0^{E[|f_1(X_1)f_2'(X_2)|]}\{\int_{[x_2:g_{f_1(X_1),f_2'(X_2)}(x_2)>u]} 1 \ \ dx_2\}du. \end{aligned}$$

Let

$$H_{f_1(X_1),f_2'(X_2)}(u) = \inf\{x_2 : g_{f_1(X_1),f_2'(X_2)}(x_2) \leq u\}].$$

Then it follows that

$$(3.\ 8) \qquad E(|f_1(X_1)f_2(X_2)|) \leq \int_0^{E[|f_1(X_1)f_2'(X_2)|]} H_{f_1(X_1),f_2'(X_2)}(u)du.$$

An analogous inequality holds by interchanging $f_1(X_1)$ and $f_2(X_2)$:

$$(3.\ 9) \qquad E(|f_1(X_1)f_2(X_2)|) \leq \int_0^{E[|f_1'(X_1)f_2(X_2)|]} H_{f_1'(X_1),f_2(X_2)}(u)du.$$

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