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December 30, 2003
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The structure of finite clusters in high intensity Poisson Boolean stick process

Rahul Roy
Hideki Tanemura

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

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Rahul Roy[1] and Hideki Tanemura
Indian Statistical Institute and *Chiba University*

Abstract Sticks at one of different orientation are placed in an i.i.d. fashion at points of a Poisson point process of intensity λ. Sticks of the same direction have the same length, while sticks in different directions may have different lengths. We study the geometry of finite cluster as $\lambda \to \infty$. The asymptotic shape of the custer being determined by the probabilities of the sticks in various direction and their lengths and orientations. We also obtain the limiting geometric structure of this component.

1 Introduction

Consider one dimensional sticks placed at random locations and with random orientations in the two dimensional plane. In the language of stochastic geometry we have a planar fibre process whose *grains* are two dimensional linear segments and whose germs are the random locations. The most commonly studied fibre process model which incorporates these features is when the germs arise as realisations of a Poisson point process of intensity λ on $\mathbb{R}^2$ and each germ is the centre of a stick of either fixed length or a random length and having a random orientation, with the distribution of the length and orientation of a stick being independent of the underlying Poisson process. This is the Poisson Boolean stick process, a particular instance of the more general planar Boolean fibre process. Hall [1990] (Chapter 4), Stoyan Kendall and Mecke [1995] (Chapter 9) discuss the geometric and statistical aspects of this process.

While the stochastic geometry study of these processes was motivated by its application in geology, viz., the subterranean earthquake faults are modelled as a Poisson Boolean stick process (see, e.g., Weber [1977]); the interest in the physics community of this model led to a probabilistic study of its percolative properties. Suppose mirrors are placed randomly on the plane and we are interested in the path of a ray of light in this set-up. Clearly the geometry of the mirrors on the plane determine the trajectory of the ray of light. This model is a modern equivalent of the Ehrenfest wind-tree model which was introduced by Ehrenfest [1957] to study the Lorentz lattice gas model (see Grimmett [1998] for an exposition of the mathematical study of this model). This model has also been studied for its percolative properties (in particular,

[1]RR is grateful to Chiba University for its warm hospitality and acknowledges the financial support of JSPS.
Keywords: Poisson process, Boolean model. **AMS Classification**: 60K35

the critical phenomenon it exhibits and the corresponding critical parameters) by Domany and Kinzel [1984], Hall [1985], Menshikov [1986], Roy [1991] and Harris [1997].

Here we study the geometric features of finite clusters in the Poisson Boolean stick process when the intensity of the underlying Poisson process is high. More particularly, consider a Poisson point process of intensity λ on $\mathbb{R}^2$ conditioned to have a point at the origin. At each point x_i we centre a stick of length r_i and orientation θ_i measured anticlockwise w.r.t. the x-axis. We suppose that

(i) $r_1, r_2, \ldots$ is an i.i.d. sequence of random variables,

(ii) $\theta_1, \theta_2, \ldots$ is an i.i.d. sequence of random variables, and

(iii) the sequences $\{r_i\}$ and $\{\theta_i\}$ and the underlying Poisson process are independent of each other.

Consider the cluster of the origin (which is the connected component formed by sticks containing the stick at the origin). For the above model Hall [1985] has shown that if the random variable r_1 is bounded, and the random variable θ_1 is non-degenerate then there is a critical intensity λ_c such that, for $\lambda > \lambda_c$, with positive probability the cluster defined above is unbounded. Moreover this probability goes to 1 as $\lambda \to \infty$. Given the rare event that this cluster contains exactly m sticks, we investigate its structure as the intensity $\lambda \to \infty$.

In the case of the Boolean model which consists of an underlying Poisson point process of intensity λ on $\mathbb{R}^d$ and each point of the process is the centre of a d-dimensional ball of radius r, Alexander [1991] showed that conditional on the cluster of the origin (i.e. the connected component of balls containing a ball which covers the origin) being finite and consisting of m balls, the event that these balls are centred in a small region of radius $O(\lambda^{-1})$ has a probability which tends to 1 as $\lambda \to \infty$. This region where the balls are centred has volume $O(\lambda^{-d})$ whereas the ambient density is λ, thereby giving rise to the phenomenon of compression wherein many more Poisson points are accomodated in this region than the ambient density allows. Sarkar [1998] showed that in case the balls forming the Boolean model are allowed to be of varying sizes, then given that the cluster of the origin contains m balls, not all of the same size, the phenomenon of rarefaction occurs, wherein the biggest sized balls remain compressed in a very small region, but the other balls are sparsely placed in the region covered by the biggest sized balls.

In our model the phenomenon of compression also occurs, however that is of secondary interest. Instead we look at the geometry and the distribution of the sticks of various orientation in the finite cluster.

In this paper we restrict ourselves to the study of the model when the sticks have exactly two or three possible orientations and sticks of the same orientation have the same length. In the case of two possible orientations the asymptotic distribution was shown to be independent of the angle and the length of the sticks – a result which is not surprising in view of the affine invariance of the model. However, if three or more orientations are allowed then the affine invariance breaks down and the asymptotic distribution do depend on the angles. In this case we show that the asymptotic shape consists of sticks with only two orientations. The orientations which "survive" are chosen according to the lengths and angles of the possible orientations and the probabilities of the sticks in various directions.

The paper is organised as follows:– in the next section we present a formal defintion of the process as well as the statements of our results and in Sections 3 and 4 we prove the results.

2 Preliminaries and statement of results

2.1 Notation

Let $\mathcal{R} = \mathbb{R}^2 \times [0, \pi) \times (0, \infty)$, and

$$\mathcal{M} = \mathcal{M}(\mathcal{R}) := \{\xi = \{\xi_i, i \in \mathbb{N}\} : \xi_i = (x_i, \theta_i, r_i) \in \mathcal{R}\}.$$

For $(x, \theta, r) \in \mathcal{R}$, $S(x, \theta, r) = \{x + ue_\theta, u \in [-r, r]\}$ is the stick with centre x, angle θ and length $2r$, where $e_\theta = (\cos\theta, \sin\theta)$. We define the collection of sticks for $\xi \in \mathcal{M}$ as $S(\xi) = \{S(x, \theta, r) : (x, \theta, r) \in \xi\}$.

We say two sticks S and S' are connected and write $S \overset{\xi}{\leftrightarrow} S'$ if there exists $S_1, S_2, \dots S_k \in S(\xi)$ such that $S \cap S_1 \neq \emptyset$, $S' \cap S_k \neq \emptyset$ and $S_i \cap S_{i+1} \neq \emptyset$ for every $i = 1, 2, \dots, k-1$. If $S(\xi)$ contains a stick $S_{\mathbf{0}}$ centred at the origin $\mathbf{0}$, we denote by $C_{\mathbf{0}}(\xi)$ the cluster of sticks containing $S_{\mathbf{0}}$, i.e.

$$C_{\mathbf{0}}(\xi) = \{y \in S : S \in S(\xi), S \overset{\xi}{\leftrightarrow} S_{\mathbf{0}}\}.$$

(We put $C_{\mathbf{0}}(\xi) = \emptyset$, if $S(\xi)$ does not contain any stick with centre $\mathbf{0}$).

