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On a Berry-Esseen type bound for the maximum likelihood estimator of a parameter for some stochastic partial differential equations

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On a Berry-Esseen type bound for the maximum likelihood estimator of a parameter for some stochastic partial differential equations

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Abstract: This paper is concerned with the study of the rate of convergence of the distribution of the maximum likelihood estimator (MLE) of parameter appearing linearly in the drift coefficient of two types of stochastic partial differential equations (SPDE's).

Key words and phrases : Stochastic partial differential equations, Berry-Esseen type bound, Maximum likelihood estimator, Inference for Stochastic processes.

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1 Introduction

Maximum likelihood estimation of a parameter appearing linearly in some stochastic partial differential equations (SPDE) has been considered by Hubner et al. (1993). Detail discussion of these SPDE's and some interesting phenomena arising out of the parameter estimation have been considered by them in two examples. In this paper, we study the rate of convergence of the distribution of the MLE $\hat{\theta}_{N,\epsilon}$ of the parameter θ occuring linearly in such SPDE's. Bounds on the difference $\left|\hat{\theta}_{N,\epsilon} - \theta_0\right|$, where θ_0 is the true value of the parameter, can be obtained using these results as in Mishra and Prakasa Rao (1985).

In Section 2, we describe a SPDE with parameter θ such that the corresponding stochastic process u_e generate the measures $\{P_\theta^\epsilon, \theta \in \Theta\}$ which are mutually absolutely continuous, and the main results pertaining to this section have been described in the Section 3. In Section 4, we describe a SPDE with parameter θ such that the corresponding stochastic process u_e generate the measures which form a family of probability measures $\{P_\theta^\epsilon, \theta \in \Theta\}$ which are singular with respect to each other and this section also contains the main results connected to this problem. A corresponding survey on statistical inference for such classes of SPDE's is given in Prakasa Rao (1999, 2002).

Throughout the paper, we shall denote by C positive constant different at different places of occurence possibly dependent on the initial conditions of the SPDE's.

2 Stochastic PDE with linear drift (Absolutely continuous case) : Estimation

Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider the process $u_\epsilon(t,x),\ 0 \le x \le 1,\ 0 \le t \le T$ governed by the stochastic partial differential equation

$$du_\epsilon(t,x) = (\Delta u_\epsilon(t,x) + \theta u_\epsilon(t,x))\, dt + \epsilon dW_Q(t,x) \qquad (2.\ 1)$$

where $\Delta = \frac{\partial^2}{\partial x^2}$. Let $\epsilon \to 0$ and $\theta \in \Theta \subset I\!R$. Suppose that the initial and boundary conditions are given by

$$\begin{aligned} u_\epsilon(0,x) &= f(x),\ f \in L_2[0,1], \\ u_\epsilon(t,0) &= u_\epsilon(t,1) = 0,\ 0 \le t \le T \end{aligned} \qquad (2.\ 2)$$

and Q is the nuclear covariance operator for the Wiener process $W_Q(t,x)$ taking values in $L_2[0,1]$, so that $W_Q(t,x) = Q^{\frac{1}{2}} W(t,x)$ and $W(t,x)$ is a cylindrical Brownian motion in $L_2[0,1]$.

Then it is known that (cf. Rozovskii (1990))

$$W_Q(t,x) = \sum_{i=1}^{\infty} q_i^{\frac{1}{2}} e_i(x) W_i(t) \quad \text{a.s.} \tag{2. 3}$$

where $\{W_i(t),\ 0 \le t \le T\}$, $i \ge 1$ are independent one dimensional standard Wiener processes and $\{e_i\}$ is a complete orthonormal system in $L_2[0,1]$ consisting of eigen vectors of Q and $\{q_i\}$ eigen values of Q.

Let us consider a special covariance operator Q with $e_k = \sin k\pi x, k \ge 1$ and $\lambda_k = (\pi k)^2, k \ge 1$. Then $\{e_n\}$ is a CONS with the eigen values $q_i = (1+\lambda_i)^{-1}$, $i \ge 1$ for the operator Q where $Q = (I - \Delta)^{-1}$. Furthermore, $dW_Q = Q^{\frac{1}{2}} dW$. we define a solution $u_\epsilon(t,x)$ of (2.1) as a formal sum

$$u_\epsilon(t,x) = \sum_{i=1}^{\infty} u_{i\epsilon}(t) e_i(x), \tag{2. 4}$$

(cf. Rozovskii (1990)). It is known that the Fourier coefficients $u_{i\epsilon}(t)$ satisfies the stochastic differential equation

$$du_{i\epsilon}(t) = (\theta - \lambda_i) u_{i\epsilon}(t) dt + \frac{\epsilon}{\sqrt{\lambda_i + 1}} dW_i(t),\ \ 0 < t \le T \tag{2. 5}$$

with the initial conditions

$$u_{i\epsilon}(0) = v_i, \quad v_i = \int_0^1 f(x) e_i(x) dx. \tag{2. 6}$$

It is further known that $u_\epsilon(t,x)$ as defined above belongs to $L_2\left([0,T] \times \Omega; L_2[0,1]\right)$ together with its derivative in t. Furthermore, $u_\epsilon(t,x)$ is the only solution of (2.1) under the boundary condition (2.2). Let P_θ^ϵ be the measure generated by u_ϵ on $C[0,T]$ when θ is the true parameter. It has been shown by Hubner et al. (1993) that the family of measures $\left\{P_\theta^{(\epsilon)}, \theta \in \Theta\right\}$ are mutually absolutely continuous and

$$\begin{aligned} &\log \frac{dP_\theta^\epsilon}{dP_{\theta_0}}(u_\epsilon) \\ &= \sum_{i=1}^{\infty} \frac{\lambda_i + 1}{\epsilon^2} \left[(\theta - \theta_0) \int_0^T u_{i\epsilon}(t) du_{i\epsilon}(t) - \frac{1}{2} \left\{ (\theta - \lambda_i)^2 - (\theta_0 - \lambda_i)^2 \right\} \int_0^T u_{i\epsilon}^2(t) dt \right]. \end{aligned}$$

