isid/ms/2003/30
October 25, 2003
http://www.isid.ac.in/~statmath/eprints

Hölder norm estimates for elliptic operators on finite and infinite dimensional spaces

Siva R. Athreya
Richard F. Bass
Edwin A. Perkins

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

Hölder norm estimates for elliptic operators on finite and infinite dimensional spaces

Siva R. Athreya[1], Richard F. Bass[2], and Edwin A. Perkins[3]

Abstract

We introduce a new method for proving the estimate

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{C^\alpha} \leq c\|f\|_{C^\alpha},$$

where u solves the equation $\Delta u - \lambda u = f$. The method can be applied to the Laplacian on $\mathbb{R}^\infty$. It also allows us to obtain similar estimates when we replace the Laplacian by an infinite dimensional Ornstein-Uhlenbeck operator or other elliptic operators. These operators arise naturally in martingale problems arising from measure-valued branching diffusions and from stochastic partial differential equations.

Contents

1. Research supported in part by a NBHM travel grant.
2. Research supported in part by NSF grant DMS0244737.
3. Research supported in part by an NSERC Research Grant.
Keywords: Semigroups, Schauder estimates, Hölder spaces, Perturbations, Resolvents, Elliptic operators, Laplacian, Ornstein-Uhlenbeck processes, Infinite dimensional stochastic differential equations
Classification: Primary 35J15. Secondary 35R15, 47D07, 60J35
Running Head: Hölder norm estimates

1. Introduction.

Let Δ be the Laplacian on $\mathbb{R}^d$ and for $\alpha \in (0,1)$ define the usual Hölder norms by

$$\|f\|_{C^\alpha} = \sup_x |\, f(x)\, | + \sup_{x,\, h\neq 0} \frac{|\, f(x+h) - f(x)\, |}{|h|^\alpha} \equiv \|f\|_\infty + |f|_{C^\alpha}. \tag{1.1}$$

A classical estimate is that if $\lambda > 0$ and u is the solution in $\mathbb{R}^d$ to

$$\Delta u - \lambda u = f, \tag{1.2}$$

then we have the inequality

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{C^\alpha} \le c_1 \|f\|_{C^\alpha}, \tag{1.3}$$

where $1 \le i, j \le d$ and c_1 is a constant not depending on f. Two of the more important applications of this result are that it allows one to prove the existence of solutions to certain elliptic partial differential equations with variable coefficients and to prove uniqueness in law of solutions to certain stochastic differential equations.

In this paper we investigate the analogue of (1.3) when the Laplacian is replaced by other elliptic operators. In particular we:

(1) introduce a new method, which we call the semigroup method, for proving (1.3);

(2) use our method to obtain an analogue of (1.3) for the case of infinite dimensional Ornstein-Uhlenbeck operators; and

(3) lastly show how the semigroup method allows one to determine the appropriate substitute for the norms given in (1.1).

In work in preparation ([ABP]) we use some of the above results to prove uniqueness for an infinite dimensional system of Ornstein-Uhlenbeck type stochastic differential equations with Hölder continuous coefficients. The semigroup method is particularly simple in the case of the Laplacian, even if we replace $\mathbb{R}^d$ by $\mathbb{R}^\infty$. We need one elementary calculation, namely, that

$$\int \left| \frac{\partial p_t(x,y)}{\partial x} \right| dy \le \frac{c_2}{\sqrt{t}},$$

where $p_t(x,y) = (2\pi t)^{-1/2} \exp(-(y-x)^2/2t)$ for $x, y \in \mathbb{R}$. We use this and the fact that P_t, the semigroup corresponding to the Laplacian, factors to see that

$$\left\| \frac{\partial P_t f}{\partial x_i} \right\|_\infty \le \frac{c_2}{\sqrt{t}} \|f\|_\infty.$$

Some manipulations of semigroups then lead to (1.3). A key step is to define the semigroup norm

$$\|f\|_{S^\alpha} = \|f\|_\infty + \sup_{t>0} \frac{\|P_t f - f\|_\infty}{t^{\alpha/2}}. \tag{1.4}$$

This norm was also used in the argument of [CD].

In the case of the Laplacian in finite dimensions, there are a number of proofs of (1.3). See, for example, [GT], Chapter 4, or [Ba], Section II.3. Another proof can be found in [Ba], Section IV.3 or [S], Section V.4. This latter proof is at the basis of the semigroup method.

The proof of (1.3) for the Laplacian in infinitely many dimensions is relatively recent and is due to Cannarsa and Da Prato [CD]. Their method involves interpolation spaces. It is well suited to the Laplacian, but perhaps less so for other operators. Our results in Section 3 give a new proof for the infinite dimensional Laplacian.

We use the semigroup method to obtain an analogue to (1.3) when the Laplacian is replaced by the operator $\mathcal{L}$ defined by

$$\mathcal{L}f(x) = \sum_{i,j=1}^{\infty} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \sum_{i,j=1}^{\infty} V_{ij} x_j \frac{\partial f}{\partial x_i}(x), \tag{1.5}$$

where a is positive definite and V is nonnegative definite (See Theorem 5.6). This operator is a generalization of the infinite dimensional Ornstein-Uhlenbeck operator. It is well known that the infinite dimensional Ornstein-Uhlenbeck operator arises when using Fourier transforms to study parabolic stochastic partial differential equations (see [W]) and this was in fact the motivation for considering this problem. One principal difference from the Laplacian case is that the operators $\partial/\partial x_i$ and P_t no longer commute. Related results for the Ornstein-Uhlenbeck case have been obtained by [D], [L], [Z]. In Remark 5.8 we discuss them briefly and compare them to our results Theorem 5.6 and Corollary 5.7.

When one considers operators other than the Laplacian, it turns out that the C^α norms defined by (1.1) may not be the most appropriate. In fact, the semigroup norm given in (1.4) is in some cases the natural one. In the case of certain degenerate elliptic operators, we discovered this after the fact. In [BP] two of the authors investigated Hölder norm inequalities for an operator that arises in the study of branching measure-valued diffusions. There the estimates were proved by hand, and we were forced to replace the use of the C^α norms by weighted Hölder norms. In this paper we prove that these weighted Hölder norms are precisely the S^α norms used by the semigroup method. This suggests the potential for a more unified approach to such norms in the study of degenerate stochastic differential equations in both finite and infinite dimensions and avoids having to guess the appropriate norm through ad hoc methods.

Layout of the paper: Here is the plan for the rest of the paper. In Section 2 we define the semigroup norm and establish some preliminary facts. In Section 3 we present the semigroup method in the case of the infinite dimensional Laplacian (Proposition 3.3). Although the estimates in the Laplacian case are known, we present this case separately for clarity. In Section 4 we give some connections between the semigroup norm and the usual Hölder norms (Proposition 4.1 and 4.2). Next, in Section 5, we consider the Ornstein-Uhlenbeck operator, and establish the analogue of (1.3) in Theorem 5.6 and Corollary 5.7. Section 6 considers geometrical aspects of the semigroup norm, analogous to section 4. Many of these results will be used in the in the uniqueness proof for infinite dimensional stochastic equations in [ABP]. In Section 7 we establish the equivalence of the semigroup norm with weighted Hölder norms in the context of the operator considered in [BP].

Acknowledgment.

We would like to thank L. Zambotti for patiently answering our many questions concerning norms related to the Ornstein-Uhlenbeck operator.

2. The semigroup norm.

We use the following notation. If $E = \mathbb{R}^d, \mathbb{R}^d_+, \mathbb{R}^\infty$, or a separable Hilbert space H, and $f : E \to \mathbb{R}$, $D_w f(x)$ is the directional derivative of f at $x \in E$ in the direction w; we do not require w to be a unit vector. We write D_i for D_{ϵ_i} and D_{ij} for $D_i D_j$, where ϵ_i denotes the ith unit vector in a convenient orthonormal system; for $\mathbb{R}^d$ or $\mathbb{R}^\infty$, ϵ_i will be the i^{th} coordinate direction.

The inner product in E is denoted $\langle \cdot, \cdot \rangle$, and $|\cdot|$ denotes the norm generated by this inner product. $C_b = C_b(E)$ is the collection of $\mathbb{R}$-valued bounded continuous functions on E and for $\alpha \in (0,1)$, C^α is the set of functions in C_b for which $\|f\|_{C^\alpha} = \|f\|_\infty + |f|_{C^\alpha}$, defined as in (1.1) by replacing $\mathbb{R}^d$ with E, is finite. Finally C_b^2 is the set of functions in C_b for which the first and second order partials are also in C_b.

We use the letter c with subscripts for finite positive constants whose value is unimportant and which may vary from line to line.

Given an operator $\mathcal{L}$ that is the infinitesimal generator of a semigroup P_t on the space of bounded measurable functions on E, we let $R_\lambda = \int_0^\infty e^{-\lambda s} P_s \, ds$ be the corresponding resolvent. We define the semigroup norm (the "S" stands for "semigroup") $\|\cdot\|_{S^\alpha}$ for $\alpha \in (0,1)$ by

$$\|f\|_{S^\alpha} = \|f\|_\infty + \sup_{t>0} t^{-\alpha/2} \|P_t f - f\|_\infty. \tag{2.1}$$

Let S^α denote the space of measurable functions on E for which this norm is finite. We set $|f|_{S^\alpha}$ equal to the last term in (2.1), so

$$\|f\|_{S^\alpha} = \|f\|_\infty + |f|_{S^\alpha}.$$

In a number of places we will use a similar convention: $|f|_B$ will denote a seminorm in some Banach space B $\|f\|_B$ will then be $\|f\|_\infty + |f|_B$.

Remark 2.1. Since $\|P_t f - f\|_\infty \le 2\|f\|_\infty$, then

$$\|f\|_{S^\alpha} \le 3\|f\|_\infty + \sup_{0<t\le 1} t^{-\alpha/2} \|P_t f - f\|_\infty. \tag{2.2}$$

We will use the following result a number of times.

Lemma 2.2. *There exists $c_2(\alpha)$ such that if for some $w \in E$ and $0 < c_1 < \infty$,*

$$\|D_w P_t f\|_\infty \leq \frac{c_1|w|}{\sqrt{t}}\|f\|_\infty$$

for all bounded measurable f, then for all $f \in S^\alpha$,

$$\|D_w P_t f\|_\infty \leq c_1 c_2 |w| t^{(\alpha-1)/2} |f|_{S^\alpha}.$$

Proof. Note

$$D_w P_{2u} f - D_w P_u f = D_w P_u [P_u f - f].$$

The sup norm of the expression inside the brackets is bounded by $u^{\alpha/2}|f|_{S^\alpha}$. Therefore by our hypothesis,

$$\|D_w P_{2u} f - D_w P_u f\|_\infty \leq c_1 |w| u^{(\alpha-1)/2} |f|_{S^\alpha}. \tag{2.3}$$

Using the hypothesis again,

$$\|D_w P_{t2^k} f\|_\infty \leq c_1 |w| (t2^k)^{-1/2} \|f\|_\infty \to 0$$

as $k \to \infty$. Therefore

$$D_w P_t f = \sum_{k=0}^\infty (D_w P_{t2^k} f - D_w P_{t2^{k+1}} f).$$

Using (2.3) and the triangle inequality,

$$\|D_w P_t f\|_\infty \leq \sum_{k=0}^\infty c_1 |w| (t2^k)^{(\alpha-1)/2} |f|_{S^\alpha} \leq c_1 |w| c_2(\alpha) t^{(\alpha-1)/2} |f|_{S^\alpha}.$$

□

Lemma 2.3. *Assume*

$$\|D_w P_t f\|_\infty \leq \frac{c_1|w|}{\sqrt{t}}\, \|f\|_\infty \tag{2.4}$$

for all bounded measurable f on E and all $w \in E$. Then $S^\alpha \subset C^\alpha$ and

$$\|f\|_{C^\alpha} \leq (c_1 c_2(\alpha) + 2)\|f\|_{S^\alpha},$$

where $c_2(\alpha)$ is as in Lemma 2.2.