Let ρ be the Radon measure on $\mathcal{R}$ defined by

$$\rho(dx d\theta dr) = dx \sum_{j=1}^{d} p_j \delta_{\alpha_j}(d\theta)\delta_{R_j}(dr), \tag{2.1}$$

where $\alpha_1 = 0 < \alpha_2 < \alpha_3 < \cdots < \alpha_d < \pi$, $p_j \geq 0, \sum_{j=1}^d p_j = 1$, $R_j > 0$, $j = 1, 2, \dots, d$ and δ_* denotes the usual Dirac delta measure. We denote by μ_ρ the Poisson point process on $\mathcal{M}(\mathcal{R})$ with intensity measure ρ. Let

$$\Gamma_0 := \{\xi \in \mathcal{M} : (\mathbf{0}, \alpha_j, R_j) \in \xi \text{ for some } j = 1, 2, \dots, d\}. \tag{2.2}$$

For $w_i = (x_i, \theta_i, r_i)$, $i = 1, 2, \dots, m$, let

$$\mathbf{w}_m := (w_1, w_2, \dots, w_m), \{\mathbf{w}_m\} := \{w_1, w_2, \dots, w_m\}, C_{\mathbf{0}}(\mathbf{w}_m) := C_{\mathbf{0}}(\{\mathbf{w}_m\}). \tag{2.3}$$

For $\mathbf{k} = (k_1, k_2, \dots, k_d) \in (\mathbb{N} \cup \{0\})^d$, we denote by $\Lambda(\mathbf{k})$ the set of clusters containing exactly $|\mathbf{k}| = \sum_j^d k_j$ sticks with k_j sticks at an orientation α_j, $j = 1, 2, \dots, d$.

For $\alpha, \beta > 0$, $R_\alpha, R_\beta > 0$, $e_\alpha = (\cos\alpha, \sin\alpha)$, and $\mathbf{x}_m = (x_1, x_2, \cdots, x_m) \in (\mathbb{R}^2)^m$, we define the following regions:-

$$\begin{aligned}
B^{\alpha,\beta}_{R_\alpha,R_\beta} &:= \{x^\alpha e_\alpha + x^\beta e_\beta : (x^\alpha, x^\beta) \in [-R_\alpha, R_\alpha] \times [-R_\beta, R_\beta]\}, \\
B^{\alpha,\beta}_{R_\alpha,R_\beta}(x) &:= B^{\alpha,\beta}_{R_\alpha,R_\beta} + x, \quad x \in \mathbb{R}^2, \\
B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{x}_m) &:= \bigcup_{j=1}^{m} B^{\alpha,\beta}_{R_\alpha,R_\beta}(x_j).
\end{aligned}$$

2.2 Sticks of two types

In this subsection we assume that
(i) *there are sticks with only two orientations, and*
(ii) *sticks of the same orientation are of the same length but sticks along different directions could be of different lengths.*
Without loss of generality we assume that sticks are either horizontal or at an angle $\alpha \in (0, \pi]$. Sticks which are horizontal are of length R_0 and sticks at an angle α are of length R_α.

In this case $\Lambda(k, \ell)$ is the set of clusters containing k horizontal sticks and ℓ sticks at an angle α with respect to the x-axis. We show that

Theorem 2.1 *Let $m = k + \ell,\ k, \ell \geq 1,\ \alpha \in (0, \pi)$ and $0 < R_0, R_\alpha$. As $\lambda \to \infty$, we have*
(i) $\quad \mu_{\lambda\rho}(C_0 \in \Lambda(k, \ell) \mid \Gamma_0)$

$$\sim \left(\frac{1}{\lambda |B^{0,\alpha}_{R_0,R_\alpha}|} \right)^{m-3} e^{-\lambda |B^{0,\alpha}_{R_0,R_\alpha}|} (pq)^{-2(m-1)} m p^{3k} k! q^{3l} \ell!,$$

where $a(\lambda) \sim b(\lambda)$ means that $\frac{a(\lambda)}{b(\lambda)} \to 1$ as $\lambda \to \infty$;
(ii) $\quad p_{\lambda,m}(k, \ell) := \mu_{\lambda\rho}(\#C_0 = (k, \ell) \mid \#C_0 = (k', \ell'),\ k' + \ell' = m)$

$$\sim \frac{p^{3k} k! q^{3\ell} \ell!}{\sum_{k+\ell=m} p^{3k} k! q^{3\ell} \ell!}.$$

An interesting observation from (ii) above is that asymptotically, as $\lambda \to \infty$, the conditional probability $p_{\lambda,m}(k, \ell)$ of the sticks comprising the *finite* cluster C_0, is independent of both the angle α as well as R_0 and R_α, the lengths of the sticks. This is not surprising because the model is invariant under affine transformations. Now let $p_m(k, \ell) := \lim_{\lambda \to \infty} p_{\lambda,m}(k, \ell)$. We also observe from Theorem 2.1 (ii) that, as $m \to \infty$,

$$\begin{aligned} p_m(m-1, 1) &\to 1 && \text{for } p > q, \\ p_m(1, m-1) &\to 1 && \text{for } p < q, \\ p_m(1, m-1) = p_m(m-1, 1) &\to \frac{1}{2} && \text{for } p = q. \end{aligned}$$

Moreover, let k and m both approach infinity in such a way that $(k/m) \to s$, for some $s \in [0, 1]$, then we have

$$\lim_{\substack{m \to \infty \\ (k/m) \to s}} \frac{1}{m} \log p_m(k, \ell) = H(s),$$

where

$$H(s) = s \log s + (1-s) \log(1-s) + \begin{cases} 3(1-s) \log(q/p), & \text{if } p > q, \\ 3s \log(p/q), & \text{if } p < q, \\ 0, & \text{if } p = q, \end{cases}$$

from which we may deduce that as $m \to \infty$, for $0 \leq a \leq b \leq 1$,
P(the proportion (k/m) of horizontal sticks in the cluster lies between a and b)
$\sim \exp\{\sup_{s \in (a,b)} H(s)\}$.

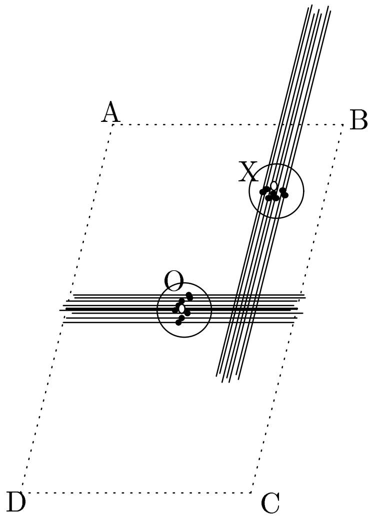

Figure 1: *The finite cluster for large λ. The region X which contains the centres of the sticks at an angle α w.r.t. the x-axis is uniformly distributed in the parallelogram $ABCD$.*

From the proof of the above theorem we also observe that the centres of the horizontal sticks comprising the cluster C_0 lie in a neighbourhood whose area is of the order $o(\lambda^{-1+(\delta/2)})$. Similarly the centres of the sticks of orientation α comprising the cluster C_0 lie in another neighbourhood whose area is of the order $o(\lambda^{-1+(\delta/2)})$. (See Figure 1.)

2.3 Sticks of three types

In this subsection we assume that
(i) *there are sticks with only three orientations* – $0, \alpha$ *and* β,
(ii) *sticks of the same orientation are of the same length.*
Here the results are significantly different from those obtained in the previous section. In particular the absence of any affine invariance leads to the dependence of the results on both the length and orientation of the sticks through the following quantities

$$H_\alpha = \frac{R_\alpha}{\sin\beta}, \qquad H_\beta = \frac{R_\beta}{\sin\alpha}, \qquad H_0 = \frac{R_0}{\sin(\beta-\alpha)}. \tag{2.4}$$

By a suitable scaling we take

$$H_0 = 1 \text{ and let } H_\alpha = a, \;\; H_\beta = b \text{ after the scaling.} \tag{2.5}$$

As the following theorem exhibits, the asymptotic (as $\lambda \to \infty$) composition of the finite cluster contains sticks of only two distinct orientation, while the third does not figure at all. Here we use the shorthand "$A(x, y)$ occurs" to mean that as $\lambda \to \infty$ the asymptotic shape of C_0 consists of sticks only in the directions x and y.