The log likelihood ratio of the projection of the solution $u_\epsilon(t,x)$ onto the subspace π^N spanned by $\{e_1, e_2, \ldots, e_N\}$ (see [Liptser, Shiryayev (1978)]) is given by $u_\epsilon^N(t,x) = \sum_{i=1}^N u_{i\epsilon}(t) e_i(x)$ is as follows :

$$\begin{aligned} &\log \frac{dP_\theta^{\epsilon,N}}{dP_{\theta_0}^{\epsilon,N}} \\ &= \sum_{i=1}^{N} \frac{\lambda_i + 1}{\epsilon^2} \left[(\theta - \theta_0) \int_0^T u_{i\epsilon}(t) du_{i\epsilon}(t) - \frac{1}{2} \left\{ (\theta - \lambda_i)^2 - (\theta_0 - \lambda_i)^2 \right\} \int_0^T u_{i\epsilon}(t) dt \right]. \end{aligned} \tag{2. 7}$$

Here $P_\theta^{\epsilon,N}$ is the probability measure generated by the process $u_\epsilon^N(t,x)$ on $C[0,T]$ when θ is the true parameter.

The Maximum likelihood estimator (MLE) of θ has the form

$$\hat{\theta}_{N,\epsilon} = \frac{\sum_{i=1}^N (\lambda_i + 1) \int_0^T u_{i\epsilon}(t)(du_{i\epsilon}(t) + \lambda_i u_{i\epsilon}(t)dt)}{\sum_{i=1}^N (\lambda_i + 1) \int_0^T u_{i\epsilon}^2(t)dt} \tag{2. 8}$$

(cf. Hubner et. al. (1993), p.154).

3 Stochastic PDE with linear drift (Absolutely continuous case) : Berry-Esseen type bound

In this section we prove the following theorem for which we need few lemmas. It can be checked that $E_{\theta_0} \int_0^T u_{i\epsilon}^2(t)dt < \infty$. We assume that $\theta_0 < \pi^2$ where θ_0 is the true parameter.

Let $\Phi(.)$ denote the standard normal distribution function and define

$$Q_{N,T}^{(\epsilon)} = \sum_{i=1}^N \frac{\lambda_i + 1}{2(\theta - \lambda_i)} \left(v_i^2 \left(e^{2(\theta - \lambda_i)T} - 1 \right) - T \frac{\epsilon^2}{\lambda_i + 1} \right).$$

Theorem 3.1 : For any $0 < \delta < 1$,

$$\sup_y \left| P_{\theta_0}^{\epsilon,N} \left\{ \sqrt{Q_{N,T}^{(\epsilon)}} \epsilon^{-1} \left(\hat{\theta}_{N,\epsilon} - \theta_0 \right) \le y \right\} - \Phi(y) \right| \le 2P_{\theta_0}^{\epsilon,N} \left\{ \left| \sum_{i=1}^N \frac{\lambda_i + 1}{Q_{N,T}^{(\epsilon)}} \int_0^T u_{i\epsilon}^2(t)dt - 1 \right| \ge \delta \right\} + 3\sqrt{\delta}.$$

Lemma 3.1 : Let $(\Omega, \mathcal{F}, P)$ be a probability space and f and g be $\mathcal{F}$-measurable functions. Then, for any $\delta > 0$,

$$\sup_x \left| P\left\{ \omega : \frac{f(\omega)}{g(\omega)} \le x \right\} - \Phi(x) \right| \le \sup_y |P\{\omega : f(\omega) \le y\} - \Phi(y)| + P\{\omega : |g(\omega) - 1| \ge \delta\} + \delta.$$

Proof : See Michael and Pfanzagl (1971).

Lemma 3.2: Let $\{W(t),\ t \ge 0\}$ be a standard Wiener process and Z be a nonnegative random variable. Then, for every $x \in I\!\!R$ and $\delta > 0$,

$$|P\{W(Z) \le x\} - \Phi(x)| \le (2\delta)^{\frac{1}{2}} + P\{|Z - 1| \ge \delta\}.$$

Proof: See Hall and Heyde (1980, p.85).

Theorem 3.2: There exists a constant C depending on θ_0, $\|f\|$ and T such that, for any $0 < \delta \leq 1$,

$$P_{\theta_0}^{\epsilon,N}\left\{\left|\sum_{i=1}^{N}\frac{\lambda_i+1}{Q_{N,T}^{(\epsilon)}}\int_0^T u_{i\epsilon}^2(t)dt-1\right|\geq\delta\right\}\leq C\frac{\epsilon}{\delta}\left(\frac{1+T^{\frac{1}{2}}}{\|f\|^2+\epsilon^2 T}\right)$$

as $\epsilon \to 0$ (N fixed).

Proof of Theorem 3.1: We can write, using (2.8),

$$\sqrt{Q_{N,T}^{(\epsilon)}}\epsilon^{-1}\left(\hat{\theta}_{N,\epsilon}-\theta_0\right) = \frac{\left\{\sum_{i=1}^N\sqrt{\lambda_i+1}\int_0^1 u_{i\epsilon}(t)dW_i(t)\right\}/\sqrt{Q_{N,T}^{(\epsilon)}}}{\left\{\sum_{i=1}^N(\lambda_i+1)\int_0^T u_{i\epsilon}^2(t)dt\right\}/Q_{N,T}^{(\epsilon)}}$$

where

$$Q_{N,T}^{(\epsilon)}=\sum_{i=1}^N\left[\frac{\lambda_i+1}{2(\theta-\lambda_i)}\left(v_i^2(e^{2(\theta-\lambda_i)T}-1)-T\frac{\epsilon^2}{\lambda_1+1}\right)\right].$$