Proof. By (2.4), Lemma 2.2 and the mean value theorem, if $w \in E$ then

$$|P_t f(x+w) - P_t f(x)| \leq c_1 c_2 |w| t^{(\alpha-1)/2} \|f\|_{S^\alpha}.$$

We also have

$$|P_tf(x+w)-f(x+w)|\le t^{\alpha/2}\|f\|_{S^\alpha},\qquad |P_tf(x)-f(x)|\le t^{\alpha/2}\|f\|_{S^\alpha},$$

by the definition of S^α. By the triangle inequality,

$$|f(x+w)-f(x)|\le t^{\alpha/2}(c_1c_2|w|t^{-1/2}+2)\|f\|_{S^\alpha}.$$

If we take $t=|w|^2$, we see that $\|f\|_{C^\alpha}\le (c_1c_2+2)\|f\|_{S^\alpha}$. □

Lemma 2.4. *Let $\{X_t,t\ge 0\}$ be an E-valued Markov process with semigroup P_t and laws $\{\mathbb{P}^x,x\in E\}$. Assume (2.4) and also*

$$\mathbb{E}^x(|X_t-\mathbb{E}^x(X_t)|^2)\le c_0t^{1/2}\ \text{for all}\ t\le 1. \tag{2.5}$$

If $f,g\in S^\alpha$, then $fg\in S^\alpha$ and for some $c_1=c_1(c_0,\alpha)$,

$$|fg|_{S^\alpha}\le c_1[\|f\|_\infty|g|_{S^\alpha}+|f|_{S^\alpha}\|g\|_\infty+|f|_{C^\alpha}|g|_{C^\alpha}+\|f\|_\infty\|g\|_\infty], \tag{2.6}$$

and

$$\|fg\|_{S^\alpha}\le c_1\|f\|_{S^\alpha}\|g\|_{S^\alpha}. \tag{2.7}$$

Proof. Let $R_tx=\mathbb{E}^x(X_t)\in E$ (by hypothesis). Note that

$$\begin{aligned}P_tfg(x)-fg(x)=&\mathbb{E}^x((f(X_t)-f(R_tx))(g(X_t)-g(R_tx))+g(R_tx)(P_tf(x)-f(x))\\&+f(R_tx)(P_tg(x)-g(x))-(f(R_tx)-f(x))(g(R_tx)-g(x)).\end{aligned} \tag{2.8}$$

Note also that for $t\le 1$,

$$\begin{aligned}|f(R_tx)-f(x)|&\le |P_tf(x)-f(x)|+|\mathbb{E}^x(f(X_t)-f(R_tx))|\\&\le |f|_{S^\alpha}t^{\alpha/2}+|f|_{C^\alpha}\mathbb{E}^x(|X_t-R_tx|^\alpha)\\&\le |f|_{S^\alpha}t^{\alpha/2}+|f|_{C^\alpha}c_0^{\alpha/2}t^{\alpha/4},\end{aligned} \tag{2.9}$$

the latter by (2.5) and Jensen's inequality. We put this into (2.8) and use Hölder's inequality to conclude that for all $t\le 1$,

$$\begin{aligned}|P_tfg(x)-fg(x)|\le&|f|_{C^\alpha}|g|_{C^\alpha}\mathbb{E}^x(|X_t-R_tx|^2)^\alpha\\&+(\|g\|_\infty|f|_{S^\alpha}+\|f\|_\infty|g|_{S^\alpha})t^{\alpha/2}\\&+|f(R_tx)-f(x)|\Big(\big[|g|_{S^\alpha}t^{\alpha/2}+c_0^{\alpha/2}|g|_{C^\alpha}t^{\alpha/4}\big]\wedge 2\|g\|_\infty\Big)\\\le&[c|f|_{C^\alpha}|g|_{C^\alpha}+\|g\|_\infty|f|_{S^\alpha}+3\|f\|_\infty|g|_{S^\alpha}]t^{\alpha/2}\\&+|f(R_tx)-f(x)|c_3[(|g|_{C^\alpha}t^{\alpha/4})\wedge\|g\|_\infty].\end{aligned}$$

We use (2.9) again to bound the last term by

$$c_4[|f|_{S^\alpha}\|g\|_\infty + |f|_{C^\alpha}|g|_{C^\alpha}]t^{\alpha/2}.$$

Substituting this into the above, we see that for $t \le 1$,

$$|P_t fg(x) - fg(x)| \le c_1[\|f\|_\infty |g|_{S^\alpha} + |f|_{S^\alpha}\|g\|_\infty + |f|_{C^\alpha}|g|_{C^\alpha}].$$

If $t > 1$, the left-hand side is at most $2\|f\|_\infty\|g\|_\infty$ and (2.6) follows. This and Lemma 2.3 now imply (2.7). $\square$

3. Hölder estimates – the Laplacian case.

Let ℓ^2 be the space of real square summable sequences $\{x_i : i \in \mathbb{N}\}$ equipped with the norm $|x| = (\sum_i x_i^2)^{1/2}$ and take ϵ_i to be the unit vector in the ith coordinate direction. We study perturbations of

$$\mathcal{L} = \tfrac{1}{2}\sum_i a_i^2 D_{ii}.$$

Here we assume each $a_i > 0$ and $|a|^2 = \sum_i a_i^2 < \infty$. The reader interested only in the finite dimensional case may restrict all indices to the range 1 to d and take each $a_i = 1$ but we will be implicitly working in ℓ^2 below.

Lemma 3.1. *There exists c_1 such that for any bounded measurable f,*

$$\|D_i P_t f\|_\infty \le \frac{c_1}{a_i\sqrt{t}}\|f\|_\infty.$$

Proof. Let

$$p_t^j(x_j, dy_j) = \frac{1}{a_j\sqrt{2\pi t}}e^{-(y_j-x_j)^2/2a_j^2 t}dy_j$$

be the transition density of one dimensional Brownian motion with parameter a_j^2. Let

$$q_t^j(x_j, dy_j) = D_j p_t^j(x_j, dy_j) = \frac{1}{a_j\sqrt{2\pi t}}\frac{y_j - x_j}{a_j^2 t}e^{-(y_j-x_j)^2/2a_j^2 t}dy_j.$$

Note that

$$\int |q_t^j(x_j, dy_j)| = \int_{-\infty}^{\infty}\frac{1}{a_j\sqrt{2\pi t}}\frac{|y_j - x_j|}{a_j^2 t}e^{-(y_j-x_j)^2/2a_j^2 t}dy_j = \frac{c_2}{a_j\sqrt{t}}.$$

Now fix i and let

$$F(y_i; x, t, i) = \int \prod_{j\ne i} p_t^j(x_j, dy_j) f(y_1, y_2, \ldots).$$

Then

$$D_i P_t f(x) = \int D_i\Big(\prod_j p_t^j(x_j, dy_j)\Big) f(y) = \int\int q_t^i(x_i, dy_i) \prod_{j\neq i} p_t^j(x_j, dy_j) f(y)$$
$$= \int q_t^i(x_i, dy_i) F(y_i; x, t, i).$$

Since $p_t^j(x_j, dy_j)$ integrates to one for each j, we see that $\|F\|_\infty \le \|f\|_\infty$. Therefore

$$|D_i P_t f(x)| \le \|F\|_\infty \int |q_t^i(x_i, dy_i)| \le \frac{c_2}{a_i\sqrt{t}} \|f\|_\infty.$$

□

Remark 3.2. The conclusion of Lemma 3.1 is not the same as (2.4) because of the presence of the a_i.

Proposition 3.3. *There exists c_1 not depending on λ and $c_2 = c_2(\lambda)$ such that for all $f \in S^\alpha$,*

(*a*) $$\|D_i D_j R_\lambda f\|_\infty \le \frac{c_1}{a_i a_j} \lambda^{-\alpha/2} \|f\|_{S^\alpha},$$

(*b*) $$\|D_i D_j R_\lambda f\|_{S^\alpha} \le \frac{c_2(\lambda)}{a_i a_j} \|f\|_{S^\alpha},$$

(*c*) $$\|D_i R_\lambda f\|_\infty \le \frac{c_1}{a_i} \lambda^{-1/2} \|f\|_{S^\alpha},$$

and

(*d*) $$\|D_i R_\lambda f\|_{S^\alpha} \le \frac{c_2(\lambda)}{a_i} \|f\|_{S^\alpha}.$$

Proof. (a) By the translation invariance of Brownian motion, D_i and P_t commute. By the semigroup property we have

$$D_i D_j R_\lambda f(x) = \int_0^\infty e^{-\lambda s} D_i D_j P_s f(x)\, ds = \int_0^\infty e^{-\lambda s} D_i P_{s/2} D_j P_{s/2} f(x)\, ds.$$

(The interchange of the integration and differentiation follows easily by dominated convergence.) By Lemmas 3.1 and 2.2, $\|D_j P_{s/2} f\|_\infty \le c_3 a_j^{-1} s^{(\alpha-1)/2} \|f\|_{S^\alpha}$. Using Lemma 3.1 again

$$\|D_i D_j R_\lambda f\|_\infty \le \frac{c_4}{a_i} \int_0^\infty e^{-\lambda s} \frac{1}{a_j\sqrt{s}} s^{(\alpha-1)/2} ds\, \|f\|_{S^\alpha} \le \frac{c_5}{a_i a_j} \lambda^{-\alpha/2} \|f\|_{S^\alpha}. \tag{3.1}$$

(b) In view of Remark 2.1, we need only consider $t \leq 1$. We write

$$\begin{aligned} P_t(D_iD_jR_\lambda f) - (D_iD_jR_\lambda f) &= e^{\lambda t}\int_t^\infty e^{-\lambda s} D_iD_jP_sf\,ds - \int_0^\infty e^{-\lambda s} D_iD_jP_sf\,ds \qquad (3.2) \\ &= (e^{\lambda t}-1)\int_0^\infty e^{-\lambda s} D_iD_jP_sf\,ds - e^{\lambda t}\int_0^t e^{-\lambda s} D_iD_jP_sf\,ds. \end{aligned}$$

Since $t \leq 1$, then $|e^{\lambda t} - 1| \leq c_6(\lambda)t \leq c_6 t^{\alpha/2}$, and so the L^∞ norm of the first term on the last line is bounded by $c_6 t^{\alpha/2}\|D_iD_jR_\lambda f\|_\infty$. Applying (3.1), we bound the first term by $c_7(\lambda)(a_ia_j)^{-1}t^{\alpha/2}\|f\|_{S^\alpha}$.

Since $t \leq 1$, then $e^{\lambda t}$ is bounded. By Lemmas 3.1 and 2.2,

$$\|D_iD_jP_sf\|_\infty = \|D_iP_{s/2}D_jP_{s/2}f\|_\infty \leq \frac{c_8}{a_i}s^{-1/2}\|D_jP_{s/2}f\|_\infty \leq \frac{c_9}{a_ia_j}s^{-1/2}s^{(\alpha-1)/2}\|f\|_{S^\alpha}.$$

Integrating from 0 to t, the second term on the last line of (3.2) is bounded by

$$\frac{c_{10}}{a_ia_j}\|f\|_{S^\alpha}\int_0^t s^{\frac{\alpha}{2}-1}ds = \frac{c_{11}}{a_ia_j}t^{\alpha/2}\|f\|_{S^\alpha}.$$

(c) The first derivative estimates are similar but easier. Using Lemma 3.1,

$$\begin{aligned} \|D_iR_\lambda f\|_\infty &\leq \int_0^\infty e^{-\lambda s}\|D_iP_sf\|_\infty\,ds \qquad (3.3) \\ &\leq \frac{c_{12}}{a_i}\int_0^\infty e^{-\lambda s}s^{-1/2}ds\|f\|_\infty \leq \frac{c_{13}}{a_i}\lambda^{-1/2}\|f\|_\infty. \end{aligned}$$

(d) For $t \leq 1$, we write

$$P_t(D_iR_\lambda f) - (D_iR_\lambda f) = (e^{\lambda t}-1)D_iR_\lambda f + e^{\lambda t}\int_0^t e^{-\lambda s}D_iP_sf\,ds$$

as in (3.2). The first term on the right is bounded by $c_{14}(\lambda)a_i^{-1}t\|f\|_\infty$, which is fine since $t < 1$. Use Lemmas 2.2 and 3.1 to bound the second term on the right by

$$\frac{c_{15}}{a_i}\|f\|_{S^\alpha}\int_0^t s^{(\alpha-1)/2}ds \leq \frac{c_{16}}{a_i}t^{(\alpha+1)/2}\|f\|_{S^\alpha} \leq \frac{c_{17}}{a_i}t^{\alpha/2}\|f\|_{S^\alpha}.$$