Theorem 2.2 *Given that C_0 consists of m sticks,*

(1) *for* $a, b \geq 2$;

(i) *if* $(ab - a + 1/4)p_\beta + a < (ab - b + 1/4)p_\alpha + b$, *then* $A(0, \alpha)$ *occurs,*

(ii) *if* $(ab - a + 1/4)p_\beta + a > (ab - b + 1/4)p_\alpha + b$, *then* $A(0, \beta)$ *occurs, and*

(iii) *if* $(ab - a + 1/4)p_\beta + a = (ab - b + 1/4)p_\alpha + b$, *then both* $A(0, \alpha)$ *and* $A(0, \beta)$ *have positive probabilities of occurrence;*

(2) *for* $1/2 < \min\{a, b\} < 2$ *and* $a \neq b$, $a, b \neq 1$ *and for* $x, y, z \in \{0, \alpha, \beta\}$ *let*

$$f(x, y, z) := p_x H_x \max\{H_y, H_z\} + p_x \min\{H_y, H_z\}^2/4 + (1 - p_x)H_y H_z,$$

(i) $A(\alpha, \beta)$ *occurs when* $f(0, \alpha, \beta) < \min\{f(\beta, 0, \alpha), f(\alpha, \beta, 0)\}$

(ii) $A(0, \alpha)$ *and* $A(0, \beta)$ *have positive probabilities of occurrence, when* $f(\beta, 0, \alpha) = f(\alpha, \beta, 0) < f(0, \alpha, \beta)$, *and*

(iii) $A(\alpha, \beta)$, $A(0, \alpha)$ *and* $A(0, \beta)$ *all have positive probabilities of occurrence when* $f(\beta, 0, \alpha) = f(\alpha, \beta, 0) = f(0, \alpha, \beta)$;

(3) *for* $0 < a = b < 1$, *and,*

(i) *for* $p_0 \leq \min\{p_\alpha, p_\beta\}$, $A(\alpha, \beta)$ *occurs,*

(ii) *for* $p_0 > \min\{p_\alpha, p_\beta\}$,
if $a < \mathbf{l}_1(p_0, p_\alpha, p_\beta) := 1 - \frac{p_0 - \min\{p_\alpha, p_\beta\}}{4 - 3p_0 - \min\{p_\alpha, p_\beta\}}$, *then* $A(\alpha, \beta)$ *and fixation occurs, while,*
if $a \geq \mathbf{l}_1(p_0, p_\alpha, p_\beta)$, $A(0, \alpha)$ *occurs for* $p_\alpha > p_\beta$ *and both* $A(0, \alpha)$ *and* $A(0, \beta)$ *have positive probability of occurrence for* $p_\alpha = p_\beta$;

(4) *for* $1 < a = b < 2$, *and,*

(i) *for* $p_0 < \min\{p_\alpha, p_\beta\}$,
if $a < \mathbf{l}_2(p_0, p_\alpha, p_\beta) := \frac{2\max\{p_\alpha, p_\beta\} + \sqrt{4\max\{p_\alpha, p_\beta\}^2 + 4p_\alpha p_\beta + p_0 \min\{p_\alpha, p_\beta\}}}{4\max\{p_\alpha, p_\beta\} + p_0}$, *then* $A(\alpha, \beta)$ *and fixation occurs, while,*
if $a \geq \mathbf{l}_2(p_0, p_\alpha, p_\beta)$, $A(0, \alpha)$ *occurs for* $p_\alpha > p_\beta$ *and both* $A(0, \alpha)$ *and* $A(0, \beta)$ *have positive probability of occurrence for* $p_\alpha = p_\beta$,

(ii) *for* $\min\{p_\alpha, p_\beta\} \leq p_0$, $A(0, \alpha)$ *occurs for* $p_\alpha > p_\beta$ *and both* $A(0, \alpha)$ *and* $A(0, \beta)$ *have positive probability of occurrence for* $p_\alpha = p_\beta$;

(5) *for* $a = b = 1$, *fixation always occurs and*

(i) $A(x, y)$ *occurs when* $p_z < \min\{p_x, p_y\}$,

(ii) *with equal probability* $A(x, y)$ *and* $A(x, z)$ *occur when* $p_y = p_z < p_x$, *and*

(iii) *with equal probability* $A(x, y)$, $A(y, z)$ *and* $A(z, x)$ *occur when* $p_x = p_y = p_z$;

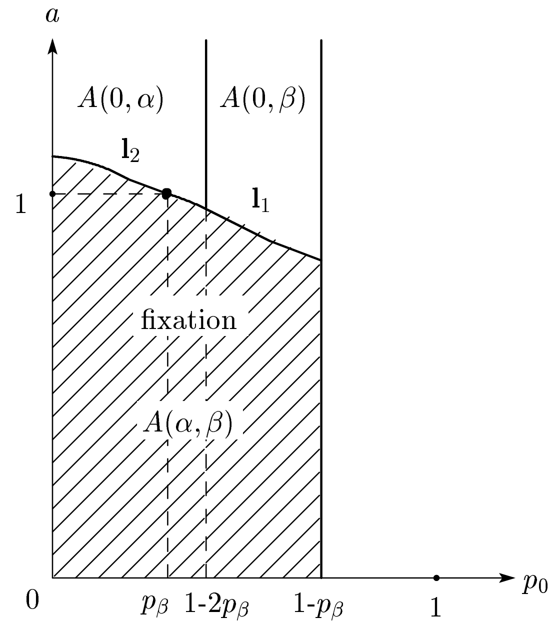

Figure 2: *The diagram in the case that $a = b$ and $p_\beta \in (0, 1/3)$. The curved line is the line $\mathbf{l}_1 1_{\{0 \le \mathbf{l}_1 \le 1\}} + \mathbf{l}_2 1_{\{1 \le \mathbf{l}_1 \le 2\}}$. For $p_0 > 0$ and a below this line $A(\alpha, \beta)$ occurs, while for a above the line $A(0, \beta)$ occurs when $p_\alpha < p_\beta$. At $p_0 = 0$, only $A(\alpha, \beta)$ occurs.*

Observe that for $\min a, b \le 1/2$:

(A) If $b, 1 \ge 2a$, then by the scaling which transforms a to 1, b to b/a and 1 to $1/a$, the resulting asymptotic cluster may be read from (1) of Theorem 2.2. Similarly if $a, 1 \ge 2b$, we may scale suitably to obtain a situation as in (1) of Theorem 2.2.

(B) If either $a/2 < \min\{1, b\} < 2a, a \neq b,\ a, b \neq 1$, or $b/2 < \min\{1, a\} < 2b, a \neq b,\ a, b \neq 1$, then scaling shows that (2) of Theorem 2.2 may be used to yield the asymptotic shape.

(C) If either $0 < b = 1 < a$ or $0 < a = 1 < b$, then scaling shows that (3) of Theorem 2.2 may be used to yield the asymptotic shape.

(D) If either $a < b = 1 < 2a$ or $b < a = 1 < 2b$, then scaling shows that (4) of Theorem 2.2 may be used to yield the asymptotic shape.

Thus the above four observations demonstrate that Theorem 2.2 yields the asymptotic shapes for all possible values of a and b.

To prove the above theorem we need to know the conditional probability of the composition of a cluster given that it is finite. This is obtained in the next two sections.

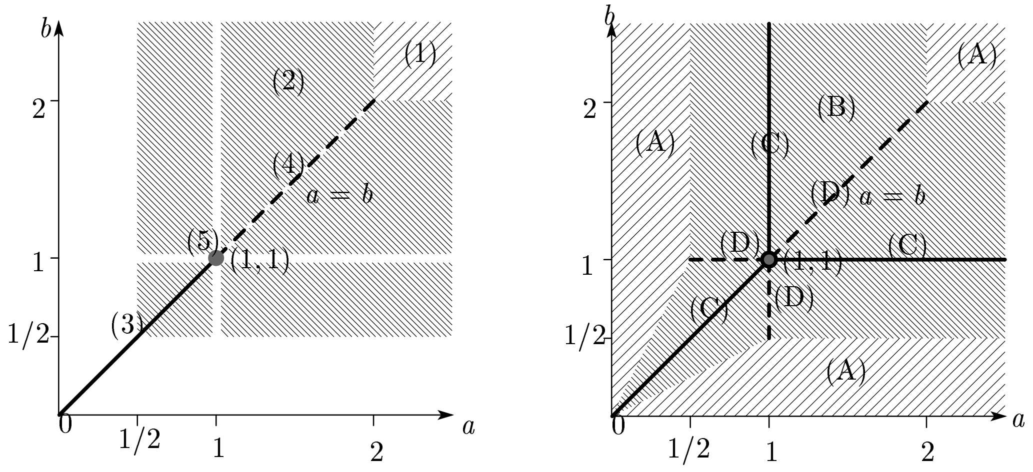

Figure 3: *The various regions where Theorem the various parts of Theorem 2.2 hold.*