Now, for any $y \in R$,

$$\begin{aligned}
&\left|P_{\theta_0}^{\epsilon,N}\left\{\sqrt{Q_{N,T}^{(\epsilon)}}\epsilon^{-1}\left(\hat{\theta}_{N,\epsilon}-\theta_0\right)\leq y\right\}-\Phi(y)\right|\\
\leq\ &\left|P_{\theta_0}^{\epsilon,N}\left\{\frac{\sum_{i=1}^N\sqrt{\lambda_i+1}\int_0^T u_{i\epsilon}(t)dW_i(t)/\sqrt{Q_{N,\epsilon}^{(\epsilon)}}}{\sum_{i=1}^N(\lambda_i+1)\int_)^T u_{i\epsilon}^2(t)dt/Q_{N,T}^{(\epsilon)}}\leq y\right\}-\Phi(y)\right|\\
\leq\ &\sup_x\left|p_{\theta_0}^{\epsilon,N}\left\{\frac{\sum_{i=1}^N\sqrt{\lambda_i+1}\int_0^T u_{i\epsilon}(t)dW_i(t)}{\sqrt{Q_{N,T}^{(\epsilon)}}}\leq x\right\}-\Phi(x)\right|\\
&+P_{\theta_0}^{\epsilon,N}\left\{\left|\frac{\sum_{i=1}^N(\lambda_i+1)\int_0^T u_{i,\epsilon}^2(t)dt}{Q_{N,T}^{(\epsilon)}}-1\right|\geq\delta\right\}+\delta\\
&\text{(by Lemma 3.1)}\\
=\ &\sup_x\left|P_{\theta_0}^{\epsilon,N}\left\{\tilde{W}\left(\sum_{i=1}^N\frac{\lambda_i+1}{Q_{N,T}^{(\epsilon)}}\int_0^T u_{i\epsilon}^2(t)dt\right)\leq x\right\}-\Phi(x)\right|\\
&+P_{\theta_0}^{\epsilon,N}\left\{\left|\sum_{i=1}^N\frac{\lambda_i+1}{Q_{N,T}^{(\epsilon)}}\int_0^T u_{i\epsilon}^2(t)dt-1\right|\geq\delta\right\}+\delta
\end{aligned}$$

where $\tilde{W}(.)$ is an independent standard Wiener process, by using the Theorem 2.3 in Feigin (1976) (due to Kunita-Watanabe) and the fact that $\int_0^T u_{i\epsilon}^2(t)dW_i(t)$, $1 \leq i \leq n$ are independent square integrable martingales. Hence

$$\begin{aligned}\left|P_{\theta_0}^{\epsilon,N}\left\{\sqrt{Q_{N,T}^{(\epsilon)}}\epsilon^{-1}\left(\hat{\theta}_{N,\epsilon}-\theta_0\right)\le y\right\}-\Phi(y)\right| &\le \sqrt{2\delta}+2P_{\theta_0}^{\epsilon,N}\left\{\left|\sum_{i=1}^{N}\frac{\lambda_i+1}{Q_{N,T}^{(\epsilon)}}\int_0^T u_{i\epsilon}^2(t)dt-1\right|\ge\delta\right\} \\ &\quad +\delta(\text{by Lemma 3.2}) \\ &\le 2\left[P_{\theta_0}^{\epsilon,N}\left\{\left|\sum_{i=1}^{N}\frac{\lambda_i+1}{Q_{N,T}^{(\epsilon)}}\int_0^T u_{i\epsilon}^2(t)dt-1\right|\right\}\right]+3\sqrt{\delta}\end{aligned}$$

for $0<\delta\le 1$.

Proof of Theorem 3.2: From (2.5) we obtain,

$$\begin{aligned} du_{i\epsilon}(s) &= (\theta-\lambda_i)\,u_{i\epsilon}(s)\,ds+\frac{\epsilon}{\sqrt{\lambda_1+1}}dW_i(s), \quad 0<t\le T, \\ u_{i\epsilon}(0) &= v_i.\end{aligned}$$

By the Ito formula, we have

$$d\left(u_{i\epsilon}(s)\bar{e}^{(\theta-\lambda_i)s}\right)=\frac{\epsilon}{\sqrt{\lambda_i+1}}\bar{e}^{(\theta-\lambda_i)s}dW_i(s)$$

or

$$u_{i\epsilon}(t)\bar{e}^{(\theta-\lambda_i)t}-v_i=\int_0^t\frac{\epsilon}{\sqrt{\lambda_i+1}}\bar{e}^{(\theta-\lambda_i)s}dW_i(s). \tag{3. 1}$$

Further more

$$d(u_{i\epsilon}^2(t))=2(\theta-\lambda_i)u_{i\epsilon}^2(t)dt+\frac{\epsilon}{\sqrt{\lambda_i+1}}u_{i\epsilon}(t)dW_i(t)+\frac{\epsilon^2}{\lambda_i+1}dt$$

or equivalently

$$\begin{aligned}&\frac{\lambda_i+1}{2(\theta-\lambda_i)}u_{i\epsilon}^2(T)-\frac{\lambda_i+1}{2(\theta-\lambda_i)}v_i^2 \\ &= \int_0^T(\lambda_i+1)u_{i\epsilon}^2(t)dt+\frac{\epsilon\sqrt{\lambda_i+1}}{2(\theta-\lambda_i)}\int_0^T u_{i\epsilon}(t)dW_i(t)+\frac{\epsilon^2}{2(\theta-\lambda_i)}T.\end{aligned} \tag{3. 2}$$

We know, from (3.1), that

$$u_{i\epsilon}^2(T)=v_i^2e^{2(\theta-\lambda_i)T}+e^{2(\theta-\lambda_i)T}\left(\frac{\epsilon}{\sqrt{\lambda_i+1}}\int_0^T\bar{e}^{(\theta-\lambda_i)s}dW_i(s)\right)^2+2v_ie^{2(\theta-\lambda_i)T}\int_0^1\frac{\epsilon}{\sqrt{\lambda_i+1}}\bar{e}^{(\theta-\lambda_i)s}dW_i(s).$$