$\square$

4. Relationship between norms – the Laplacian case.

Proposition 4.1. *If $f \in C^\alpha$ and $g \in S^\alpha$, then*

$$\|fg\|_{S^\alpha} \leq (|a|^\alpha + 1)\|f\|_{C^\alpha}\|g\|_{S^\alpha}.$$

In fact,

$$|fg|_{S^\alpha} \leq [\|f\|_\infty |g|_{S^\alpha} + |a|^\alpha |f|_{C^\alpha}\|g\|_\infty].$$

Proof. The L^∞ norm of fg is clearly bounded by the product of the L^∞ norms of f and g. Fix x. We need to obtain a bound on

$$|P_t(fg)(x) - (fg)(x)|.$$

Let $\widetilde{f}(y) = f(y) - f(x)$; clearly $\widetilde{f}(x) = 0$. Then

$$P_t(fg)(x) - fg(x) = P_t(\widetilde{f}g)(x) + f(x)P_t g(x) - f(x)g(x),$$

so

$$|P_t(fg)(x) - fg(x)| \leq |P_t(\widetilde{f}g)(x)| + |f(x)|\,|P_t g - g| \leq |P_t(\widetilde{f}g)(x)| + t^{\alpha/2}\|f\|_\infty |g|_{S^\alpha}.$$

The first term on the right hand side is

$$\begin{aligned}
|\mathbb{E}\,(\widetilde{f}g)(x + X_t)| &\leq \|g\|_\infty \mathbb{E}\,|f(x + X_t) - f(x)| \\
&\leq \|g\|_\infty |f|_{C^\alpha} \mathbb{E}\,(|X_t|^\alpha) \\
&\leq \|g\|_\infty |f|_{C^\alpha} (\mathbb{E}\,(|X_t|^2))^{\alpha/2} \\
&= \|g\|_\infty |f|_{C^\alpha} |a|^\alpha t^{\alpha/2},
\end{aligned}$$

where X_t is the Brownian motion associated with the semigroup P_t. The required bound follows. $\square$

Clearly the function that is identically one is in S^α, and hence the above proposition implies that $C^\alpha \subset S^\alpha$. Here is a partial converse, which also shows that these spaces coincide and have equivalent norms in the finite-dimensional case. Incidentally, this and Proposition 3.3 provide a new proof for (1.3) as well.

Note that because of the presence of the a_i in the conclusion of Lemma 3.1, we cannot conclude that S^α and C^α are equivalent in the infinite dimensional case. Let us set

$$|f|_{\alpha,i} = \sup_{x,h} \frac{|f(x + h\epsilon_i) - f(x)|}{|h|^\alpha}. \tag{4.1}$$

Proposition 4.2. *There exists $c_1(\alpha)$ such that for each i, $|f|_{\alpha,i} \le c_1 a_i^{-\alpha} \|f\|_{S^\alpha}$.*

Proof. By Lemmas 2.2 and 3.1

$$|P_t f(x + h\epsilon_i) - P_t f(x)| \le |h| \, \|D_i P_t f\|_\infty \le c_2 |h| a_i^{-1} t^{(\alpha-1)/2} \|f\|_{S^\alpha}.$$

We also have

$$|P_t f(x) - f(x)| \le t^{\alpha/2} \|f\|_{S^\alpha},$$

and the same with x replaced by $x + h\epsilon_i$. Using the triangle inequality,

$$|f(x + h\epsilon_i) - f(x)| \le (2t^{\alpha/2} + c_2 |h| a_i^{-1} t^{(\alpha-1)/2}) \|f\|_{S^\alpha}.$$

Taking $t = a_i^{-2} |h|^2$ yields our result. □

Remark 4.3. Consider the d-dimensional case with all the a_i's equal to 1. For each positive integer $J \le d^{\alpha/2}$ it is not hard to construct an example where $\|f\|_{C^\alpha} = 1$, $|f|_{\alpha,i} = 1$ for each i, yet $\|f\|_{S^\alpha} = J$. So there does not appear to be a simple characterization of S^α in terms of the $|f|_{\alpha,i}$. On the other hand, if we write

$$\|f\|_{S^\alpha} = \sup_t t^{-\alpha/2} \sup_x \left| \int_{\mathbb{R}^d} P(t, 0, y - x)[f(y) - f(x)] dy \right|$$

where $P(t, x, y)$ is the transition density for P_t in $\mathbb{R}^d$, we see that S^α does have a geometric characterization in terms of a weighted average of $f(y) - f(x)$.

5. Hölder estimates – the generalized Ornstein-Uhlenbeck case.

In this section we obtain Hölder norm estimates for perturbations of an appropriate Ornstein-Uhlenbeck operator. Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $V : \mathcal{D}(V) \to H$ be a (densely defined) self-adjoint non-negative definite operator on H such that

$$V^{-1} \text{ is a trace class operator on } H, \tag{5.1}$$

Then there is a complete orthonormal system $\{\epsilon_n : n \in \mathbb{N}\}$ of eigenvectors of V^{-1} with corresponding eigenvalues λ_n^{-1}, $\lambda_n > 0$, satisfying

$$\sum_{n=1}^\infty \lambda_n^{-1} < \infty, \quad \lambda_n \uparrow \infty, \quad V\epsilon_n = \lambda_n \epsilon_n$$

(see, e.g. Section 120 in [RN]). Let $Q_t = e^{-tV}$ be the semigroup of contraction operators on H with generator $-V$. If $w \in H$, let $w_n = \langle w, \epsilon_n \rangle$ and, as discussed in Section 2, we will write

$D_i f$ and $D_{ij} f$ for $D_{\epsilon_i} f$ and $D_{\epsilon_i} D_{\epsilon_j} f$, respectively. (In the example from the theory of SPDEs that motivated us, V is given by $V\epsilon_i = c_1 i^2 \epsilon_i$, and clearly V^{-1} is of trace class.)

Assume $a : H \to H$ is a bounded self-adjoint positive definite operator on H and set $a_{ij} = \langle a\epsilon_i, \epsilon_j \rangle$. Therefore for some $\gamma > 0$,

$$\gamma^{-1}|z|^2 \geq \sum_{i,j} a_{ij} z_i z_j \geq \gamma |z|^2, \qquad z \in H. \tag{5.2}$$

We consider the H-valued process which, with respect to the coordinates $\langle x, \epsilon_i \rangle$, is associated with the generator

$$\mathcal{L} f(x) = \tfrac{1}{2} \sum_{i,j=1}^{\infty} a_{ij} D_{ij} f(x) - \sum_{i=1}^{\infty} \lambda_i x_i D_i f(x). \tag{5.3}$$

The definition is as follows.

Let $(W_t, t \geq 0)$ be the cylindrical Brownian motion on H with covariance a. Recall (see section 3.2 of [KX]) this means if σ is the positive definite square root of a, then W_t is an $\mathbb{R}^\infty$-valued process such that for some sequence of independent 1-dimensional Brownian motions $\{B_j\}$,

$$W^i(t) \equiv W_t(\epsilon_i) = \sum_j \sigma_{ij} B_j(t),$$

and so more generally,

$$W_t(h) = \sum_i \langle h, \epsilon_i \rangle W_t(\epsilon_i), \qquad h \in H, t \geq 0$$

is a mean zero Gaussian process with covariance

$$\mathbb{E}\left(W_s(h) W_t(h') \right) = \langle h, ah' \rangle (s \wedge t).$$

As usual we may extend the definition of $(W_t(h), t \leq T)$ to measurable paths $h : [0,T] \to H$ such that $\int_0^T \|h_s\|^2 ds < \infty$. Then $(W_t(h), t \leq T, h \in H)$ is again a mean zero Gaussian process with covariance

$$\mathbb{E}\left(W_t(h) W_s(g) \right) = \int_0^{s \wedge t} \langle h_r, a g_r \rangle dr.$$

We often will write $\int_0^t h_s dW_s$ for $W_t(h)$. $\mathcal{F}_t$ denotes the right-continuous filtration generated by W.

Consider the stochastic differential equation

$$dX_t = -V X_t dt + dW_t.$$

A continuous H-valued $\mathcal{F}_t$-adapted process is a solution of this stochastic differential equations if and only if for all $h \in \mathcal{D}(V)$ we have

$$\langle X_t, h \rangle = \langle X_0, h \rangle - \int_0^t \langle X_s, Vh \rangle ds + W_t(h) \qquad t \geq 0, \qquad \text{a.s.} \tag{5.4}$$

One easily checks that such a solution is a continuous H-valued $\mathcal{F}_t$-adapted process which solves the mild form of (5.4) with initial condition $X_0 \in H$, that is

$$\langle X_t, h\rangle = \langle X_0, Q_t h\rangle + \int_0^t Q_{t-s} h dW_s \qquad \text{a.s. for all } t \geq 0 \text{ and } h \in H. \tag{5.5}$$

There is a pathwise unique solution of (5.5) (which also solves (5.4)) whose laws $\{\mathbb{P}^x, x \in H\}$ define a unique homogeneous strong Markov process on the space of continuous H-valued paths (see, e.g. Section 5.2 of [KX]). We let $P_t f(x) = \mathbb{E}^x(f(X_t))$ denote the associated semigroup. Clearly $\{X_t, t \geq 0\}$ is an H-valued Gaussian process satisfying

$$\mathbb{E}(\langle X_t, h\rangle) = \langle X_0, Q_t h\rangle \text{ for all } h \in H, \tag{5.6}$$

and

$$\mathrm{Cov}(\langle X_t, g\rangle\langle X_t, h\rangle) = \int_0^t \langle Q_{t-s}h, aQ_{t-s}g\rangle ds \equiv C_t(g,h). \tag{5.7}$$

Our reason for introducing (5.4) is that it shows that X will solve a martingale problem associated with $\mathcal{L}$. More precisely if $f : H \to \mathbb{R}$ is a bounded C^2 function of $(x_1, \dots, x_n)$ with bounded first and second partials, then $f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$ is an $\mathcal{F}_t$-martingale. Our objective in this section is to obtain bounds on $D_i D_j R_\lambda$ in the S^α norm associated with P_t, where R_λ is the λ-resolvent corresponding with P_t. We start by noting that P_t no longer commutes with the differential operators D_w.

Proposition 5.1. *Assume $t \geq 0$, $w \in H$, and $f : H \to \mathbb{R}$ is a bounded measurable function such that $D_{Q_t w} f$ is bounded and continuous (on H). Then*

$$D_w P_t f(x) = P_t(D_{Q_t w} f)(x), \qquad x \in H.$$

Proof. Let $Z_t \in H$ denote a mean zero Gaussian random vector with covariance C_t. Then $\mathbb{P}^x(X_t \in \cdot) = \mathbb{P}(Q_t x + Z_t \in \cdot)$. Therefore if $r \in \mathbb{R}$,

$$\frac{P_t f(x + rw) - P_t f(x)}{r} = \mathbb{E}\Big(\frac{f(Z_t + Q_t(x + rw)) - f(Z_t + Q_t x)}{r}\Big). \tag{5.8}$$

Use the mean value theorem to see that for some r' between 0 and r the integrand on the right side of (5.8) equals $D_{Q_t w} f(Z_t + Q_t x + r' Q_t w)$, which approaches $D_{Q_t w} f(Z_t + Q_t x)$ as r approaches 0 by the assumed continuity of $D_{Q_t w} f$. The result now follows by dominated convergence. □

The next step is the analogue of Lemma 3.1, which will require considerably more work in the present Ornstein-Uhlenbeck setting. Recall that $C_b(H)$ is the space of bounded continuous real-valued functions on H.

We introduce the following notation. Let

$$h(t) = \begin{cases} 2t/(e^{2t}-1) & \text{if } t > 0 \\ 1 & \text{if } t = 0. \end{cases}$$

For $t \geq 0$ and $w \in H$ set $|w|_t = (\sum_i w_i^2 h(\lambda_i t))^{1/2}$. Clearly $h(t)$ and $|w|_t$ are decreasing functions of t and $|w|_0 = |w|$.