3 Proof of Theorem 2.1

3.1 General set-up

For $\mathbf{k} \in (\mathbb{N} \cup 0)^d$, $d \geq 2$, with $|\mathbf{k}| = m$, let $\Lambda(\mathbf{k})$ and Γ_0 be as in Section 2.1. First we calculate $\mu_{\lambda\rho}(C_\mathbf{0} \in \Lambda(\mathbf{k}) | \Gamma_0)$. Suppose that $w_m = (\mathbf{0}, \alpha_{j_0}, R_{j_0})$ for some $j_0 \in \{1, 2, \ldots, d\}$. We have

$$\begin{aligned}
&\mu_{\lambda\rho}(C_\mathbf{0} \in \Lambda(\mathbf{k}) \mid \; w_m \in \xi) \\
&= \int\limits_{\mathcal{M}} \mu_{\lambda\rho}(d\xi) \sum_{\{\mathbf{w}_{m-1}\} \subset \xi} 1_{\Lambda(\mathbf{k})}(C_\mathbf{0}(\mathbf{w}_m)) 1_{\{S(\xi \setminus \{\mathbf{w}_m\}) \cap S(\{\mathbf{w}_m\}) = \emptyset\}},
\end{aligned}$$

where $\mathbf{w_m}$, $\{\mathbf{w_m}\}$ and $C_\mathbf{0}(\mathbf{w}_m)$ are as defined in (2.3). Thus,

$$\begin{aligned}
&\mu_{\lambda\rho}(C_\mathbf{0} \in \Lambda(\mathbf{k}) \mid \; w_m \in \xi) \\
&= \frac{\lambda^{m-1}}{(m-1)!} \int\limits_{\mathcal{M}} \mu_{\lambda\rho}(d\eta) \int\limits_{\mathcal{R}^{m-1}} \rho^{\otimes(m-1)}(d\mathbf{w}_{m-1}) 1_{\Lambda(\mathbf{k})}(C_\mathbf{0}(\mathbf{w}_m)) 1_{\{S(\eta) \cap S(\{\mathbf{w}_m\}) = \emptyset\}} \\
&= \frac{\lambda^{m-1}}{(m-1)!} \int\limits_{\mathcal{R}^{m-1}} \rho^{\otimes(m-1)}(d\mathbf{w}_{m-1}) 1_{\Lambda(\mathbf{k})}(C_\mathbf{0}(\mathbf{w}_m)) e^{-\lambda\rho(w : S(w) \cap S(\{\mathbf{w}_m\}) \neq \emptyset)}.
\end{aligned}$$

Note that $S(x, \theta, r) \cap S(\{\mathbf{w}_m\}) \neq \emptyset$ if and only if $x \in \cup_{i=1}^m B_{r_i, r}^{\theta_i, \theta}(x_i)$ where $w_i = (x_i, \theta_i, r_i)$, $i = 1, 2, \ldots, m$. Hence,

$$\rho(w : S(w) \cap S(\{\mathbf{w}_m\}) \neq \emptyset) = \sum_{j=1}^d p_j |\bigcup_{i=1}^m B_{r_i, R_j}^{\theta_i, \alpha_j}(x_i)|,$$

and so

$$\mu_{\lambda\rho}(C_{\mathbf{0}} \in \Lambda(\mathbf{k}) \mid \ w_m \in \xi) = \frac{\lambda^{m-1}}{(m-1)!} \int\limits_{\mathcal{R}^{m-1}} \rho^{\otimes(m-1)}(d\mathbf{w}_{m-1}) 1_{\Lambda(\mathbf{k})}(C_{\mathbf{0}}(\mathbf{w}_m))$$
$$\times \exp\left[-\lambda \sum_{j=1}^{d} p_j |\bigcup_{i=1}^{m} B^{\theta_i,\alpha_j}_{r_i,R_j}(x_i)|\right].$$

Let

$$F^{\alpha_{j_0}}_{\lambda}(\mathbf{k}) = \int\limits_{(\mathbb{R}^2)^{k_1}} d\mathbf{x}_{1,k_1} \int\limits_{(\mathbb{R}^2)^{k_2}} d\mathbf{x}_{2,k_2} \cdots \int\limits_{(\mathbb{R}^2)^{k_{j_0}-1}} d\mathbf{x}_{j_0,k_{j_0}-1} \cdots \int\limits_{(\mathbb{R}^2)^{k_d}} d\mathbf{x}_{d,k_d}$$
$$\times 1_{\Lambda(\mathbf{k})}(C_{\mathbf{0}}(\mathbf{x})) \exp\left[-\lambda \sum_{j=1}^{d} p_j | \bigcup_{i=1, k_i \neq 0}^{d} B^{\alpha_i,\alpha_j}_{R_i,R_j}(\mathbf{x}_{i,k_i})|\right],$$

where $C_{\mathbf{0}}(\mathbf{x}) = C_{\mathbf{0}}(\mathbf{x}_{1,k_1}, \mathbf{x}_{2,k_2}, \dots, \mathbf{x}_{d,k_d}) = C_{\mathbf{0}}(\bigcup_{j=1}^{d}\{(x_{j,i}, \alpha_j, R_j) : i = 1, \dots, k_j\})$. From the translation invariance of Lebesgue measure it is obvious that if $k_j, k_{j'} \geq 1$, then $F^{\alpha_j}_{\lambda}(\mathbf{k}) = F^{\alpha_{j'}}_{\lambda}(\mathbf{k})$. Thus writing $F_\lambda(\mathbf{k})$ for $F^{\alpha_j}_{\lambda}(\mathbf{k})$, since $\mu_{\lambda\rho}((0, \alpha_j, R_j) \in \xi \mid \ \Gamma_{\mathbf{0}}) = p_j$, we have

$$\mu_{\lambda\rho}(C_{\mathbf{0}} \in \Lambda(\mathbf{k}) \mid \ \Gamma_{\mathbf{0}}) = \frac{\lambda^{m-1}}{(m-1)!} \prod_{j=1}^{d} \frac{m!}{k_j!} p_j^{k_j} F_\lambda(\mathbf{k}) = \lambda^{|\mathbf{k}|-1} |\mathbf{k}| \prod_{j=1}^{d} \frac{p_j^{k_j}}{k_j!} F_\lambda(\mathbf{k}). \tag{3.1}$$

3.2 Proof of Theorem 2.1

To prove Theorem 2.1, observe first that in the case when we have sticks with only two orientations, the Radon measure ρ is given by

$$\rho(dx\ d\theta\ dr) = dx\{p\delta_0(d\theta)\delta_{R_0}(dr) + q\delta_\alpha(d\theta)\delta_{R_\alpha}(dr)\}. \tag{3.2}$$

¿From (3.1) we have

$$\begin{aligned} \mu_{\lambda\rho}(C_{\mathbf{0}} \in \Lambda(k,\ell) \mid \ \Gamma_{\mathbf{0}}) &= \lambda^{k+\ell-1}(k+\ell)\frac{p^k q^\ell}{k!\ell!} F^0_\lambda((k,\ell)) \\ &= \lambda^{k+\ell-1}(k+\ell)\frac{p^k q^\ell}{k!\ell!} e^{-\lambda |B^{0,\alpha}_{R_0,R_\alpha}|} f_\lambda(k,\ell), \end{aligned}$$

where

$$f_\lambda(k,\ell) := \int\limits_{(\mathbb{R}^2)^{k-1}} d\mathbf{x}_{k-1} \int\limits_{(\mathbb{R}^2)^{l}} d\mathbf{y}_\ell \ 1_{\Lambda(k,\ell)}(C_{\mathbf{0}}(\mathbf{x}_k, \mathbf{y}_\ell)) \chi^{0,\alpha}_{p\lambda}(\mathbf{y}_\ell) \chi^{0,\alpha}_{q\lambda}(\mathbf{x}_k),$$

$$\chi^{\theta_1,\theta_2}_{c}(\mathbf{x}) = \exp\left[-c\{|B^{\theta_1,\theta_2}_{R_{\theta_1},R_{\theta_2}}(\mathbf{x})| - |B^{\theta_1,\theta_2}_{R_{\theta_1},R_{\theta_2}}|\}\right] \tag{3.3}$$