¿From (3.1) and (3.2), we obtain that

$$\sum_{i=1}^{N}\frac{\lambda_i+1}{2(\theta-\lambda_i)}\left\{v_i\left(e^{2(\theta-\lambda_i)T}-1\right)-\frac{\epsilon^2}{\lambda_i+1}T\right\}-\sum_{i=1}^{T}(\lambda_i+1)u_{i\epsilon}^2(t)dt$$

$$= \sum_{i=1}^{N}\frac{\epsilon\sqrt{\lambda_i+1}}{2(\theta-\lambda_i)}\int_0^T u_{i\epsilon}(t)dW_i(t)-$$

$$\sum_{i=1}^{N}\frac{\lambda_i+1}{2(\theta-\lambda_i)}\left[e^{2(\theta-\lambda_i)T}\left(\frac{\epsilon}{\sqrt{\lambda_i+1}}\int_0^T \bar{e}^{(\theta-\lambda_i)s}dW_i(s)\right)^2-\frac{2v_i\epsilon}{\sqrt{\lambda_i+1}}e^{2(\theta-\lambda_i)T}\int_0^T \bar{e}^{(\theta-\lambda_i)s}dW_i(s)\right].$$

Since $Q_{N_1T}^{(\epsilon)}=\sum_{i=1}^{N}\frac{\lambda_i+1}{2(\theta-\lambda_i)}\left(v_i^2\left(e^{2(\theta-\lambda_i)T}-1\right)-T\frac{\epsilon^2}{\lambda_i+1}\right)$, We have

$$\begin{aligned}
&P_\theta^{\epsilon,N}\left\{\left|\frac{\int_0^T\left(\sum_{i=1}^{N}(\lambda_i+1)u_{i\epsilon}^2(t)\right)dt}{Q_{N,T}^{(\epsilon)}}-1\right|\geq\delta\right\}\\
\leq\;& P_\theta^{\epsilon,N}\left\{\left|\sum_{i=1}^{N}\frac{\frac{\epsilon\sqrt{\lambda_i+1}}{2(\theta-\lambda_i)}\int_0^T u_{i\epsilon}(t)dW_i(t)}{Q_{N,T}^{(\epsilon)}}\right|\geq\frac{\delta}{3}\right\}\\
&+P_{\theta_0}^{\epsilon,N}\left\{\left|\frac{\sum_{i=1}^{N}\frac{2v_i\epsilon}{\sqrt{\lambda_i+1}}e^{2(\theta-\lambda_i)T}\int_0^T\bar{e}^{(\theta-\lambda_i)S}dW_i(s)}{Q_{N,T}^{(\epsilon)}}\right|\geq\frac{\delta}{3}\right\}\\
&+P_\theta^{\epsilon,N}\left\{\left|\frac{\sum_{i=1}^{N}\frac{\epsilon^2}{2(\theta-\lambda_i)}e^{2(\theta-\lambda_i)T}\left(\int_0^T\bar{e}^{(\theta-\lambda_i)s}dw_i(s)\right)^2}{Q_{N,T}^{(\epsilon)}}\right|\geq\frac{\delta}{3}\right\}\\
=\;& I_1+I_2+I_3(\text{say}).
\end{aligned}$$

Now

$$
\begin{aligned}
I_1 &\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \left\{ \sum_{k=1}^{N} \frac{\lambda_k+1}{(\theta-\lambda_k)^2} E_{\theta_0} \int_0^T u_{k\epsilon}^2(t)dt \right\}^{\frac{1}{2}} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \left\{ \sum_{k=1}^{N} \frac{\lambda_k+1}{(\theta-\lambda_k)^2} \left(\frac{1}{2(\theta-\lambda_k)} v_k^2 (1-e^{2(\theta-\lambda_k)T}) \right) \right. \\
&\quad \left. + \frac{\epsilon^2}{2} \frac{1}{(\lambda_k-\theta)^3} \left(T - \frac{1-\bar{e}^{2(\lambda_k-\theta)T}}{2(\lambda_k-\theta)} \right) \right\}^{\frac{1}{2}} \text{ (following Hubner et al. (1993), p.154)} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{\lambda_k+1}{(\lambda_k-\theta)^3} v_k^2 \left(1-\bar{e}^{2(\lambda_k-\theta)T}\right) + \frac{\epsilon^2}{2} \frac{T}{(\lambda_k-\theta)^3} \right\}^{\frac{1}{2}} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{\lambda_k+1}{(\lambda_k-\theta)^3} v_k^2 + \frac{\epsilon^2 T}{(\lambda_k-\theta)^3} \right\}^{\frac{1}{2}} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \left\{ \sum_{k=1}^{N} \frac{\|f\|}{k^2} + \epsilon T^{\frac{1}{2}} \sum_{k=1}^{N} \frac{1}{k^3} \right\} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} + \frac{C\epsilon T^{\frac{1}{2}}}{\delta Q_{N,T}^{(\epsilon)}}. \qquad (3.\ 3)
\end{aligned}
$$

Next,

$$
\begin{aligned}
I_2 &\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{v_k^2}{\lambda_k+1} \bar{e}^{2(\lambda_k-\theta)T} \int_0^T e^{2(\lambda_k-\theta)s} ds \right\}^{\frac{1}{2}} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{v_k^2}{\lambda_k+1} \frac{1}{2(\lambda_k-\theta)} \left(1-\bar{e}^{2T(\lambda_k-\theta)}\right) \right\}^{\frac{1}{2}} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{v_k^2}{(\lambda_k+1)(\lambda_k-\theta)} \right\}^{\frac{1}{2}} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \frac{\|f\|}{k^2} \leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}}.
\end{aligned}
\qquad (3.\ 4)
$$