The next result is closely related to (6.2.10) and (6.4.14) of [DZ].

Proposition 5.2. *If $f : H \to \mathbb{R}$ is bounded and measurable and $w \in H$, then for all $t > 0$, $P_t f$ is Lipschitz continuous on H, $D_w P_t f \in C_b(H)$ and*

$$\|D_w P_t f\|_\infty \leq \frac{|w|_t \, \|f\|_\infty}{\sqrt{\gamma t}}.$$

Proof. First consider $f \in C_b(H)$. Let π_n be the projection operator of H onto $\mathbb{R}^n$ given by $\pi_n y = (\langle y, \epsilon_i \rangle)_{i \leq n}$. Then under $\mathbb{P}^x$, $\pi_n X_t$ is an n-dimensional Gaussian variable with mean $\pi_n Q_t x$ and covariance matrix

$$C_t^n(i,j) = \int_0^t \langle Q_{t-s}\epsilon_i, a Q_{t-s}\epsilon_j \rangle ds = \int_0^t e^{-(\lambda_i + \lambda_j)s} ds \, a_{ij}, \qquad i, j \leq n.$$

Here of course $a_{ij} = \langle \epsilon_i, a\epsilon_j \rangle$. If $x \in \mathbb{R}^n$, then for some $\varepsilon_{n,t} > 0$,

$$\langle x, C_t^n x \rangle = \int_0^t \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j e^{-\lambda_i s} e^{-\lambda_j s} ds \geq \int_0^t \gamma \sum_{i=1}^n x_i^2 e^{-2\lambda_i s} ds \geq \varepsilon_{n,t} |x|^2.$$

This shows C_t^n is non-degenerate and so $\pi_n X_t$ has a Gaussian density

$$p_t^n(z) = (2\pi)^{-n/2} (\det C_t^n)^{-1/2} \exp(-\langle z - \pi_n Q_t x, (2C_t^n)^{-1}(z - \pi_n Q_t x) \rangle).$$

Let $f_n(y) = f(\sum_1^n \langle y, \epsilon_i \rangle \epsilon_i) \equiv \tilde{f}_n(\pi_n y)$. Then

$$\frac{P_t f_n(x + rw) - P_t f_n(x)}{r} = \int \tilde{f}_n(y) \Big[\frac{p_t^n(y - r\pi_n Q_t w) - p_t^n(y)}{r} \Big] dy. \tag{5.9}$$

By the mean value theorem, there is an $r' = r'(y)$ between 0 and r such that the expression in square brackets is

$$\begin{aligned} &-D_{\pi_n Q_t w} p_t^n (y - r'\pi_n Q_t w) \\ &\qquad = p_t^n(y - r'\pi_n Q_t w) \langle (C_t^n)^{-1} \pi_n Q_t w, y - \pi_n Q_t x - r'\pi_n Q_t w) \rangle, \end{aligned} \tag{5.10}$$

by an easy calculation. As $r \to 0$ the above converges to

$$p_t^n(y) \langle (C_t^n)^{-1} \pi_n Q_t w, y - \pi_n Q_t x) \rangle.$$

It is easy to see that the integral of the right side of (5.10) over $|y| > K$ is small uniformly in $|r| < 1$ for K large due to the Gaussian tail of p_t^n. It is therefore easy to use dominated convergence to take the limit as $r \to 0$ through the integral in (5.9) and conclude that

$$\begin{aligned} D_w P_t f_n(x) &= \int \tilde{f}_n(y) p_t^n(y) \langle (C_t^n)^{-1} \pi_n Q_t w, y - \pi_n Q_t x) \rangle dy \\ &= \mathbb{E}^x (f_n(X_t) \langle (C_t^n)^{-1} \pi_n Q_t w, \pi_n(X_t - Q_t x) \rangle). \end{aligned}$$

Introduce $U_n = (C_t^n)^{-1/2} \pi_n Q_t w$, $Z_n = (C_t^n)^{-1/2} \pi_n (X_t - Q_t x)$ and $R_n = \langle U_n, Z_n \rangle$. The above may now be rewritten as

$$D_w P_t f_n(x) = \mathbb{E}^x (f_n(X_t) R_n). \tag{5.11}$$

We need the following lemma whose proof is provided at the end of the current argument.

Lemma 5.3.

$$|U_n| \leq \frac{|w|_t}{\sqrt{\gamma t}}. \tag{5.12}$$

The coordinates of Z_n are i.i.d. standard normal random variables and so Lemma 5.3 implies that

$$\mathbb{E}^x (R_n^2) = |U_n|^2 \leq \frac{|w|_t^2}{\gamma t}. \tag{5.13}$$

If $Y_t = X_t - Q_t x$, then the joint laws of $(Y_t, Z_n), n \in \mathbb{N}$, are independent of x (recall $Z_n = (C_t^n)^{-1/2} \pi_n Y_t$) and the same is therefore true of the joint laws of (Y_t, R_n) on $H \times \mathbb{R}$. This sequence of laws is tight by (5.13) and so we may choose a subsequence $\{n_k\}$ (independent of x and f) such that $(Y_t, R_{n_k}) \Rightarrow (Y_t^\infty, R)$ with respect to weak convergence in $H \times \mathbb{R}$. As Y_t^∞ clearly is equal in law to Y_t we will drop the superscript. Using (5.11), we have

$$D_w P_t f_{n_k}(x) = \mathbb{E}^x (f(Q_t x + Y_t) R_{n_k}) + \mathbb{E}^x ((f_{n_k}(X_t) - f(X_t)) R_{n_k}) \qquad (5.14).$$

The second term is bounded in absolute value by $\mathbb{E}^x ((f_{n_k}(X_t) - f(X_t))^2)^{1/2} \mathbb{E}^x (R_{n_k}^2)^{1/2}$ which approaches 0 as $k \to \infty$ by (5.13), the continuity of f and dominated convergence. The above weak convergence along with the continuity of f and (5.13) show that as $k \to \infty$ the first term in (5.14) converges to $\mathbb{E} (f(Q_t x + Y_t) R)$, and Fatou's lemma and (5.13) show that

$$\mathbb{E} (R^2) \leq \frac{|w|_t^2}{\gamma t}. \tag{5.15}$$

We have proved that

$$\lim_{k \to \infty} D_w P_t f_{n_k}(x) = \mathbb{E} (f(Q_t x + Y_t) R) \equiv J(x).$$

Clearly J is continuous on H by the continuity of f, (5.15) and dominated convergence. Dominated convergence also shows that $P_t f_{n_k}(x) \to P_t f(x)$ as $k \to \infty$. An elementary argument using the fundamental theorem of calculus now shows that

$$D_w P_t f(x) \text{ exists and equals } J(x).$$

In particular, $D_w P_t f$ is continuous. The required bound on the sup norm of $D_w P_t f$ is now immediate from (5.15) and Cauchy-Schwarz.

Consider now the case when f is only bounded and measurable. We have shown above that for a fixed $w \in H$ and all $g \in C_b(H)$,

$$P_t g(x+w) - P_t g(x) = \int_0^1 \mathbb{E}\left(g(Q_t(x+sw) + Y_t)R\right) ds, \qquad x \in H. \tag{5.16}$$

Let $\mathcal{S}$ be the set of all bounded measurable (real-valued) maps on H for which (5.16) is valid. $\mathcal{S}$ is clearly a vector space containing $C_b(H)$ and is closed under bounded pointwise limits. A standard result (e.g., p. 11 of [M]) now shows that $\mathcal{S}$ contains all bounded measurable functions. This, together with (5.15), proves that for f as above,

$$|P_t f(x+w) - P_t f(x)| \leq \frac{\|f\|_\infty |w|_t}{\sqrt{\gamma t}}$$

and in particular $P_t f$ is Lipschitz continuous on H.

Finally if $0 < \varepsilon < t$, we may apply the bound obtained in the continuous case to the continuous map $P_\varepsilon f$ and conclude that $D_w P_t f(x) = D_w P_{t-\varepsilon}(P_\varepsilon f)(x)$ exists, is continuous and is bounded in absolute value by

$$\frac{\|P_\varepsilon f\|_\infty |w|_{t-\varepsilon}}{\sqrt{\gamma(t-\varepsilon)}} \leq \frac{\|f\|_\infty |w|_{t-\varepsilon}}{\sqrt{\gamma(t-\varepsilon)}}.$$

Let $\varepsilon \downarrow 0$ to obtain the required bound. $\square$

Proof of Lemma 5.3. Note that $\pi_n Q_t w = (e^{-\lambda_i t} w_i)_{i \leq n}$ where $(w_1, \ldots, w_n) = \pi_n w$ and so by replacing w with $\sum_1^n w_i \epsilon_i$, we may assume $\langle w, \epsilon_i \rangle = 0$ for $i > n$. We may consider Q_t as an operator on $\mathbb{R}^n$ via $Q_t = \operatorname{diag}(e^{-\lambda_i t})_{i \leq n}$ and the required result then becomes

$$|(C_t^n)^{-1/2} Q_t w|^2 \leq \frac{|w|_t^2}{\gamma t}, \qquad w \in \mathbb{R}^n.$$

Define $D_t : \mathbb{R}^n \to \mathbb{R}^n$ by $D_t w = \left(\frac{e^{-\lambda_i t} w_i}{\sqrt{h(\lambda_i t)}}\right)_{i \leq n}$. Then we claim the above follows from

$$|(C_t^n)^{-1/2} D_t u|^2 \leq \frac{|u|^2}{\gamma t}, u \in \mathbb{R}^n. \tag{5.17}$$

To see this set $u_i = w_i\sqrt{h(\lambda_i t)}$ so that $D_t u = Q_t w$ and (5.17) would then imply the required inequality. If $B_t^n = D_t^{-1} C_t^n D_t^{-1}$ (all operators now are on $\mathbb{R}^n$) then

$$B_t^n(i,j) = \int_0^t \sqrt{h(\lambda_i t)} e^{\lambda_i (t-s)} a_{ij} \sqrt{h(\lambda_j t)} e^{\lambda_j (t-s)} ds.$$

If γ is as in (5.2) then one easily sees that

$$\langle z, B_t^n z\rangle \geq \int_0^t \gamma \sum_{i=1}^n z_i^2 h(\lambda_i t) e^{2\lambda_i (t-s)} ds = \gamma t |z|^2.$$

Therefore B_t^n is symmetric positive definite matrix with all eigenvalues no smaller that γt. If the eigenvectors of B_t^n are τ_i with corresponding eigenvalues μ_i, then

$$\begin{aligned}
\langle z, B_t^n z\rangle &= \sum \mu_i \langle z, \tau_i\rangle^2 \\
&\leq \sum \frac{\mu_i^2}{\gamma t} \langle z, \tau_i\rangle^2 \\
&= (\gamma t)^{-1} \langle z, (B_t^n)^2 z\rangle = (\gamma t)^{-1} |B_t^n z|^2.
\end{aligned}$$

Therefore if $z = (C_t^n)^{-1} D_t u$, then

$$\begin{aligned}
|(C_t^n)^{-1/2} D_t u|^2 &= \langle z, C_t^n z\rangle \\
&= \langle z, D_t B_t^n D_t z\rangle \\
&= \langle D_t z, B_t^n D_t z\rangle \\
&\leq (\gamma t)^{-1} |B_t^n D_t z|^2 \qquad \text{(by the above with } D_t z \text{ in place of } z) \\
&= (\gamma t)^{-1} |u|^2.
\end{aligned}$$

Thus (5.17) holds and the proof is complete. □

Now that we have Proposition 5.2, we obtain the Hölder norm estimates by making suitable modifications to what we did in Section 3. The main difference is the lack of commutativity between P_t and D_w.