(note here that $x_k = \mathbf{0}$). Now consider the event $A(\mathbf{x}_k, \mathbf{y}_\ell, k, \ell) := \{C_0$ contains exactly m sticks $(\mathbf{0}, 0, 1/2), (x_1, 0, 1/2), \dots, (x_{k-1}, 0, 1/2), (y_1, \frac{\pi}{2}, 1/2), \dots, (y_\ell, \frac{\pi}{2}, 1/2)\}$. By the affine invariance of the Lebesgue measure

$$\begin{aligned}
f_\lambda(k,\ell) &= |B^{0,\alpha}_{R_0,R_\alpha}|^{m-1} \int\limits_{(\mathbb{R}^2)^{k-1}} d\mathbf{x}_{k-1} \int\limits_{(\mathbb{R}^2)^{\ell}} d\mathbf{y}_\ell \; 1_{A(\mathbf{x}_k,\mathbf{y}_\ell,k,\ell)} \\
&\quad \times \exp[-\lambda p |B^{0,\alpha}_{R_0,R_\alpha}| \{|B_{\frac{1}{2}}(\mathbf{y}_\ell)| - |B_{\frac{1}{2}}|\}] \\
&\quad \times \exp[-\lambda q |B^{0,\alpha}_{R_0,R_\alpha}| \{|B_{\frac{1}{2}}(\mathbf{x}_k)| - |B_{\frac{1}{2}}|\}], \qquad (3.4)
\end{aligned}$$

where $B_R = [-R,R]^2$, $B_R(x) = B_R + x$ and $B_R(\mathbf{x}_k) = \cup_{i=1}^k B_R(x_i)$.

For the proof of Theorem 2.1 we will obtain lower and upper bounds of $f_\lambda(k,l)$ which we later show to agree as $\lambda \to \infty$. To this end we need the following lemma whose proof is given in the appendix. For each $x \in \mathbb{R}^2$ we take $x^\alpha, x^\beta \in \mathbb{R}$ such that $x = x^\alpha e_\alpha + x^\beta e_\beta$. Note that (x^α, x^β) is just the representation of $x \in \mathbb{R}^2$ in the base given by the axes parallel to the orientation of the sticks. Let $h_\alpha(x) = \frac{x^\alpha}{\sin\beta}$, $h_\beta(x) = \frac{x^\beta}{\sin\alpha}$ and

$$h_\theta(\mathbf{x}_k) = (h_\theta(x_1), h_\theta(x_2), \ldots, h_\theta(x_k)), \quad \mathbf{x}_k = (x_1, x_2, \ldots, x_k) \in (\mathbb{R}^2)^k.$$

We put

$$M(\mathbf{u_k}) = \max_{1\le i,j\le k} |u_i - u_j|, \quad \mathbf{u}_k = (u_1, u_2, \ldots, u_k) \in (\mathbb{R})^k.$$

and $C_{\alpha,\beta} = \sin\alpha \sin\beta \sin(\alpha-\beta)$.

Lemma 3.1 *Let $\mathbf{x}_k = (x_1, x_2, \cdots, x_k) \in (\mathbb{R}^2)^k$ with $x_k = \mathbf{0}$. Then*

$$\begin{aligned}
|B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{x}_k) \backslash B^{\alpha,\beta}_{R_\alpha,R_\beta}| &\le 2C_{\alpha,\beta}\{H_\alpha M(h_\beta(\mathbf{x}_k)) + H_\beta M(h_\alpha(\mathbf{x}_k))\} \\
&+ C_{\alpha,\beta} M(h_\beta(\mathbf{x}_k)) M(h_\alpha(\mathbf{x}_k)), \qquad (3.5)
\end{aligned}$$

and, if $B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{x}_k)$ is connected, then we have

$$|B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{x}_k) \backslash B^{\alpha,\beta}_{R_\alpha,R_\beta}| \ge C_{\alpha,\beta}\{H_\alpha M(h_\beta(\mathbf{x}_k)) + H_\beta M(h_\alpha(\mathbf{x}_k))\}, \qquad (3.6)$$

$$\begin{aligned}
|B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{x}_k) \backslash B^{\alpha,\beta}_{R_\alpha,R_\beta}| &\ge 2C_{\alpha,\beta}\{H_\alpha M(h_\beta(\mathbf{x}_k)) + H_\beta M(h_\alpha(\mathbf{x}_k))\} \\
&- C_{\alpha,\beta} M(h_\beta(\mathbf{x}_k)) M(h_\alpha(\mathbf{x}_k)). \qquad (3.7)
\end{aligned}$$

Now we evaluate the bounds of $f_\lambda(k,\ell)$.

LOWER BOUND : By (3.5) of Lemma 3.1, taking $x_k = \mathbf{0}$ we have

$$\begin{aligned}
f_\lambda(k,\ell) &\ge |B^{0,\alpha}_{R_0,R_\alpha}|^{m-1} \int\limits_{(\mathbb{R}^2)^{k-1}} d\mathbf{x}_{k-1} \int\limits_{(\mathbb{R}^2)^{\ell}} d\mathbf{y}_\ell \; 1_{A(\mathbf{x}_k,\mathbf{y}_\ell,k,\ell)} \\
&\quad \times \exp[-\lambda q |B^{0,\alpha}_{R_0,R_\alpha}| (M(\mathbf{x}^1_k) + M(\mathbf{x}^2_k))] \\
&\quad \times \exp[-\lambda p |B^{0,\alpha}_{R_0,R_\alpha}| (M(\mathbf{y}^1_\ell) + M(\mathbf{y}^2_\ell))] \\
&\quad \times \exp[-\lambda |B^{0,\alpha}_{R_0,R_\alpha}| \{q M(\mathbf{x}^1_k) M(\mathbf{x}^2_k) + p M(\mathbf{y}^1_\ell) M(\mathbf{y}^2_\ell)\}]. \qquad (3.8)
\end{aligned}$$

Let $L(\lambda)$ be such that, as $\lambda \to \infty$, $\lambda L(\lambda) \to \infty$ and $\lambda(L(\lambda))^2 \to 0$. If $\{x_i\}_{i=1}^{k-1} \subset B_{L(\lambda)}$ and $\{y_i\}_{i=1}^{\ell-1} \subset B_{L(\lambda)}(y_\ell)$, then, for $x_k = \mathbf{0}$, $y_\ell \in B_{R-L(\lambda)}$ and for λ sufficiently large, we have $A(\mathbf{x}_k, \mathbf{y}_\ell, k, \ell)$ occurs, and so the expression on the right of the inequality (3.8) is bounded from below by