In addition,

$$\begin{aligned}
I_3 &\leq \frac{C\epsilon^2}{\delta Q^{(\epsilon)}_{N,T}} \sum_{k=1}^{N} \left\{ \frac{1}{(\lambda_k-\theta)^2} \bar{e}^{4(\lambda_k-\theta)T} E\left[\int_0^T e^{(\lambda_k-\theta)s} dW_k(s)\right]^4 \right\}^{\frac{1}{2}} \\
&\leq \frac{C\epsilon^2}{\delta Q^{(\epsilon)}_{N,T}} \sum_{k=1}^{N} \left\{ \frac{1}{(\lambda_k-\theta)^2} \frac{\bar{e}^{4(\lambda_k-\theta)T}}{4(\lambda_k-\theta)^2} \left[e^{2(\lambda_k-\theta)T}-1\right]^2 \right\}^{\frac{1}{2}} \\
&= \frac{C\epsilon^2}{\delta Q^{(\epsilon)}_{N,T}} \sum_{k=1}^{N} \left\{ \frac{1}{(\lambda_k-\theta)^2} \bar{e}^{2(\lambda_k-\theta)T} \left[e^{2(\lambda_k-\theta)T}-1\right] \right\} \\
&= \frac{C\epsilon^2}{\delta Q^{(\epsilon)}_{N,T}} \sum_{k=1}^{N} \left\{ \frac{1}{(\lambda_k-\theta)^2} \left(1-\bar{e}^{2(\lambda_k-\theta)T}\right) \right\} \\
&\leq \frac{C\epsilon^2}{\delta Q^{(\epsilon)}_{N,T}} \sum_{k=1}^{N} \frac{1}{(\lambda_k-\theta)^2} \leq \frac{C\epsilon^2}{\delta Q^{(\epsilon)}_{N,T}} \sum_{k=1}^{N} \frac{1}{k^4} \\
&\leq \frac{C\epsilon^2}{\delta Q^{(\epsilon)}_{N,T}}. \qquad (3.\ 5)
\end{aligned}$$

Note that

$$\begin{aligned}
Q^{(\epsilon)}_{N,T} &= \sum_{k=1}^{N} \frac{\lambda_k+1}{2(\theta-\lambda_k)} \left\{ v_k^2 \left(e^{2(\theta-\lambda_k)T}-1\right) - \frac{\epsilon^2 T}{\lambda_k+1} \right\} \\
&= \sum_{k=1}^{N} \frac{\lambda_k+1}{2(\lambda_k-\theta)} \left\{ v_k^2 \left(1-\bar{e}^{2(\lambda_k-\theta)T}\right) + \frac{\epsilon^2 T}{\lambda_k+1} \right\},
\end{aligned}$$

it follows that

$$Q^{(\epsilon)}_{N,T} \geq C\left[\epsilon^2 T + \|f\|^2\right] \text{ for large } N \text{ depending on } \theta \text{ and } T \text{ and for all } \epsilon.$$

Using (3.3), (3.4) and (3.5) and $Q^{(\epsilon)}_{N,T}$, we get that

$$\begin{aligned}
I_1+I_2+I_3 &\leq \frac{C_1\epsilon}{\delta Q^{(\epsilon)}_{N,T}} + \frac{C_2\epsilon T^{\frac{1}{2}}}{\delta Q^{(\epsilon)}_{N,T}} + \frac{C_3\epsilon^2}{\delta Q^{(\epsilon)}_{N,T}} \\
&\leq \frac{C\epsilon}{\delta(\epsilon^2 T+\|f\|^2)}\left(1+T^{\frac{1}{2}}\right)
\end{aligned}$$

for $0<\epsilon<1$. Choosing $\delta=\epsilon^{1-r}$, for some $0<r<1$, we get that the bound is of the order

$$\frac{C\epsilon^r(1+T^{\frac{1}{2}})}{(\epsilon^2 T+\|f\|^2)}.$$

Using Theorems 3.1 and 3.2, we have the following main Theorem for any fixed $N \geq 1$.

Theorem 3.3: There exists a constant C depending on θ_0, $\|f\|^2$ and T such that, for any

$0 < r < 1$,

$$\sup_y \left| P_{\theta_0}^{\epsilon,N} \left\{ \sqrt{Q_{N,T}^{(\epsilon)}} \epsilon^{-1} \left(\hat{\theta}_{N,\epsilon} - \theta_0 \right) \leq y \right\} - \Phi(y) \right| \leq \frac{C\epsilon^r (1 + T^{\frac{1}{2}})}{\epsilon^2 T + \|f\|^2} + 3\sqrt{\epsilon^{1-r}}$$

for $0 < \epsilon < 1$.

Remarks: Observe that the bound in Theorem 3.3 is of the order $O(\epsilon^r) + O(\epsilon^{\frac{1-r}{2}})$. Choosing $r = \frac{1}{3}$, we note that the bound is of the order $O(\epsilon^{\frac{1}{3}})$.

4 Stochastic PDE with linear drift (Singular case) : Estimation and Berry-Esseen type bound

Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider the process $u_{i\epsilon}(t,x)$, $0 \leq x \leq 1$, $0 \leq t \leq T$ governed by the stochastic partial differential equation

$$du_\epsilon(t,x) = \theta \Delta u_\epsilon(t,x)dt + \epsilon (I - \Delta)^{-\frac{1}{2}} dW(t,x) \tag{4. 1}$$

where $\theta > 0$ satisfying the initial and boundary conditions

$$u_\epsilon(0,x) = f(x), 0 < x < 1, f \epsilon L_2[0,1],$$

$$u_\epsilon(t,0) = u_\epsilon(t,1) = 0, \quad 0 \leq t \leq T \tag{4. 2}$$

Here I is the identity operator, $\Delta = \frac{\partial^2}{\partial x^2}$ as defined in the Section 3 and the process $W(t,x)$ is the cylindrical Brownian motion in $L_2[0,1]$. In analogy with (2.5) in Section 2, the Fourier co-efficients $u_{i\epsilon}(t)$ satisfy the stochastic differential equation

$$du_{i\epsilon}(t) = -\theta \lambda_i u_{i\epsilon}(t) dt + \frac{\epsilon}{\sqrt{\lambda_1 + 1}} dW_i(t), \quad 0 < t \leq T \tag{4. 3}$$

with

$$u_{i\epsilon}(0) = v_i \quad \text{where} \quad v_i = \int_0^1 f(x) e_i(x) dx. \tag{4. 4}$$

Let P_θ^ϵ be the measure generated by the process u_ϵ on $C[0,T]$ when θ is the true parameter. It can be shown that the family of measures $\{P_\theta^\epsilon, \ \theta \in \Theta\}$ do not form a family of equivalent probability measures. In fact P_θ^ϵ is singular with respect to $P_{\theta'}^\epsilon$, when $\theta \neq \theta'$ in Θ (cf. Huebner et. al (1993)).