Proposition 5.4. *Let $f : H \to \mathbb{R}$ be in S^α and let $u, w \in H$. Then $D_w P_t f$ and $D_u D_w P_t f$ are in $C_b(H)$ and for some constant $c_1(\alpha, \gamma)$, satisfy*

$$\|D_w P_t f\|_\infty \leq c_1 |w|_t t^{\frac{\alpha-1}{2}} \|f\|_{S^\alpha} \tag{5.18}$$

and

$$\|D_u D_w P_t f\|_\infty \leq c_1 |u|_{t/2} |Q_{t/2} w|_{t/2} t^{\alpha/2-1} \|f\|_{S^\alpha}. \tag{5.19}$$

Moreover

$$f \in C^\alpha \text{ and } \|f\|_{C^\alpha} \le c_1 \|f\|_{S^\alpha}. \tag{5.20}$$

Proof. Using Proposition 5.2 we have by Lemma 2.2 (with $c_1 = |w|_t/|w|$ in that result) that

$$\|D_w P_t f\|_\infty \le c_2 |w|_t t^{(\alpha-1)/2} \|f\|_{S^\alpha}. \tag{5.21}$$

The continuity of $D_w P_t f$ is given by Proposition 5.2.

Use (5.21) with Propositions 5.1 and 5.2 to conclude that for $t > 0$ and $u, w \in H$,

$$D_u D_w P_t f = D_u P_{t/2} D_{Q_{t/2} w} P_{t/2} f \text{ exists, is continuous,} \tag{5.22}$$

and satisfies

$$\begin{aligned}\|D_u D_w P_t f\|_\infty &\le (\gamma t/2)^{-1/2} |u|_{t/2} \|D_{Q_{t/2} w} P_{t/2} f\|_\infty \\ &\le c_3 t^{-1/2} |u|_{t/2} |Q_{t/2} w|_{t/2} (t/2)^{(\alpha-1)/2} \|f\|_{S^\alpha}\end{aligned}$$

which gives (5.19).

The last result follows from Proposition 5.2 and Lemma 2.3.

□

Lemma 5.5. *If $r > 0, \beta < 1$, there is a $c_1(\beta, r)$ such that for any $\lambda > 0$,*

$$\int_0^\infty e^{-\lambda t} |w|_{t/r}^2 t^{-\beta}\, dt \le c_1 \sum_{i=1}^\infty (\lambda + \lambda_i)^{\beta-1} w_i^2.$$

Proof. If $I_i = \int_0^\infty e^{-\lambda t} h(\lambda_i t/r) t^{-\beta} dt$, Fubini's theorem shows that

$$\int_0^\infty e^{-\lambda t} |w|_{t/r}^2 t^{-\beta} dt = \sum_{i=1}^\infty w_i^2 I_i. \tag{5.23}$$

Note that if $\lambda_i > 0$, then

$$\begin{aligned}I_i &\le \int_0^\infty \frac{2\lambda_i t/r}{e^{2\lambda_i t/r} - 1} (2\lambda_i t/r)^{-\beta} 2\lambda_i/r\, dt (2\lambda_i/r)^{\beta-1} \\ &\le c_2(r) \lambda_i^{\beta-1} \int_0^\infty \frac{v^{1-\beta}}{e^v - 1} dv = c_3(r) \lambda_i^{\beta-1}.\end{aligned}$$

Moreover for all λ we have

$$I_i \le \int_0^\infty e^{-\lambda t} (\lambda t)^{-\beta} \lambda^{\beta-1} \lambda dt \le c_4 \lambda^{\beta-1}.$$

Therefore $I_i \le c_5(r)(\lambda + \lambda_i)^{\beta-1}$, and if this is used in (5.23), the desired result follows. □

If $w \in H$, set $\|w\|_{H,1} = \sum_{i=1}^\infty |w_i|$.

Theorem 5.6. *There exists a constant $c_1(\alpha,\gamma)$ and for $\varepsilon \in (0,\alpha/2)$ there exist constants $c_2(\alpha,\gamma,\varepsilon)$ such that for $\lambda > 0$, for any $f : H \to \mathbb{R}$ in S^α, and any $u, w \in H$, the functions $D_w R_\lambda f$ and $D_u D_w R_\lambda f$ are bounded and continuous on H and satisfy:*

$$\|D_w R_\lambda f\|_\infty \le c_2 \lambda^{-(1+\alpha)/4} \Big(\sum_{i=1}^\infty w_i^2 (\lambda+\lambda_i)^{\varepsilon-\alpha-1}\Big)^{1/2} \|f\|_{S^\alpha}, \tag{5.24}$$

$$\|D_u D_w R_\lambda f\|_\infty \le c_2 \Big(\sum_{i=1}^\infty w_i^2 (\lambda+\lambda_i)^{-\varepsilon}\Big)^{1/2} \Big(\sum_{i=1}^\infty u_i^2 (\lambda+\lambda_i)^{\varepsilon-\alpha}\Big)^{1/2} \|f\|_{S^\alpha}, \tag{5.25}$$

$$|D_w R_\lambda f|_{S^\alpha} \le c_1 \Big(\sum_{i=1}^\infty |w_i| (\lambda+\lambda_i)^{-1/2}\Big) \|f\|_{S^\alpha}, \tag{5.26}$$

$$|D_u D_w R_\lambda f|_{S^\alpha} \le c_1 (|u|\, \|w\|_{H,1} + |w|\, \|u\|_{H,1}) \|f\|_{S^\alpha}. \tag{5.27}$$

Proof. A use of Proposition 5.4 allows us to differentiate through the time integral and see that $D_w R_\lambda f(x) = \int_0^\infty e^{-\lambda s} D_w P_s f(x) ds$ and $D_u D_w R_\lambda f(x) = \int_0^\infty e^{-\lambda s} D_u D_w P_s f(x) ds$ are both continuous on H. Moreover by (5.19),

$$\begin{aligned}
&\|D_u D_w R_\lambda f\|_\infty \\
&\quad \le c_4 \|f\|_{S^\alpha} \int_0^\infty |u|_{s/2} |Q_{s/2} w|_{s/2} s^{\alpha/2-1} e^{-\lambda s} ds \qquad (5.28)\\
&\quad \le c_4 \|f\|_{S^\alpha} \Big(\int_0^\infty |u|_{s/2}^2 s^{\alpha-\varepsilon-1} e^{-\lambda s} ds\Big)^{1/2} \Big(\int_0^\infty |Q_{s/2} w|_{s/2}^2 s^{\varepsilon-1} e^{-\lambda s} ds\Big)^{1/2}.
\end{aligned}$$

Use Lemma 5.5 and the trivial bound $|Q_{s/2} w|_{s/2} \le |w|_{s/2}$ to conclude from the above that

$$\|D_u D_w R_\lambda f\|_\infty \le c_5 \|f\|_{S^\alpha} \Big(\sum_{i=1}^\infty (\lambda+\lambda_i)^{\varepsilon-\alpha} u_i^2\Big)^{1/2} \Big(\sum_{i=1}^\infty (\lambda+\lambda_i)^{-\varepsilon} w_i^2\Big)^{1/2}.$$

This gives (5.25) and the derivation of (5.24) is similar.

Now consider (5.26). As in Remark 2.1 we may assume that $0 < t \le 1$. Use (5.18) to see that

$$\begin{aligned}
&\|D_w P_t R_\lambda f - D_w R_\lambda f\|_\infty \\
&\le (e^{\lambda t} - 1) \Big\|\int_t^\infty e^{-\lambda s} D_w P_s f ds\Big\|_\infty + \Big\|\int_0^t e^{-\lambda s} D_w P_s f ds\Big\|_\infty \\
&\le c_6(\alpha,\gamma) \|f\|_{S^\alpha} \Big[(e^{\lambda t} - 1) \int_t^\infty e^{-\lambda s} |w|_s s^{(\alpha-1)/2} ds + \int_0^t e^{-\lambda s} |w|_s s^{(\alpha-1)/2} ds\Big] \\
&= c_6 \|f\|_{S^\alpha} \Big[I_1 + I_2\Big]. \qquad (5.29)
\end{aligned}$$

First bound I_1 by

$$
\begin{aligned}
&(e^{\lambda t}-1)\int_t^\infty e^{-\lambda s}\sum_{i=1}^\infty |w_i|\sqrt{2\lambda_i s}(e^{2\lambda_i s}-1)^{-1/2}s^{(\alpha-1)/2}ds\\
&\quad\le (e^{\lambda t}-1)\sum_{\lambda_i>\lambda}|w_i|e^{-\lambda t}\int_t^\infty \sqrt{2\lambda_i s}(e^{2\lambda_i s}-1)^{-1/2}(\lambda_i s)^{(\alpha-1)/2}\lambda_i\,ds\,\lambda_i^{(-1-\alpha)/2}\\
&\qquad+(e^{\lambda t}-1)\sum_{\lambda_i\le\lambda}|w_i|\int_t^\infty e^{-\lambda s}(\lambda s)^{(\alpha-1)/2}\lambda\,ds\,\lambda^{(-1-\alpha)/2}.
\end{aligned}
\tag{5.30}
$$

A substitution shows the integral in the first term in (5.30) is bounded uniformly in i and so this first term is at most

$$c_7(\alpha)(1-e^{-\lambda t})\sum_{\lambda_i>\lambda}|w_i|\lambda_i^{(-1-\alpha)/2}. \tag{5.31}$$

The integral in the second term in (5.30) is at most $c_8(\alpha)e^{-\lambda t}$ and so the second term in (5.30) is at most

$$c_8(1-e^{-\lambda t})\sum_{\lambda_i\le\lambda}|w_i|\lambda^{(-1-\alpha)/2}. \tag{5.32}$$

Use (5.31) and (5.32) in (5.30) to conclude that

$$
\begin{aligned}
I_1&\le c_9(1-e^{-\lambda t})\sum_{i=1}^\infty|w_i|(\lambda+\lambda_i)^{(-1-\alpha)/2}\\
&\le c_9 t^{\alpha/2}\sum_{i=1}^\infty |w_i|(\lambda+\lambda_i)^{-1/2}.
\end{aligned}
\tag{5.33}
$$

Next bound I_2 by

$$
\begin{aligned}
&\int_0^t e^{-\lambda s}\sum_{i=1}^\infty |w_i|\sqrt{2\lambda_i s}(e^{2\lambda_i s}-1)^{-1/2}s^{(\alpha-1)/2}ds\\
&\quad\le\sum_{\lambda_i>\lambda}|w_i|\int_0^t\sqrt{2\lambda_i s}(e^{2\lambda_i s}-1)^{-1/2}(\lambda_i s)^{(\alpha-1)/2}\lambda_i\,ds\,\lambda_i^{(-\alpha-1)/2}\\
&\qquad+\sum_{\lambda_i\le\lambda}|w_i|\int_0^t e^{-\lambda s}(\lambda s)^{(\alpha-1)/2}\lambda\,ds\,\lambda^{(-1-\alpha)/2}\\
&\quad\le\sum_{\lambda_i>\lambda}|w_i|\int_0^{\lambda_i t}\sqrt{2}(e^{2u}-1)^{-1/2}u^{\alpha/2}du\,\lambda_i^{(-1-\alpha)/2}\\
&\qquad+\sum_{\lambda_i\le\lambda}|w_i|\int_0^{\lambda t}e^{-u}u^{(\alpha-1)/2}du\,\lambda^{(-1-\alpha)/2}
\end{aligned}
\tag{5.34}
$$

The integral in the first summation is at most

$$c_{10}\int_0^{\lambda_i t} u^{\alpha/2-1}du \le c_{10}(\lambda_i t)^{\alpha/2}$$

and the integral in the second summation in (5.34) is at most

$$\int_0^{\lambda t} e^{-u/2}u^{\alpha/2-1}du \le c_{10}(\lambda t)^{\alpha/2}.$$

Use these bounds in (5.34) to see that I_2 is also bounded by the right hand side of (5.33). Use this and (5.33) in (5.29) to conclude that

$$\|D_w P_t R_\lambda f - D_w R_\lambda f\|_\infty \le c_{11}\Big(\sum_{i=1}^{\infty}(\lambda+\lambda_i)^{-1/2}|w_i|\Big)t^{\alpha/2}\|f\|_{S^\alpha}. \tag{5.35}$$

Proposition 5.1 and (5.18) imply that

$$\begin{aligned}\|P_t D_w P_s f - D_w P_t P_s f\|_\infty &= \|P_t D_{w-Q_t w} P_s f\|_\infty \\ &\le c_{12}|w-Q_t w|_s s^{(\alpha-1)/2}\|f\|_{S^\alpha}. \end{aligned}\tag{5.36}$$

Note that

$$\begin{aligned}&\int_0^\infty e^{-\lambda s}|w-Q_t w|_s s^{(\alpha-1)/2}ds \\ &\qquad\le \sum_{i=1}^{\infty}|w_i|(1-e^{-\lambda_i t})\int_0^\infty e^{-\lambda s}(2\lambda_i s)^{1/2}(e^{2\lambda_i s}-1)^{-1/2}s^{(\alpha-1)/2}ds \\ &\qquad\le \sum_{\lambda_i>\lambda}|w_i|(1-e^{-\lambda_i t})\int_0^\infty (2\lambda_i s)^{1/2}(e^{2\lambda_i s}-1)^{-1/2}(\lambda_i s)^{(\alpha-1)/2}\lambda_i ds\lambda_i^{(-1-\alpha)/2} \\ &\qquad\quad+\sum_{\lambda_i\le\lambda}|w_i|(1-e^{-\lambda_i t})\int_0^\infty e^{-\lambda s}(\lambda s)^{(\alpha-1)/2}\lambda ds\lambda^{(-1-\alpha)/2} \\ &\qquad\le c_{13}\sum_{i=1}^{\infty}|w_i|(1-e^{-\lambda_i t})(\lambda+\lambda_i)^{(-1-\alpha)/2} \\ &\qquad\le c_{13}\sum_{i=1}^{\infty}|w_i|(\lambda_i+\lambda)^{-1/2}t^{\alpha/2}.\end{aligned}$$

Integrate (5.36) with respect $e^{-\lambda s}ds$, use the above bound, and combine the resulting inequality with (5.35) to derive (5.26).