$$\begin{aligned}
& |B^{0,\alpha}_{R_0,R_\alpha}|^{m-1} \int\limits_{(B_{L(\lambda)})^{k-1}} d\mathbf{x}_{k-1} \int\limits_{B_{1/2-L(\lambda)}} dy_\ell \int\limits_{(B_{L(\lambda)}(y_\ell))^{\ell-1}} d\mathbf{y}_{\ell-1} \\
&\qquad \times \exp[-\lambda q |B^{0,\alpha}_{R_0,R_\alpha}|(M(\mathbf{x}^1_k) + M(\mathbf{x}^2_k))] \\
&\qquad \times \exp[-\lambda p |B^{0,\alpha}_{R_0,R_\alpha}|(M(\mathbf{y}^1_k) + M(\mathbf{y}^2_k))] \\
&\qquad \times \exp[-\lambda |B^{0,\alpha}_{R_0,R_\alpha}|\{qM(\mathbf{x}^1_k)M(\mathbf{x}^2_k) + pM(\mathbf{y}^1_k)M(\mathbf{y}^2_k)\}] \\
\geq\ & |B^{0,\alpha}_{R_0,R_\alpha}|^{m-1} e^{-4(p+q)(L(\lambda))^2} |B_{1/2-L(\lambda)}| \int\limits_{(B_{L(\lambda)})^{k-1}} d\mathbf{x}_{k-1} \int\limits_{(B_{L(\lambda)})^{\ell-1}} d\mathbf{y}_{\ell-1} \\
&\qquad \times \exp[-\lambda q |B^{0,\alpha}_{R_0,R_\alpha}|(M(\mathbf{x}^1_k) + M(\mathbf{x}^2_k))] \\
&\qquad \times \exp[-\lambda p |B^{0,\alpha}_{R_0,R_\alpha}|(M(\mathbf{y}^1_k) + M(\mathbf{y}^2_k))] \\
=\ & e^{-4\lambda(L(\lambda))^2} |B_{1/2-L(\lambda)}| (q\lambda)^{-2(k-1)} (p\lambda)^{-2(\ell-1)} |B^{0,\alpha}_{R_0,R_\alpha}|^{-(m-3)} \\
&\qquad \times \int\limits_{(B_{q\lambda_\alpha L(\lambda)})^{k-1}} d\mathbf{u}_{k-1} \exp[-M(\mathbf{u}^1_k) - M(\mathbf{u}^2_k)] \\
&\qquad \times \int\limits_{(B_{p\lambda_\alpha L(\lambda)})^{\ell-1}} d\mathbf{v}_{\ell-1} \exp[-M(\mathbf{v}^1_k) - M(\mathbf{v}^2_k)]
\end{aligned} \tag{3.9}$$

where $\mathbf{u}_k = (u_1, \ldots, u_k)$ and $\mathbf{v}_\ell = (v_1, \ldots, v_\ell)$ with $v_\ell = u_k = \mathbf{0}$, and $\lambda_\alpha = |B^{0,\alpha}_{R_0,R_\alpha}|\lambda$. Then we have

$$\begin{aligned}
f_\lambda(k,\ell) \ \geq\ & e^{-4\lambda(L(\lambda))^2} |B_{R-L(\lambda)}| \lambda^{-2(m-2)} |B^{0,\alpha}_{R_0,R_\alpha}|^{-(m-3)} q^{-2(k-1)} p^{-2(\ell-1)} \\
\times & \left[\int_{-q\lambda_\alpha L(\lambda)}^{q\lambda_\alpha L(\lambda)} da_1 \cdots \int_{-q\lambda_\alpha L(\lambda)}^{q\lambda_\alpha L(\lambda)} da_{k-1}\ \exp\{-\max_{1\leq i,j\leq k} |a_i - a_j|\} \right]^2 \\
\times & \left[\int_{-p\lambda_\alpha L(\lambda)}^{p\lambda_\alpha L(\lambda)} db_1 \cdots \int_{-p\lambda_\alpha L(\lambda)}^{p\lambda_\alpha L(\lambda)} db_{\ell-1}\ \exp\{-\max_{1\leq i,j\leq \ell} |b_i - b_j|\} \right]^2 .
\end{aligned} \tag{3.10}$$

Since $e^{-4\lambda(L(\lambda))^2} = 1 - O(\lambda(L(\lambda))^2)$ as $\lambda \to 0$, by (3.10) and the above lemma we obtain that, as $\lambda \to 0$,

$$f_\lambda(k,\ell) \geq \left[\left(\frac{1}{\lambda}\right)^{2(m-2)} \left(\frac{1}{|B^{0,\alpha}_{R_0,R_\alpha}|}\right)^{m-3} q^{-2(k-1)} p^{-2(\ell-1)} (k!)^2 (\ell!)^2 \right] (1 - O(\lambda(L(\lambda))^2)). \tag{3.11}$$

Now we will obtain the upper bound of $f_\lambda(k,\ell)$.

UPPER BOUND: For $L(\lambda)$ as earlier, consider the event
$E := \{x_1, \ldots, x_{k-1} \in B_{L(\lambda)}, y_1, \ldots, y_{\ell-1} \in B_{L(\lambda)}(y_\ell)\}$.

If $x_k = \mathbf{0}$, for $E \cap A(\mathbf{x}_k, \mathbf{y}_\ell, k, \ell)$ to occur, we must have $y_\ell \in B_{(1/2)+L(\lambda)}$. Thus from (3.4) we have

$$\begin{aligned} f_\lambda(k,\ell) \quad \leq \quad & |B^{0,\alpha}_{R_0,R_\alpha}|^{m-1} \int\limits_{(\mathbb{R}^2)^{k-1}} d\mathbf{x}_{k-1} \int\limits_{\mathbb{R}^2} dy_\ell \int\limits_{(\mathbb{R}^2)^\ell} d\mathbf{y}_{\ell-1} \\ \times \quad & (1_{E\cap\{y_\ell\in B_{(1/2)+L(\lambda)}\}} + 1_{E^c} 1_{A(\mathbf{x}_k,\mathbf{y}_\ell,k,\ell)}) \\ & \times \exp[-\lambda p |B^{0,\alpha}_{R_0,R_\alpha}| \{|B_{\frac{1}{2}}(\mathbf{y}_\ell)| - |B_{\frac{1}{2}}|\}] \\ & \times \exp[-\lambda q |B^{0,\alpha}_{R_0,R_\alpha}| \{|B_{\frac{1}{2}}(\mathbf{x}_k)| - |B_{\frac{1}{2}}|\}]. \end{aligned} \tag{3.12}$$

On opening the parenthesis $(1_{E\cap\{y_\ell\in B_{(1/2)+L(\lambda)}\}} + 1_{E^c} 1_{A(\mathbf{x}_k,\mathbf{y}_\ell,k,\ell)})$ in the expression on the right of the inequality (3.12) above the term involving $1_{E\cap\{y_\ell\in B_{(1/2)+L(\lambda)}\}}$, for large λ, may be bounded from above by

$$\begin{aligned} & e^{4\lambda(L(\lambda))^2} |B_{1/2+L(\lambda)}| (q\lambda)^{-2(k-1)} (p\lambda)^{-2(\ell-1)} |B^{0,\alpha}_{R_0,R_\alpha}|^{-(m-3)} \\ & \times \int\limits_{(B_{q\lambda_\alpha L(\lambda)})^{k-1}} d\mathbf{u}_{k-1} \exp[-M(\mathbf{u}^1_k) - M(\mathbf{u}^2_k)] \\ & \times \int\limits_{(B_{p\lambda_\alpha L(\lambda)})^{\ell-1}} d\mathbf{v}_{\ell-1} \exp[-M(\mathbf{v}^1_k) - M(\mathbf{v}^2_k)]. \end{aligned} \tag{3.13}$$

(Here we have used the inequality (3.7) of Lemma 3.1 and calculations similar to those leading to (3.9).)