Let $u_\epsilon^{(N)}(t,x)$ be the projection of $u_\epsilon(t,x)$ onto the subspace spanned by $\{e_1, e_2, \ldots, e_N\}$ in $L_2[0,1]$. In other words,

$$u_\epsilon^{(N)}(t,x) = \sum_{i=1}^{N} u_{i\epsilon}(t) e_i(x). \tag{4. 5}$$

Let $P_\theta^{\epsilon,N}$ be the probability measure generated by $u_\epsilon^{(N)}$ on the subspace spanned by $\{e_1, \ldots e_N\}$ in $L_2[0,1]$. It can be shown that the measures $\left\{P_\theta^{\epsilon,N},\ \theta \in \Theta\right\}$ form an equivalent family and

$$\begin{aligned}
&\log \frac{dP_\theta^{\epsilon,N}}{dP_{\theta_0}^{(\epsilon,N)}}(u_\epsilon^{(N)}) \\
= \quad & -\frac{1}{\epsilon^2} \sum_{i=1}^{N} \lambda_i(\lambda_i+1)\left[(\theta-\theta_0)\int_0^T u_{i\epsilon}(t)(du_{i\epsilon}(t) + \theta_0\lambda_i u_{i\epsilon}(t)dt)\right. \\
& \left. +\frac{1}{2}(\theta-\theta_0)^2\lambda_i \int_0^T u_{i\epsilon}^2(t)dt\right] \qquad (4.\ 6)
\end{aligned}$$

It can be checked that the MLE $\hat{\theta}_{N,\epsilon}$ of θ based on $u_\epsilon^{(N)}$ satisfies the likelihood equation

$$\alpha_{\epsilon,N} = \epsilon^{-1}\left(\hat{\theta}_{N,\epsilon} - \theta_0\right)\beta_{\epsilon,N} \tag{4. 7}$$

when θ_0 is the true parameter,

$$\alpha_{\epsilon,N} = \sum_{i=1}^{N} \lambda_i\sqrt{\lambda_i+1}\int_0^T u_{i\epsilon}(t)dW_i(t)$$

and

$$\beta_{\epsilon,N} = \sum_{i=1}^{N} (\lambda_i+1)\lambda_i^2 \int_0^T u_{i,\epsilon}^2(t)dt.$$

¿From (4.7) we obtain,

$$\sqrt{R_{N,T}^{(\epsilon)}}(\hat{\theta}_{N,\epsilon}-\theta_0) = \frac{\epsilon\left\{\sum_{i=1}^{N}\lambda_i\sqrt{\lambda_i+1}\int_0^T u_{i\epsilon}(t)dW_i(t)\right\}/\sqrt{R_{N,T}^{(\epsilon)}}}{\left\{\sum_{i=1}^{N}(\lambda_i+1)\lambda_i^2\int_0^T u_{i\epsilon}^2(t)dt\right\}/R_{N,T}^{(\epsilon)}}$$

where

$$R_{N,T}^{(\epsilon)} = \sum_{i=1}^{N}\frac{\lambda_i(\lambda_i+1)}{2\theta}\left\{v_i^2(1-\bar{e}^{2\theta\lambda_i T}) + T\frac{\epsilon^2}{\lambda_i+1}\right\}.$$

It can be checked that

$$E_{\theta_0}\int_0^T u_{i\epsilon}^2(t)dt < \infty.$$

Theorem 4.1 : For any $0 < \delta < 1$,

$$\sup_y \left| P_{\theta_0}^{\epsilon,N}\left\{ \sqrt{R_{N,T}^{(\epsilon)}}(\hat{\theta}_{N,\epsilon}-\theta_0) \le y \right\} - \Phi(y) \right|$$
$$\le \; 2\left[P_{\theta_0}^{\epsilon,N}\left\{ \left| \frac{\epsilon^{-1}\sum_{i=1}^{N}(\lambda_i+1)\lambda_i^2\int_0^T u_{i\epsilon}^2(t)dt}{R_{N,T}^{(\epsilon)}} - 1 \right| \ge \delta \right\} + 3\sqrt{\delta}\;. \right.$$

We can prove the Theorem 4.1 using the Lemmas 3.1 and 3.2 and following the procedure of proof of Theorem 3.1.

Theorem 4.2 : There exists a constant C depending on θ_0, $\|f\|^2$ and T such that, for any $\delta > 0$,

$$P_{\theta_0}^{\epsilon,N}\left\{ \left| \frac{\epsilon^{-1}\sum_{i=1}^{N}(\lambda_i+1)\lambda_i^2\int_0^T u_{i\epsilon}^2(t)dt}{R_{N,T}^{(\epsilon)}} - 1 \right| \ge \delta \right\} \le \frac{CN^3(1+T^{\frac{1}{2}})}{\delta(\epsilon^2 TN^3 + \sum_{k=1}^{N} k^4 v_k^2)}.$$

Proof: By the Ito formula, we get that

$$d(u_{i\epsilon}^2(t)) = -2\theta\lambda_i u_{i\epsilon}^2(t)dt + \frac{\epsilon}{\sqrt{\lambda_i+1}}u_{i\epsilon}(t)dW_i(t) + \frac{\epsilon^2}{\lambda_i+1}dt$$

or equivalently

$$d\left(\frac{\lambda_i(\lambda_i+1)}{2\theta}u_{i\epsilon}^2(t)\right) = -\lambda_i^2(\lambda_i+1)u_{i\epsilon}^2(t)dt + \frac{\epsilon\lambda_i\sqrt{\lambda_i+1}}{2\theta}u_{i\epsilon}(t)dW_i(t) + \frac{\epsilon^2\lambda_i}{2\theta}dt$$

or

$$\frac{\lambda_i(\lambda_i+1)}{2\theta}u_{i\epsilon}^2(T) - \frac{\lambda_i(\lambda_i+1)}{2\theta}v_i^2$$
$$= \; -\int_0^T \lambda_i^2(\lambda_i+1)u_{i\epsilon}^2(t)dt + \frac{\epsilon\lambda_i\sqrt{\lambda_i+1}}{2\theta}\int_0^T u_{i\epsilon}(t)dW_i(t) + \frac{\epsilon^2\lambda_i}{2\theta}T. \tag{4. 8}$$