Finally consider (5.27). Use (5.19) to see that for $0 < t \le 1$ and $u, w \in H$,

$$\begin{aligned}
&\|D_u D_w P_t R_\lambda f - D_u D_w R_\lambda f\|_\infty \\
&\quad \le (e^{\lambda t} - 1)\Big\|\int_t^\infty e^{-\lambda s} D_u D_w P_s f ds\Big\|_\infty + \Big\|\int_0^t e^{-\lambda s} D_u D_w P_s f ds\Big\|_\infty \\
&\quad \le c_{14}\|f\|_{S^\alpha}\Big[(e^{\lambda t} - 1)\int_t^\infty e^{-\lambda s}|u|_{s/2}|Q_{s/2}w|_{s/2}s^{(\alpha/2)-1}ds \\
&\qquad + \int_0^t e^{-\lambda s}|u|_{s/2}|Q_{s/2}w|_{s/2}s^{(\alpha/2)-1}ds\Big] \\
&\quad \le c_{14}|w|\,|u|\,\|f\|_{S^\alpha}\Big[(e^{\lambda t} - 1)\int_{\lambda t}^\infty e^{-u}u^{\alpha/2-1}du\lambda^{-\alpha/2} + \int_0^t s^{\alpha/2-1}ds\Big] \\
&\quad \le c_{15}|w|\,|u|\,\|f\|_{S^\alpha}[(1 - e^{-\lambda t})\lambda^{-\alpha/2} + t^{\alpha/2}] \\
&\quad \le c_{16}|w|\,|u|\,\|f\|_{S^\alpha}t^{\alpha/2}. \qquad (5.37)
\end{aligned}$$

Now

$$\begin{aligned}
P_t D_u D_w P_s f - D_u D_w P_t P_s f &= [P_t D_u D_w P_s f - D_u P_t D_w P_s f] \qquad (5.38)\\
&\quad + [D_u P_t D_w P_s f - D_u D_w P_t P_s f].
\end{aligned}$$

By Proposition 5.1 and (5.22) (the latter to verify the hypothesis of Proposition 5.1), the first term on the right is equal to $P_t D_{u-Q_t u} D_w P_s f$ and so by (5.19) has sup norm bounded by

$$c_{17}s^{\frac{\alpha}{2}-1}|u - Q_t u|_{s/2}|Q_{s/2}w|_{s/2}\|f\|_{S^\alpha} \le c_{18}s^{\frac{\alpha}{2}-1}|u - Q_t u|_{s/2}|w|_{s/2}\|f\|_{S^\alpha}.$$

Propositions 5.1 and 5.2 show that the second term on the right-hand side of (5.38) is $D_u P_t D_{w-Q_t w} P_s f$, which by (5.22) and Proposition 5.1 equals $P_t D_{Q_t u} D_{(I-Q_t)w} P_s f$. Use (5.19) to bound the sup norm of this expression by

$$c_{19}s^{\frac{\alpha}{2}-1}|Q_t u|_{s/2}|Q_{s/2}(w - Q_t w)|_{s/2}\|f\|_{S^\alpha} \le c_{19}s^{\frac{\alpha}{2}-1}|u|_{s/2}|w - Q_t w|_{s/2}\|f\|_{S^\alpha}.$$

These bounds and (5.38) give

$$\begin{aligned}
&\|P_t D_u D_w P_s f - D_u D_w P_t P_s f\|_\infty \qquad (5.39)\\
&\qquad \le c_{20}s^{\frac{\alpha}{2}-1}[|u|_{s/2}|w - Q_t w|_{s/2} + |w|_{s/2}|u - Q_t u|_{s/2}]\|f\|_{S^\alpha}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^\infty e^{-\lambda s}s^{(\alpha/2)-1}|u|_{s/2}|w - Q_t w|_{s/2}ds\|f\|_{S^\alpha} \\
&\quad \le |u|\,\|f\|_{S^\alpha}\int_0^\infty s^{(\alpha/2)-1}\Big[\textstyle\sum_i w_i^2(1 - e^{-\lambda_i t})^2\frac{\lambda_i s}{e^{\lambda_i s} - 1}\Big]^{1/2}ds \\
&\quad \le |u|\,\|f\|_{S^\alpha}\int_0^\infty s^{(\alpha/2)-1}\textstyle\sum_i|w_i|(1 - e^{-\lambda_i t})\frac{\sqrt{\lambda_i s}}{\sqrt{e^{\lambda_i s} - 1}}ds \\
&\quad \le |u|\,\|f\|_{S^\alpha}\textstyle\sum_i|w_i|\displaystyle\int_0^\infty \frac{(\lambda_i s)^{(\alpha-1)/2}\lambda_i^{-\alpha/2}}{\sqrt{e^{\lambda_i s} - 1}}\lambda_i ds(1 - e^{-\lambda_i t}).
\end{aligned}$$

Note that $1 - e^{-\lambda_i t} \le (\lambda_i t)^{\alpha/2}$ and so the above gives

$$\int_0^\infty e^{-\lambda s} s^{(\alpha/2)-1} |u|_{s/2} |w - Q_t w|_{s/2} ds \|f\|_{S^\alpha} \le c_{21} |u| \, \|w\|_{H,1} \|f\|_{S^\alpha} t^{\alpha/2}.$$

Integrate (5.39) with respect to $e^{-\lambda s} ds$, use the above bound, and combine the resulting bound with (5.37) to conclude

$$\|P_t D_u D_w R_\lambda f - D_u D_w R_\lambda f\|_\infty \le c_{22} [|u| \, |w| + |u| \, \|w\|_{H,1} + \|u\|_{H,1} |w| \,] \, \|f\|_{S^\alpha} t^{\alpha/2}$$

and (5.27) follows. □

Corollary 5.7. *There exists a constant $c_1(\alpha, \gamma)$ such that for all $\lambda > 0$, any bounded measurable $f : H \to \mathbb{R}$, and for all $i \le j \in \mathbb{N}$,*

$$\|D_i R_\lambda f\|_\infty \le c_1 (\lambda + \lambda_i)^{-(\alpha+1)/2} \|f\|_{S^\alpha}, \tag{5.40}$$

$$\|D_{ij} R_\lambda f\|_\infty \le c_1 (\lambda + \lambda_j)^{-\alpha/2} \|f\|_{S^\alpha}, \tag{5.41}$$

$$\|D_i R_\lambda f\|_{S^\alpha} \le c_1 (\lambda + \lambda_i)^{-1/2} \|f\|_{S^\alpha}, \tag{5.42}$$

$$\|D_{ij} R_\lambda f\|_{S^\alpha} \le c_1 \|f\|_{S^\alpha}. \tag{5.43}$$

Proof. The first two inequalities follow easily from the bounds in the proof of Theorem 5.6 prior to the use of Hölder's inequality. For example, to derive (5.41), use (5.28) with $u = \epsilon_i$ and $w = \epsilon_j$ to conclude

$$\begin{aligned} |D_{ij} R_\lambda f| &\le c_3 \|f\|_{S^\alpha} \int_0^\infty \sqrt{h(\lambda_j s/2)} s^{\alpha/2 - 1} e^{-\lambda s} ds \\ &\le c_3 \|f\|_{S^\alpha} \int_0^\infty \sqrt{h(u/2)} u^{\alpha/2-1} du \lambda_j^{-\alpha/2} \\ &\le c_4 \|f\|_{S^\alpha} \lambda_j^{-\alpha/2}. \end{aligned}$$

Use $h \le 1$ to also bound the first line of the above display by $c_5 \|f\|_{S^\alpha} \lambda^{-\alpha/2}$ and (5.41) follows. A similar argument gives (5.40). The last two inequalities are now immediate from (5.26), (5.27) and the first two inequalities. □

Remark 5.8 In Corollary 5.7 we showed that the operator $D_{ij} R_\lambda$ is a bounded operator on S^α with a norm independent of i and j. It is also known that $D_{ij} R_\lambda$ is a bounded operator with respect to the usual C^α norm, again with a norm independent of i and j; see [D], [L], [Z], or especially Section 6.4.1 of [DZ]. Neither of these results contains the other. The C^α norm emphasizes the local continuity, while the S^α norm also gives weight to the behavior of $f(x)$ when $|x|$ is large. Both results are of interest.

6. Relationship between norms – the generalized Ornstein-Uhlenbeck case.

We now prove the analogue of Proposition 4.1. Let $|f|_{\alpha,i}$ be defined as in (4.1) and set

$$|f|_{\alpha,i,w} = \sup_{x,h\neq 0} \frac{|f(x+h\epsilon_i) - f(x)|\,|x_i|^{\alpha/2}}{|h|^{\alpha/2}}. \tag{6.1}$$

Let

$$\|f\|_{E^\alpha} = \|f\|_\infty + \sum_i |f|_{\alpha,i} + \sum_i \lambda_i^{\alpha/2} |f|_{\alpha,i,w} \equiv \|f\|_\infty + |f|_{E^\alpha}, \tag{6.2}$$

and let E^α be the space of continuous functions with $\|f\|_{E^\alpha} < \infty$. In Proposition 6.3 below we introduce a norm $\|\cdot\|_{F^\alpha}$ which is equivalent to $\|\cdot\|_{S^\alpha}$ in finite dimensions. This norm could be used in place of $\|\cdot\|_{E^\alpha}$ in the statement of Proposition 6.1; we use $\|\cdot\|_{E^\alpha}$ in the next proposition because of its simpler form.

Proposition 6.1. *There exists $c_1(\alpha,\gamma)$ such that if $f \in E^\alpha$ and $g \in S^\alpha$, then*

$$\|fg\|_{S^\alpha} \le c_1 \|f\|_{E^\alpha} \|g\|_{S^\alpha}.$$

In fact,

$$\|fg\|_{S^\alpha} \le c_1[\|f\|_\infty |g|_{S^\alpha} + |f|_{E^\alpha} \|g\|_\infty].$$

In particular $E^\alpha \subset S^\alpha \subset C^\alpha$.