Using the inequality (3.6) of Lemma 3.1 we bound the expression involving $1_{E^c} 1_{A(\mathbf{x}_k,\mathbf{y}_\ell,k,\ell)}$ in the right of the inequality (3.12) by $|B^{0,\alpha}_{R_0,R_\alpha}|^{m-1}\{I_1 + I_2\}$, where

$$\begin{aligned} I_1 \quad = \quad & \int\limits_{(\mathbb{R}^2)^{k-1}\setminus(B_{L(\lambda)})^{k-1}} d\mathbf{x}_{k-1} \int\limits_{B_m} dy_\ell \int\limits_{(\mathbb{R}^2)^{\ell-1}} d\mathbf{y}_{\ell-1} \\ & \times \exp\{-(q/2)\lambda(M(\mathbf{x}^1_k) + M(\mathbf{x}^2_k))\} \exp\{-(p/2)\lambda(M(\mathbf{y}^1_\ell) + M(\mathbf{y}^2_\ell))\} \end{aligned}$$

and

$$\begin{aligned} I_2 \quad = \quad & \int\limits_{(\mathbb{R}^2)^{k-1}} d\mathbf{x}_{k-1} \int\limits_{B_m} dy_\ell \int\limits_{(\mathbb{R}^2)^{\ell-1}\setminus(B_{L(\lambda)})^{\ell-1}} d\mathbf{y}_{\ell-1} \\ & \times \exp\{-(q/2)\lambda(M(\mathbf{x}^1_k) + M(\mathbf{x}^2_k))\} \exp\{-(p/2)\lambda(M(\mathbf{y}^1_\ell) + M(\mathbf{y}^2_\ell))\}. \end{aligned}$$

Let $a_k = 0$. Then, it is easy to see that

$$\int\limits_{\mathbb{R}^{k-1}} da_1 \cdots da_{k-1} \exp\{-\max_{1\leq i,j\leq k} |a_i - a_j|\} = k!.$$

Using this equation and calculations as in (3.10) and (3.11), for $\lambda \to \infty$, the expression in (3.13) may be bounded above by

$$\left[\left(\frac{1}{\lambda}\right)^{2(m-2)} \left(\frac{1}{|B^{0,\alpha}_{R_0,R_\alpha}|}\right)^{m-3} q^{-2(k-1)} p^{-2(\ell-1)} (k!)^2 (\ell!)^2 \right] (1 + O(\lambda(L(\lambda))^2)).$$

Thus to show that, asymptotically in λ the lower bound (3.11) of $f(k,\ell)$ agrees with its upper bound it suffices to show that

$$I_1 + I_2 = O(\lambda^{-2m-3}) \text{ as } \lambda \to \infty. \tag{3.14}$$

To estimate the integrals I_1 and I_2, we use the symmetry of the integrand in I_1 to obtain

$$\begin{aligned} I_1 &\leq 4(k-1) \int\limits_{(\mathbb{R}^2)^{k-2}} d\mathbf{x}_{k-2} \int\limits_{\mathbb{R}} dx_{k-1}^1 \int\limits_{L(\lambda)}^{\infty} dx_{k-1}^2 \, |B_m| \int\limits_{(\mathbb{R}^2)^{\ell-1}} d\mathbf{y}_{\ell-1} \\ &\quad \times \exp\{-(q/2)\lambda(M(\mathbf{x}_k^1) + M(\mathbf{x}_k^2))\} \exp\{-(p/2)\lambda(M(\mathbf{y}_\ell^1) + M(\mathbf{y}_\ell^2))\} \\ &= 4(k-1)\, |B_m| \left(\frac{q\lambda}{2}\right)^{-2(k-1)} \left(\frac{p\lambda}{2}\right)^{-2(\ell-1)} k!(\ell!)^2 \\ &\quad \times \int\limits_{\mathbb{R}^{k-2}} da_1 \cdots da_{k-2} \int\limits_{q\lambda L(\lambda)}^{\infty} da_{k-1} \exp\{-\max_{1\leq i,j\leq k} |a_i - a_j|\}. \end{aligned}$$

Since $a_k = 0$, we have the inequality $\max_{1\leq i,j\leq k} |a_i - a_j| \geq \frac{1}{2} \max_{\substack{1\leq i,j\leq k \\ i,j\neq k-1}} |a_i - a_j| + \frac{1}{2}|a_{k-1}|$, which we use to obtain

$$\begin{aligned} &\int\limits_{\mathbb{R}^{k-2}} da_1 \cdots da_{k-2} \int\limits_{q\lambda L(\lambda)}^{\infty} da_{k-1} \exp\{-\max_{1\leq i,j\leq k} |a_i - a_j|\} \\ &\leq 2^{k-1} \int\limits_{\mathbb{R}^{k-1}} da_1 da_2 \cdots da_{k-2} \exp\{-\max_{1\leq i,j\leq k} |a_i - a_j|\} \int\limits_{\frac{1}{2}q\lambda L(\lambda)}^{\infty} da_{k-1} e^{-a_{k-1}} \\ &= 2^{k-1}(k-1)! e^{-\frac{1}{2}q\lambda L(\lambda)}. \end{aligned}$$

Hence

$$\begin{aligned} I_1 &\leq 2^{k+1}\, |B_m|\, \lambda^{-2(m-2)} \left(\frac{p}{2}\right)^{-2(\ell-1)} \left(\frac{q}{2}\right)^{-2(k-1)} (k!)^2 (\ell!)^2 e^{-\frac{1}{2}q\lambda L(\lambda)} \\ &= o(e^{-\frac{1}{2}q\lambda L(\lambda)}) \qquad \text{as } \lambda \to \infty. \end{aligned}$$

Similarly we obtain

$$I_2 = o(e^{-\frac{1}{2}p\lambda L(\lambda)}) \qquad \text{as } \lambda \to \infty.$$

Now fix $0 < \delta < 1/2$ and take $L(\lambda) = \lambda^{-1+(\delta/2)}$. The bounds obtained above for I_1 and I_2 show that (3.14) holds.

This proves Theorem 2.1(i). The second part of Theorem 2.1 is derived easily from the first part.

4 Proof of Theorem 2.2

We now prove Theorem 2.2. Towards this end we need some estimates on the areas of the unions of various parallelograms. These are presented in the next subsection. The proof of these results are given in the appendix.

4.1 Area estimates

Throughout this section we assume $0 < \alpha < \beta < \pi$.

Lemma 4.1 (i) *If $H_\alpha, H_\beta > 2H_0$, then*

$$|B^{0,\alpha}_{R_0,R_\alpha} \cup B^{0,\beta}_{R_0,R_\beta}| = 4C_{\alpha,\beta} H_0 (H_\alpha + H_\beta - H_0).$$

(ii) *If $\min\{H_\alpha, H_\beta\} \le 2H_0$, then*

$$|B^{0,\alpha}_{R_0,R_\alpha} \cup B^{0,\beta}_{R_0,R_\beta}| = C_{\alpha,\beta}\{4H_0 \max\{H_\alpha, H_\beta\} + \min\{H_\alpha^2, H_\beta^2\}\}.$$

Next we will estimate

$$\triangle(x) = \frac{1}{C_{\alpha,\beta}}\{|B^{0,\alpha}_{R_0,R_\alpha} \cup B^{0,\beta}_{R_0,R_\beta}(x)| - |B^{0,\alpha}_{R_0,R_\alpha} \cup B^{0,\beta}_{R_0,R_\beta}|\}, \quad x \in \mathbb{R}^2. \tag{4.1}$$

Taking

$$D^{\theta,\theta'}_{R,R'} := \begin{pmatrix} R\cos\theta & R'\cos\theta' \\ R\sin\theta & R'\sin\theta' \end{pmatrix}$$

and

$$A^{\theta,\theta'}_{R,R'} := \begin{pmatrix} R'\sin\theta' & -R'\cos\theta' \\ -R\sin\theta & R\cos\theta \end{pmatrix},$$

for $\theta, \theta' \in [0, \pi)$, $R, R' > 0$, we have $B^{\alpha,\beta}_{R_\alpha,R_\beta} = D^{\alpha,\beta}_{R_\alpha,R_\beta}[-1,1]^2$, and

$$D^{\alpha,\beta}_{R_\alpha,R_\beta}{}^{-1} = \frac{1}{\sin(\beta-\alpha)R_\alpha R_\beta} A^{\alpha,\beta}_{R_\alpha,R_\beta}.$$

In this notation we have

$$\begin{pmatrix} h_\alpha(x) \\ h_\beta(x) \end{pmatrix} = D^{\alpha,\beta}_{\sin\beta,\sin\alpha}{}^{-1} x = \frac{1}{C_{\alpha,\beta}} \begin{pmatrix} \sin\alpha\langle x, e_{\beta-\frac{\pi}{2}}\rangle \\ \sin\beta\langle x, e_{\alpha+\frac{\pi}{2}}\rangle \end{pmatrix} \tag{4.2}$$

where h_α and h_β are as defined prior to Lemma 3.1. Note that

$$(h_\alpha(x), h_\beta(x)) \in [-H_\alpha, H_\alpha] \times [-H_\beta, H_\beta], \text{ if and only if } x \in B^{\alpha,\beta}_{R_\alpha,R_\beta},$$

and

$$\overline{h}_0(x) := \frac{\langle x, e_{\frac{\pi}{2}}\rangle}{\sin\alpha\sin\beta} = h_\alpha(x) + h_\beta(x), \qquad x \in \mathbb{R}^2.$$

See Figure 4.