Again, by the Ito formula, it follows that

$$d\left(u_{i\epsilon}(t)e^{\theta\lambda_i t}\right) = \frac{\epsilon}{\sqrt{\lambda_i+1}}e^{\theta\lambda_i t}dW_i(t)$$

or

$$u_{i\epsilon}(T)e^{\theta\lambda_i T} - v_i = \int_0^T \frac{\epsilon}{\sqrt{\lambda_i+1}}e^{\theta\lambda_i t}dW_i(t)$$

or

$$u_{i\epsilon}(T) - v_i\bar{e}^{\theta\lambda_i T} = \bar{e}^{\theta\lambda_i T}\int_0^T \frac{\epsilon}{\sqrt{\lambda_i+1}}e^{\theta\lambda_i t}dW_i(t)$$

or

$$u_{i\epsilon}^2(T) = \bar{e}^{2\theta\lambda_i T}\left(\int_0^T \frac{\epsilon}{\sqrt{\lambda_i+1}} e^{\theta\lambda_i t} dW_i(t)\right)^2$$
$$+ v_i^2 \bar{e}^{2\theta\lambda_i T} + \frac{2\epsilon}{\sqrt{\lambda_i+1}} v_i \bar{e}^{2\theta\lambda_i T}\int_0^T e^{\theta\lambda_i t} dW_i(t)$$

or

$$\frac{\lambda_i(\lambda_i+1)}{2\theta} u_{i\epsilon}^2(T) = \frac{\epsilon^2\lambda_i}{2\theta}\bar{e}^{2\theta\lambda_i T}\left(\int_0^T e^{\theta\lambda_i t} dW_i(t)\right)^2$$
$$+\frac{\lambda_i(\lambda_i+1)}{2\theta} v_i^2 \bar{e}^{2\theta\lambda_i T} + \frac{\epsilon}{\theta}\lambda_i\sqrt{\lambda_i+1}\bar{e}^{2\theta\lambda_i T} v_i \int_0^T e^{\theta\lambda_i t} dW_i(t). \qquad (4.\ 9)$$

¿From (4.8) and (4.9), we get that

$$\sum_{i=1}^N \frac{\lambda_i(\lambda_i+1)}{2\theta}\left\{v_i^2\left(1-\bar{e}^{2\theta\lambda_i T}\right) + \frac{\epsilon^2}{\lambda_i+1}T\right\} - \sum_{i=1}^N \int_0^T \lambda_i^2(\lambda_i+1) u_{i\epsilon}^2(t) dt$$
$$= \frac{\epsilon^2}{2\theta}\sum_{i-1}^N \lambda_i \bar{e}^{2\theta\lambda_i T}\left(\int_0^T e^{\theta\lambda_i t} dW_i(t)\right)^2 + 2\epsilon\sum_{i=1}^N \lambda_i\sqrt{\lambda_i+1} v_i \bar{e}^{2\theta\lambda_i T}\int_0^T e^{\theta\lambda_i t} dW_i(t)$$
$$-\epsilon\sum_{i=1}^N \frac{\lambda_i\sqrt{\lambda_i+1}}{2\theta}\int_0^T u_{i\epsilon}(t) dW_i(t).$$

So

$$P_{\theta_0}^{\epsilon,N}\left\{\left|\frac{\epsilon^{-1}\sum_{i=1}^N\int_0^T \lambda_i^2(\lambda_i+1)u_{i\epsilon}^2(t)dt}{R_{N,T}^{(\epsilon)}} - 1\right| \geq \delta\right\}$$
$$\leq P_{\theta_0}^{\epsilon,N}\left\{\left|\frac{\frac{\epsilon}{2\theta}\sum_{i=1}^N \lambda_i \bar{e}^{2\theta\lambda_i T}\left(\int_0^T e^{\theta\lambda_i t} dW_i(t)\right)^2}{R_{N,T}^{(\epsilon)}}\right| \geq \frac{\delta}{3}\right\}$$
$$+P_{\theta_0}^{\epsilon,N}\left\{\left|\frac{\frac{1}{\theta}\sum_{i=1}^N \lambda_i\sqrt{\lambda_i+1} v_i \bar{e}^{2\theta\lambda_i T}\int_0^T e^{\theta\lambda_i t} dW_i(t)}{R_{N,T}^{(\epsilon)}}\right| \geq \frac{\delta}{3}\right\}$$
$$+P_{\theta_0}^{\epsilon,N}\left\{\left|\frac{\frac{1}{2\theta}\sum_{i=1}^N \lambda_i\sqrt{\lambda_i+1}\int_0^T u_{i\epsilon}(t) dW_i(t)}{R_{N,T}^{(\epsilon)}}\right| \geq \frac{\delta}{3}\right\}$$
$$= J_1 + J_2 + J_3 \text{ (say)}$$

where

$$R_{N,T}^{(\epsilon)} = \sum_{i=1}^N \frac{\lambda_i(\lambda_i+1)}{2\theta}\left\{v_i^2\left(1-\bar{e}^{2\theta\lambda_i T}\right) + \frac{\epsilon^2}{\lambda_i+1}T\right\}.$$