Proof. As in the proof of Proposition 4.1, it suffices to fix $x \in H$ and show that if $f(x) = 0$, then for some $c_2 = c_2(\alpha,\gamma)$

$$|P_t(fg)(x)| \le c_2 |f|_{E^\alpha} \|g\|_\infty t^{\alpha/2}. \tag{6.3}$$

For $y \in H$ let $z_i(y), z_i^*(y) \in H$ satisfy

$$\langle z_i(y), \epsilon_j \rangle = \langle y, \epsilon_j \rangle 1_{(j\le i)} + \langle x, \epsilon_j \rangle 1_{(j>i)}$$

and

$$\langle z_i^*(y), \epsilon_j \rangle = \langle y, \epsilon_j \rangle 1_{(j<i)} + \langle Q_t x, \epsilon_i \rangle 1_{(j=i)} + \langle x, \epsilon_j \rangle 1_{(j>i)}.$$

Let

$$f_i(y) = f(z_i(y)) - f(z_{i-1}(y)).$$

Note that $f_i(y)$ is equal to $f(z_{i-1}(y) + (y_i - x_i)\epsilon_i) - f(z_{i-1}(y))$. Therefore we see $\|f_i\|_\infty \le |f|_{\alpha,i} |y_i - x_i|^\alpha$. Our assumption $f(x) = 0$, together with dominated convergence and the continuity of f, implies $P_t(fg)(x) = \sum_{i=1}^\infty P_t(f_i g)(x)$. Then

$$|P_t(fg)(x)| \le \sum_i P_t|f_i g|(x) \le \sum_i \|g\|_\infty P_t|f_i|(x). \tag{6.4}$$

Let Z_t denote a mean zero Gaussian random vector in H with covariance C_t. Then

$$\begin{aligned}
P_t(|f_i|)(x) &= \mathbb{E}\left(|f(z_i(Q_tx+Z_t)) - f(z_{i-1}(Q_tx+Z_t))|\right) \\
&\le \mathbb{E}\left(|f(z_i(Q_tx+Z_t)) - f(z_i^*(Q_tx+Z_t))|\right) + \mathbb{E}\left(|f(z_i^*(Q_tx+Z_t)) - f(z_{i-1}(Q_tx+Z_t))|\right) \\
&\le |f|_{\alpha,i}\mathbb{E}\left(|\langle Z_t, \epsilon_i\rangle|^\alpha\right) + |f|_{\alpha,i,w}|\langle Q_tx - x, \epsilon_i\rangle|^{\alpha/2}|x_i|^{-\alpha/2}1_{(x_i\neq 0)}. \qquad (6.5)
\end{aligned}$$

Note that

$$\mathbb{E}\left(\langle Z_t, \epsilon_i\rangle^2\right) = a_{ii}(1-e^{-2\lambda_i t})(2\lambda_i)^{-1} \le \gamma^{-1}t. \tag{6.6}$$

Therefore the first term in (6.5) is at most

$$|f|_{\alpha,i}\mathbb{E}\left(\langle Z_t, e_i\rangle^2\right)^{\alpha/2} \le |f|_{\alpha,i}\gamma^{-\alpha/2}t^{\alpha/2}. \tag{6.7}$$

The second term in (6.5) is bounded by

$$|f|_{\alpha,i,w}(1-e^{-\lambda_i t})^{\alpha/2} \le |f|_{\alpha,i,w}\lambda_i^{\alpha/2}t^{\alpha/2}. \tag{6.8}$$

Put (6.7) and (6.8) into (6.5) and sum over i to conclude

$$\begin{aligned}
\sum_i P_t(|f_i|)(x) &\le \Big[\gamma^{-\alpha/2}\sum_i |f|_{\alpha,i} + \sum_i |f|_{\alpha,i,w}\lambda_i^{\alpha/2}\Big]t^{\alpha/2} \\
&\le c_2(\alpha,\gamma)|f|_{E^\alpha}t^{\alpha/2}.
\end{aligned}$$

Put this bound into (6.4) to derive (6.3) and hence complete the proof of the required inequalities.

Set $g=1$ and use (5.20) to prove the final inclusions. $\square$

Proposition 6.2. *Assume $\lambda_i \ge c_1 i^2$ for all i and some $c_1 > 0$. Then S^α is an algebra and (2.6) and (2.7) are valid.*

Proof. We verify the hypothesis of Lemma 2.4. If Z_t is as in the previous proof, by (6.6)

$$\begin{aligned}
\mathbb{E}^x(|X_t - \mathbb{E}^x(X_t)|^2) &= \sum_{i=1}^\infty \mathbb{E}\left(\langle Z_t, \epsilon_i\rangle^2\right) \\
&= \sum_{i=1}^\infty a_{ii}\frac{1-e^{-2\lambda_i t}}{2\lambda_i} \\
&\le c_2\sum_{i=1}^\infty (i^{-2}\wedge t).
\end{aligned}$$

An elementary calculation shows the above is at most $c_3\sqrt{t}$ and so the result follows now from Lemma 2.4. $\square$

Finally, we present a norm that is equivalent to S^α in the finite dimensional case. Define

$$|f|_{F^\alpha} = \sup_{t\neq 0,x} \frac{|f(Q_t x) - f(x)|}{t^{\alpha/2}}. \tag{6.9}$$

The letter F stands for "flow", as what we have here is a weighted Hölder seminorm along the flow $Q_t x$. Note Q_t is deterministic:

$$Q_t x = Q_t\Big(\sum_i x_i \epsilon_i\Big) = \sum_i e^{-\lambda_i t} x_i \epsilon_i.$$

Define

$$\|f\|_{F^\alpha} = \|f\|_{C^\alpha} + |f|_{F^\alpha}. \tag{6.10}$$

Let π_d denote the projection of H onto the subspace spanned by $\{\epsilon_1, \ldots, \epsilon_d\}$. In the next result we effectively reduce to the finite-dimensional case by considering functions which only depend on the first d coordinates.

Proposition 6.3. *There exist positive c_1 and c_2 depending on (γ, d) such that for any measurable $f : H \to \mathbb{R}$ satisfying $f = f \circ \pi_d$,*

$$c_1\|f\|_{S^\alpha} \le \|f\|_{F^\alpha} \le c_2\|f\|_{S^\alpha}.$$

Proof. Let Z_t be the Gaussian vector introduced in the previous proof. Then, using (6.6), we have

$$\begin{aligned} |P_t f(x) - f(x)| &\le |\mathbb{E}\left(f(Q_t x + Z_t) - f(Q_t x)\right)| + |f(Q_t x) - f(x)| \\ &\le |f|_{C^\alpha}\mathbb{E}\left(|\pi_d Z_t|^\alpha\right) + |f|_{F^\alpha} t^{\alpha/2} \\ &\le t^{\alpha/2}\Big[|f|_{C^\alpha}(d\gamma^{-1})^{\alpha/2} + |f|_{F^\alpha}\Big] \end{aligned} \tag{6.11}$$

and the left hand inequality is established.

Turning to the right hand inequality we have,

$$\begin{aligned} |f(Q_t x) - f(x)| &= \Big|(P_t f(x) - f(x)) - \left(\mathbb{E}\left(f(Q_t x + Z_t)\right) - f(Q_t x)\right)\Big| \\ &\le |f|_{S^\alpha} t^{\alpha/2} + |f|_{C^\alpha}\mathbb{E}\left(|\pi_d Z_t|^\alpha\right) \\ &\le t^{\alpha/2}\Big[|f|_{S^\alpha} + c_3\|f\|_{S^\alpha}(d\gamma^{-1})^{\alpha/2}\Big], \end{aligned}$$

where in the last line we have used (5.20) and (6.6) again. This together with a further application of (5.20) give the right hand inequality. □

The following gives a relationship between S^α and C^α.

Proposition 6.4. *We have*

$$|f|_{S^\alpha} \le c_1 \sum_k |f|_{\alpha,k} + |f|_{F^\alpha}.$$

Proof. As in (6.11),

$$|P_t f(x) - f(x)| \le |\mathbb{E}\, f(Q_t x + Z_t) - f(Q_t x)| + |f(Q_t x) - f(x)|.$$

The second term on the right is bounded by $|f|_{F^\alpha} t^{\alpha/2}$, so we need to bound $|\mathbb{E}\, f(y+Z_t) - f(y)|$, where we write y for $Q_t x$. Replacing $f(\cdot)$ by $f(\cdot) - f(y)$, without loss of generality we may assume $f(y) = 0$. Define random variables Y_i by

$$\langle Y_i(\omega), \epsilon_j \rangle = \langle y + Z_t(\omega), \epsilon_j \rangle 1_{(j \le i)} + \langle y, \epsilon_j \rangle 1_{(j > i)}.$$

Then

$$\begin{aligned} |\mathbb{E}\, f(y + Z_t)| &\le \sum_{i=1}^\infty \mathbb{E}\, |f(Y_i) - f(Y_{i-1})| \\ &\le \sum_{i=1}^\infty |f|_{\alpha,i} \mathbb{E}\, |\langle Z_t, \epsilon_i \rangle|^\alpha. \end{aligned}$$

Using the calculation in (6.7), this is turn is bounded by

$$\sum_i |f|_{\alpha,i} (\gamma^{-1} t)^{\alpha/2},$$

which gives the proposition. □

7. Relationship between norms: super-Markov chains.

In [BP] Hölder norm estimates were proved for the operator

$$\mathcal{L} f(x) = \sum_{i=1}^d [\gamma_i x_i D_{ii} f(x) + b_i D_i f(x)]$$

operating on functions on $\mathbb{R}_+^d$. Here $\gamma = (\gamma_1, \ldots, \gamma_d) \in (0,\infty)^d$ and $b = (b_1, \ldots, b_d) \in \mathbb{R}_+^d$. The estimates were with respect to the norm defined by

$$\|f\|_{C_w^\alpha} = \|f\|_\infty + \sum_{i=1}^d |f|_{w,\alpha,i},$$

where

$$|f|_{w,\alpha,i} = \sup_{h>0, x \in [0,\infty)^d} \frac{|f(x + h\epsilon_i) - f(x)|}{h^\alpha} x_i^{\alpha/2}.$$

Set $C_w^\alpha = \{f \in C_b(\mathbb{R}_+^d) : \|f\|_{C_w^\alpha} < \infty\}$. (Continuity of f at points in $\partial\mathbb{R}_+^d$ does not follow from $\|f\|_{C_w^\alpha} < \infty$ and hence must be assumed.) In [BP] this norm was essentially forced on us in order to get the estimates we needed. The Hölder norm estimates for this case are derived in [BP] and make up a considerable portion of that paper. So in this section we content ourselves with showing that the C_w^α norm is equivalent to the S^α norm for this operator.

Let P_t denote the semigroup associated with $\mathcal{L}$ and $\mathbb{E}^x$ denote expectation with respect to the associated Markov process $(X_t, t > 0)$ in $\mathbb{R}_+^d$, starting at $x \in \mathbb{R}_+^d$. More precisely under $\mathbb{P}^x$, X is the unique (in law) process such that $X_0 = x$ and

$$M^f(t) = f(X_t) - f(x) - \int_0^t \mathcal{L}f(X_s)ds$$

is a $\sigma(X_s, s \le t)$-martingale for all $f \in C_b^2(\mathbb{R}_+^d)$. If $d = 1$, let $\mathbb{P}^x(X_t \in dy) = p_t^{\gamma,b}(x, dy)$ and write $p_t^i(x_i, dy_i)$ for $p_t^{\gamma_i,b_i}(x_i, dy_i)$.

Remark 7.1. Functions in C_w^α are not necessarily continuous on the boundary of $\mathbb{R}_+^d$, and so we restrict statements below to functions in $S^\alpha \cap C_b$. However functions f for which $\|f\|_{C_w^\alpha} < \infty$ have an extension to a continuous function on $\mathbb{R}_+^d$ ([BP], Proposition 2.2). In view of Theorem 7.6 below, functions for which $\|f\|_{S^\alpha} < \infty$ also have such an extension.

Lemma 7.2. *Let f be a bounded Borel function on $\mathbb{R}_+^d$. If $t > 0$ then $D_iP_tf(x)$ is a continuous function in x_i satisfying*

$$|D_iP_tf(x)| \le c_1[(\gamma_i t x_i)^{-1/2} \wedge (\gamma_i t)^{-1}]\|f\|_\infty$$

for some constant c_1.

Proof. Let $\hat{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \in \mathbb{R}_+^{d-1}$ for $x \in \mathbb{R}_+^d$ and define

$$F^{\hat{x}_i}(y_i) = \int \prod_{j \ne i} p_t^j(x_j, dy_j) f(y).$$

Set $s = \gamma_i t$ for a fixed $t > 0$. Then use Lemmas 4.1(a) and 4.5(a) of [BP] (the continuity of f assumed there is not used) to see

$$\begin{aligned}
D_iP_tf(x) &= D_i \int F^{\hat{x}_i}(y_i) p_t^i(x_i, dy_i) \\
&= \sum_{k=1}^\infty e^{-x_i/s} \frac{(x_i/s)^k}{k!} \int_0^\infty F^{\hat{x}_i}(zs) e^{-z} \Big[\frac{z^{k+(b_i/\gamma_i)}}{\Gamma(k + (b_i/\gamma_i) + 1)} - \frac{z^{k+(b_i/\gamma_i)-1}}{\Gamma(k + (b_i/\gamma_i))}\Big] \frac{dz}{s} \qquad (7.1) \\
&\quad + e^{-x_i/s} \int_0^\infty F^{\hat{x}_i}(zs) e^{-z} \frac{z^{b_i/\gamma_i}}{\Gamma((b_i/\gamma_i) + 1)} \frac{dz}{s} \\
&\quad - 1_{(b_i>0)} e^{-x_i/s} \int_0^\infty F^{\hat{x}_i}(zs) e^{-z} \frac{z^{(b_i/\gamma_i)-1}}{\Gamma(b_i/\gamma_i)} \frac{dz}{s} - 1_{(b_i=0)} e^{-x_i/s} F^{\hat{x}_i}(0) \int_0^\infty e^{-z} \frac{dz}{s}.
\end{aligned}$$

If $a_k = a_k(\hat{x}_i)$ is the integral in the above summation over k, then

$$\begin{aligned}|a_k| \leq &\|F^{\hat{x}_i}\|_\infty \int_0^\infty e^{-z} \frac{z^{k+(b_i/\gamma_i)-1}}{\Gamma(k+(b_i/\gamma_i))} \frac{|z-(k+(b_i/\gamma_i))|}{k+(b_i/\gamma_i)} \frac{dz}{s} \\ \leq &c_2\|f\|_\infty \Big((k+(b_i/\gamma_i))^{1/2}+1\Big)(k+(b_i/\gamma_i))^{-1}s^{-1} \\ \leq &2c_2\|f\|_\infty (k+(b_i/\gamma_i))^{-1/2}s^{-1},\end{aligned}$$

where Lemma 3.2(a) of [BP] is used in the second inequality. It is now easy to see that the series in (7.1) converges uniformly for x_i in a compact set and so $D_iP_tf(x)$ is continuous in x_i. Moreover this bound and (7.1) also show that

$$\begin{aligned}|D_iP_tf(x)| \leq &\sum_{k=1}^\infty e^{-x_i/s}\frac{(x_i/s)^k}{k!} c_3\|f\|_\infty (k+(b_i/\gamma_i))^{-1/2}s^{-1} + 2e^{-x_i/s}\|f\|_\infty s^{-1} \\ \leq &c_4(1\wedge (x_i/s)^{-1/2})\|f\|_\infty s^{-1} + 2e^{-x_i/s}\|f\|_\infty s^{-1}\end{aligned}$$

by an elementary bound (see Lemma 3.3(a) of [BP]). Since $e^{-x_i/s} \leq 1\wedge (x_i/s)^{-1/2}$, the required result follows. □

Lemma 7.3. *If f is a bounded Borel function on $\mathbb{R}^d_+$, then*

$$|D_iP_tf(x)| \leq c_1(\alpha)\frac{t^{(\alpha-1)/2}}{\sqrt{\gamma_i x_i}}\|f\|_{S^\alpha},$$

where c_1 depends only on α.

Proof. This follows from the previous result, exactly as in the proof of Lemma 2.2. □

Proposition 7.4. *Let f be a bounded Borel function on $\mathbb{R}^d_+$. Then*

$$|f|_{w,\alpha,i} \leq c_1\gamma_i^{-\alpha/2}\|f\|_{S^\alpha}.$$

Proof. If $h>0$, then Lemma 7.2, the fundamental theorem of calculus and Lemma 7.3 show

$$\begin{aligned}|P_tf(x+h\epsilon_i) - P_tf(x)| = &\Big|\int_0^h D_iP_tf(x+h'\epsilon_i)dh'\Big| \\ \leq &c_2t^{(\alpha-1)/2}\gamma_i^{-1/2}\int_{x_i}^{x_i+h} y^{-1/2}dy\|f\|_{S^\alpha} \\ \leq &c_2t^{(\alpha-1)/2}(\gamma_i x_i)^{-1/2}h\|f\|_{S^\alpha}.\end{aligned}$$

We also have

$$|P_tf(x) - f(x)| \leq \|f\|_{S^\alpha}t^{\alpha/2}.$$

The above two inequalities imply

$$|f(x+h\epsilon_i)-f(x)| \le (2t^{\alpha/2}+c_2t^{(\alpha-1)/2}(\gamma_i x_i)^{-1/2}h)\|f\|_{S^\alpha}.$$

We optimize by setting $t=(c_2^2/4)h^2(x_i\gamma_i)^{-1}$, and so

$$|f(x+h\epsilon_i)-f(x)| \le c_3(\alpha)\gamma_i^{-\alpha/2}h^\alpha x_i^{-\alpha/2}\|f\|_{S^\alpha}.$$

□

Recall the definition of $|f|_{S^\alpha}$ from (2.1).

Proposition 7.5. *If $f\in C_b(\mathbb{R}_+^d)$, then $|f|_{S^\alpha}\le c_1(\alpha)\sum_{i=1}^d((b_i/\gamma_i)+1)\gamma_i^{\alpha/2}|f|_{w,\alpha,i}$.*

Proof. We may assume without loss of generality that $f\in C_w^\alpha$. Let $\varepsilon>0$. Results in [BP] (notably Proposition 7.2 and Lemma 7.6 there) imply $P_\varepsilon f\in C_b^2(\mathbb{R}_+^d)\cap\mathcal{D}(\mathcal{L})$ and so the fact that we are working with a solution to the martingale problem for X implies

$$\begin{aligned}|P_tf(x)-P_\varepsilon f(x)| &= \Big|\int_0^{t-\varepsilon}P_s\mathcal{L}(P_\varepsilon f)(x)ds\Big|\\ &= \Big|\int_0^{t-\varepsilon}\mathcal{L}P_{s+\varepsilon}f(x)ds\Big|\\ &\le \int_\varepsilon^t|\mathcal{L}P_sf(x)|ds.\end{aligned}$$

Use the upper bounds in Proposition 5.1 of [BP] to see that

$$\begin{aligned}|P_tf(x)-P_\varepsilon f(x)| &\le c_2\sum_{i=1}^d(b_i\gamma_i^{(\alpha/2)-1}+\gamma_i^{\alpha/2})|f|_{w,\alpha,i}\int_\varepsilon^t s^{\alpha/2-1}ds\\ &\le c_3\sum_{i=1}^d(1+b_i/\gamma_i)\gamma_i^{\alpha/2}|f|_{w,\alpha,i}t^{\alpha/2}.\end{aligned}$$

Now let $\varepsilon\downarrow 0$ to complete the proof. □

Theorem 7.6. *Assume $0<\varepsilon\le\gamma_i\le K$ and $b_i\le K$ for $i=1,\dots,d$, for some $\varepsilon\le 1\le K$. There are constants c_1 and $c_2(\alpha)$ such that for all $f\in C_b(\mathbb{R}_+^d)$,*

$$c_1\varepsilon^{\alpha/2}\max_{i\le d}|f|_{w,\alpha,i}\le|f|_{S^\alpha}\le c_2(K/\varepsilon)\sum_{i=1}^d|f|_{w,\alpha,i}$$

and therefore there are constants c_3 and c_4 such that

$$c_3d^{-1}\|f\|_{C_w^\alpha}\le\|f\|_{S^\alpha}\le c_4\|f\|_{C_w^\alpha}$$

for all $f \in C_b(\mathbb{R}_+^d)$.

Proof. This is immediate from Propositions 7.4 and 7.5. $\square$

Remark 7.7. Let D denote differentiation with respect to t, define

$$\|f\|_{G^\alpha} = \|f\|_\infty + \sup_{t>0} \|DP_t f\|_\infty t^{1-(\alpha/2)},$$

and introduce

$$G^\alpha = \{f \in C_b(\mathbb{R}_+^d) : DP_t f(x) \text{ exists and is continuous in } t > 0 \text{ for all } x, \|f\|_{G^\alpha} < \infty\}.$$

The proof of Proposition 7.5 can be easily modified to show $C_w^\alpha \subset G^\alpha$ and

$$\|f\|_{G^\alpha} \le c_1 \sum_{i=1}^d (1 + b_i/\gamma_i)\gamma_i^{\alpha/2} |f|_{w,\alpha,i} + \|f\|_\infty$$

for all $f \in C_w^\alpha$. A trivial integration shows $G^\alpha \subset \{f \in C_b(\mathbb{R}_+^d) : \|f\|_{S^\alpha} < \infty\}$ and $\|f\|_{S^\alpha} \le \frac{2}{\alpha}\|f\|_{G^\alpha}$. Combine these observations with Theorem 7.6 to conclude $C_w^\alpha = G^\alpha = S^\alpha \cap C_b$ and for ε, K as in Theorem 7.6 there are c_2 and c_3 such that

$$c_2 d^{-1} \|f\|_{C_w^\alpha} \le \|f\|_{S^\alpha} \le \frac{2}{\alpha}\|f\|_{G^\alpha} \le c_3 \|f\|_{C_w^\alpha}.$$

References.

[ABP] S. Athreya, R.F. Bass and E.A. Perkins, On uniqueness of infinite dimensional stochastic differential equations of Ornstein-Uhlenbeck type, in preparation.

[Ba] R.F. Bass, *Probabilistic techniques in analysis.* Springer-Verlag, New York, 1995.

[BP] R.F. Bass and E.A. Perkins, Degenerate stochastic differential equations with Hölder continuous coefficients and super-Markov chains. *Trans. Amer. Math. Soc.* **355**, (2003) 373–405.

[CD] P. Cannarsa and G. Da Prato, Infinite-dimensional elliptic equations with Hölder-continuous coefficients. *Adv. Differential Equations* **1** (1996) 425–452.

[D] G. Da Prato, Some results on elliptic and parabolic equations in Hilbert spaces. *Rend. Mat. Acc. Lincei* **7** (1996) 181–199.

[DZ] G. Da Prato and J. Zabczyk, *Second order partial differential equations in Hilbert spaces.* Cambridge University Press, Cambridge, 2002.

[GT] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order, Second edition.* Springer-Verlag, Berlin, 1983.

[KX] G. K. Kallianpur and J. Xiong, *Stochastic Differential Equations in Infinite Dimensional Spaces.* IMS Lecture Notes-Monograph Series, Vol. 26, IMS, Hayward, 1995.

[L] A. Lunardi, An interpolation method to characterize domains of generators of semigroups. *Semigroup Forum* **53** (1996) 321–329.

[M] P.A. Meyer, *Probability and Potentials*, Blaisdell, Waltham, Mass., 1966.

[RN] F. Riesz, B. Sz.-Nagy, *Functional Analysis*, Ungar, New York, 1955.

[S] E.M. Stein, *Singular integrals and differentiability properties of functions.* Princeton, Princeton Univ. Press, 1970.

[W] J.B. Walsh, An introduction to stochastic partial differential equations. *Ecole d'été de probabilités de Saint-Flour, XIV—1984*, 265–439. Springer-Verlag, Berlin, 1986.

[Z] L. Zambotti, An analytic approach to existence and uniqueness for martingale problems in infinite dimensions. *Probab. Theory Related Fields* **118** (2000) 147–168.

Addresses:

S.R. Athreya: Indian Statistical Institute, 7 S.J.S. Sansanwal Marg,
New Delhi 110016, India

R.F. Bass: Department of Mathematics, University of Connecticut,
Storrs, CT 06269, USA.

E.A. Perkins: Department of Mathematics, University of British Columbia,
Vancouver, B.C., Canada V6T 1Z2.