So

$$\begin{aligned} J_1 &\leq \frac{C\epsilon}{\delta R_{N,T}^{(\epsilon)}} \sum_{i=1}^{N} \lambda_i \bar{e}^{2\theta\lambda_i T} \int_0^T e^{2\theta\lambda_i t} dt \\ &= \frac{C\epsilon}{\delta R_{N,T}^{(\epsilon)}} \sum_{i=1}^{N} \lambda_i \bar{e}^{2\theta\lambda_i T} \left\{ \left(e^{2\theta\lambda_i T} - 1 \right) / 2\theta\lambda_i \right\} \\ &\leq \frac{C\epsilon N}{\delta R_{N,T}^{(\epsilon)}}, \end{aligned}$$

$$\begin{aligned} J_2 &\leq \frac{C}{\delta R_{N,T}^{(\epsilon)}} \sum_{i=1}^{N} \sqrt{\lambda_i} \sqrt{\lambda_i + 1} v_i \\ &\leq \frac{CN^3}{\delta R_{N,T}^{(\epsilon)}} \end{aligned}$$

and

$$\begin{aligned} J_3 &\leq \frac{C}{\delta R_{N,T}^{(\epsilon)}} \left(\sum_{i=1}^{N} \lambda_i^2 (\lambda_i + 1) \int_0^T E u_{i\epsilon}^2(t) dt \right)^{\frac{1}{2}} \\ &\leq \frac{C\sqrt{2\theta}}{\delta R_{N,T}^{(\epsilon)}} \left\{ \sum_{i=1}^{N} \lambda_i (\lambda_i + 1) v_i^2 (1 - \bar{e}^{2\theta\lambda_i T}) + T \sum_{i=1}^{N} \lambda_i \right\}^{\frac{1}{2}} \\ &\quad \text{(following Hubner et. al. (1993))} \\ &\leq \frac{C}{\delta R_{N,T}^{(\epsilon)}} \left\{ \sum_{i=1}^{N} \lambda_i (\lambda_i + 1) + T \sum_{i=1}^{N} \lambda_i \right\}^{\frac{1}{2}} \\ &\leq \frac{C}{\delta R_{N,T}^{(\epsilon)}} \left\{ N^{\frac{5}{2}} + T^{\frac{1}{2}} N^{\frac{3}{2}} \right\}. \end{aligned}$$

So

$$\begin{aligned} J_1 + J_2 + J_3 &\leq \frac{C}{\delta R_{N,T}^{(\epsilon)}} \left(N^3 + T^{\frac{1}{2}} N^{\frac{3}{2}} \right) \\ &\leq \frac{C}{\delta R_{N,T}^{(\epsilon)}} N^3 \left(1 + T^{\frac{1}{2}} \right). \end{aligned}$$

Now

$$
\begin{aligned}
R_{N,T}^{(\epsilon)} &= \sum_{k=1}^{N} \frac{\lambda_k(\lambda_k+1)}{2\theta} \left\{ v_k^2 \left(1 - \bar{e}^{2\theta\lambda_k T}\right) + \frac{T\epsilon^2}{\lambda_k+1} \right\} \\
&\geq C \sum_{k=k_1}^{N} k^4 \left\{ v_k^2 + \frac{T\epsilon^2}{k^2} \right\}
\end{aligned}
$$

for some k_1 depending on θ_0 and T.

Therefore,

$$
J_1 + J_2 + J_3 \leq \frac{CN^3(1+T^{\frac{1}{2}})}{\delta(\epsilon^2 TN^3 + \sum_{k=1}^{N} k^4 v_k^2)}.
$$

Choosing $\delta = N^{-\gamma}$, for some $\gamma > 0$, we get that the bound is of the order

$$
\frac{CN^3(1+T^{\frac{1}{2}})}{N^{-\gamma}\left(\epsilon^2 TN^3 + \sum_{k=1}^{N} k^4 v_k^2\right)}.
$$

Using the Theorems 4.1 and 4.2, we have the following result for any fixed $0 < \epsilon < 1$.

Theorem 4.3: There exists a constant C depending on $\theta_0, ||f||^2$ and T such that, for any $\gamma > 0$,

$$
\sup_y \left| P_{\theta_0}^{\epsilon,N} \left\{ \sqrt{R_{N,T}^{(\epsilon)}} \left(\hat{\theta}_{N,\epsilon} - \theta_0\right) \leq y \right\} - \Phi(y) \right| \leq \frac{CN^3}{N^{-\gamma}} \left(\frac{1+T^{\frac{1}{2}}}{\epsilon^2 TN^3 + \sum_{k=1}^{N} k^4 v_k^2} \right) + 3\sqrt{N^{-\gamma}}.
$$

Remarks: Observe that the bound in Theorem 4.3 is of the order $O(N^{\gamma-2}) + O(N^{-\frac{\gamma}{2}})$ provided $\sum_{k=1}^{N} k^4 v_k^2 \geq g(N) = O(N^5)$. In such a case, the bound can be obtained to be of the order $O(N^{-\frac{2}{3}})$ by choosing $\gamma = \frac{4}{3}$.

References :

1. Feigin, P. D. (1976) : *Maximum likelihood estimation for continuous time stochastic processes,* Adv. Appl. Probability, (8), 712 - 716.

2. Hall, P. and Heyde, C. C. (1980): *Martingale limit thoery and its application,* Academic Press, New York.

3. Hubner, M., Rozovskii, B. L., Khasminski, R. (1993) *Two examples of parameter estimation for stochastic partial differential equation, In : Combanis, S., Ghosh J. K., Karandikar R. L., Sen P. K. (Eds.)* Stochastic processes : A Festschrift in Honour of Gopinath Kallianpur, Springer, New York, P 149-160.

4. Liptser, R. S. and Shiryayev A. N., (1978) *Statistics of Random Processes,* Springer-verlag, New York.

5. Michael, R. and Pfanzagl, J. (1971), *The accuracy of the normal approximation for minimum contrast estimate,* Z. Wahrschieinlichkeits theorie verw. Gebiete, (18) 73-84.

6. Mishra M. N., Prakasa Rao, B. L. S. (1985) *On the Berry-Esseen bound for maximum likelihood estimator for linear homogeneous diffusion processes,* Sankhyā (47), Series A, 392-398.

7. Prakasa Rao, B. L. S. (2001) *Statistical inference for stochastic partial differential equations,* In : Selected Papers Proc. Symp. Infer. for Stochastic Proc., Ed. I. V. Basawa, C. C. Heyde and R. L. Taylor, IMS Monograph Series, Vol.37, pp.47-70.

8. Prakasa Rao, B. L. S. (2002) *On some problems of estimation for some stochastic partial differential equations.* In: Uncertainty and Optimality-Probability, Statistics and Operations Research, Ed. J. C. Misra, World Scientific, Singapore, pp. 71-154.

9. Rozovskii B. L. (1990) Stochastic Evolution Systems: Linear theory and Applications to Nonlinear Filtering, Kluwer Academic publishers.