Lemma 4.2 *Assume that $x \in \mathbb{R}^2$ with $h_\alpha(x) \in [-H_\alpha, H_\alpha]$, $h_\beta(x) \in [-H_\beta, H_\beta]$.*
(i) *Suppose that $2H_0 < H_\alpha, H_\beta$. Then*

$$\begin{aligned} \triangle(x) &= \frac{1}{2}\max\{-h_\alpha(x) + 2H_0 - H_\alpha, h_\beta(x) + 2H_0 - H_\beta, 0\}^2 \\ &+ \frac{1}{2}\max\{h_\alpha(x) + 2H_0 - H_\alpha, -h_\beta(x) + 2H_0 - H_\beta, 0\}^2. \end{aligned}$$

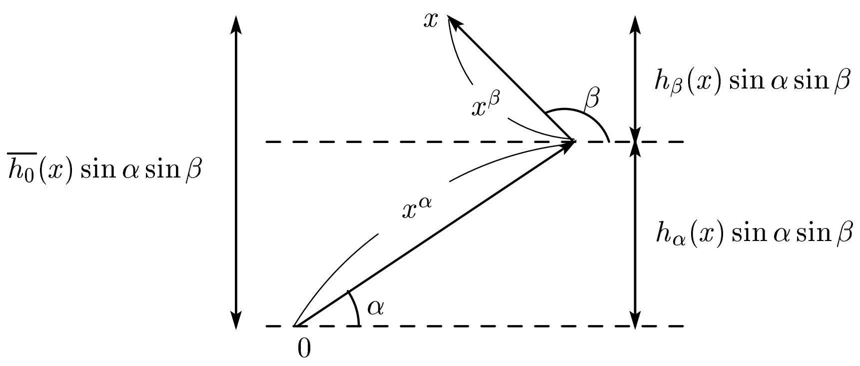

Figure 4: *The quantities h_α, h_β and $\overline{h}_0$.*

(ii) *Suppose that $2H_0 \geq \min\{H_\alpha, H_\beta\}$ and $H_\alpha \geq H_\beta$.*
(a) *When $|\overline{h}_0(x)| \leq H_\alpha - H_\beta$,*

$$\triangle(x) = \begin{cases} h_\beta(x)^2, & \text{if } |h_\beta(x)| \leq 2H_0 - H_\beta, \\ h_\beta(x)^2 - \frac{1}{2}\{|h_\beta(x)| - (2H_0 - H_\beta)\}^2, & \text{if } |h_\beta(x)| > 2H_0 - H_\beta. \end{cases}$$

(b) *When $|\overline{h}_0(x)| > H_\alpha - H_\beta$ and $|h_\beta(x)| \leq 2H_0 - H_\beta$,*

$$\begin{aligned} \triangle(x) = h_\beta(x)^2 + \frac{1}{2}\{|\overline{h}_0(x)| - (H_\alpha - H_\beta)\}^2 \\ +\{2H_0 - H_\beta - \mathrm{sgn}(\overline{h}_0(x))h_\beta(x)\}\{|\overline{h}_0(x)| - (H_\alpha - H_\beta)\}. \end{aligned}$$

(c) *When $|\overline{h}_0(x)| > H_\alpha - H_\beta$, $|h_\beta(x)| > 2H_0 - H_\beta$ and $\overline{h}_0(x)h_\beta(x) > 0$,*

$$\begin{aligned} \triangle(x) = h_\beta(x)^2 - \frac{1}{2}\{|h_\beta(x)| - (2H_0 - H_\beta)\}^2 \\ + \frac{1}{2}[2H_0 - H_\alpha + \mathrm{sgn}(h_\beta(x))h_\alpha(x)]_+^2, \end{aligned}$$

where $[a]_+ = \max\{a, 0\}$, $[a]_- = \max\{-a, 0\}$.
(d) *When $|\overline{h}_0(x)| > H_\alpha - H_\beta$, $|h_\beta(x)| > 2H_0 - H_\beta$ and $\overline{h}_0(x)h_\beta(x) < 0$,*

$$\begin{aligned} \triangle(x) &= h_\beta(x)^2 - \frac{1}{2}\{|h_\beta(x)| - (2H_0 - H_\beta)\}^2 \\ &+ \{|\overline{h}_0(x)| - (H_\alpha - H_\beta)\} \\ &\times [2H_0 - H_\beta + |h_\beta(x)| + \frac{1}{2}\{|\overline{h}_0(x)| - (H_\alpha - H_\beta)\}]. \end{aligned}$$

Remark 4.1. The area $\{x \in \mathbb{R}^2 : \triangle(x) = 0\}$ depends on angles α, β and stick lengths R_0, R_α, R_β. From the above lemma we see that

$$\{x \in \mathbb{R}^2 : \triangle(x) = 0\} = B^{\alpha,\beta}_{R_\alpha - 2R_0^\alpha, R_\beta - 2R_0^\beta}, \quad \text{when } 2H_0 < H_\alpha, H_\beta, \tag{4.3}$$

and

$$\{x \in \mathbb{R}^2 : \triangle(x) = 0\} = B^{\alpha,\beta}_{[R_\alpha - R^\alpha_\beta]_+, [R_\beta - R^\beta_\alpha]_+}, \quad \text{when } 2H_0 \geq \min\{H_\alpha, H_\beta\}, \tag{4.4}$$

where for $\theta = 0, \alpha, \beta$, $R^0_\theta = H_\theta \sin(\beta - \alpha)$, $R^\alpha_\theta = H_\theta \sin\beta$, $R^\beta_\theta = H_\theta \sin\alpha$. In particular $R^\theta_\theta = R_\theta$.

Since

$$A^{\alpha,\beta}_{R_\alpha,R_\beta} x = \begin{pmatrix} R_\beta \langle x, e_{\beta - \frac{\pi}{2}} \rangle \\ R_\alpha \langle x, e_{\alpha + \frac{\pi}{2}} \rangle \end{pmatrix},$$

we have

$$\begin{aligned} M(A^{\alpha,\beta}_{R_\alpha,R_\beta} \mathbf{x}_k(0)) &= R_\beta M(\mathbf{x}_k(\beta - \frac{\pi}{2})) = C_{\alpha,\beta} H_\beta M(h_\alpha(\mathbf{x}_k)), \\ M(A^{\alpha,\beta}_{R_\alpha,R_\beta} \mathbf{x}_k(\frac{\pi}{2})) &= R_\alpha M(\mathbf{x}_k(\alpha + \frac{\pi}{2})) = C_{\alpha,\beta} H_\alpha M(h_\beta(\mathbf{x}_k)). \end{aligned}$$

For $\mathbf{x}_k \in {\mathbb{R}^2}^k$, $\mathbf{y}_\ell \in {\mathbb{R}^2}^\ell$ and $u \in \mathbb{R}^2$ we write

$$\mathbf{x}_k \cdot \mathbf{y}_\ell = (x_1, x_2, \ldots x_k, y_1, y_2, \ldots, y_\ell) \in (\mathbb{R}^2)^{k+\ell},$$

and $\mathbf{x}_k + u = (x_1 + u, x_2 + u, \ldots, x_k + u) \in (\mathbb{R}^2)^k$. We put

$$\triangle(\mathbf{x}_k, \mathbf{y}_\ell | u) = \frac{1}{C_{\alpha,\beta}} \{|B^{0,\alpha}_{R_0,R_\alpha}(\mathbf{x}_k) \cup B^{0,\beta}_{R_0,R_\beta}(\mathbf{y}_\ell + u)| - |B^{0,\alpha}_{R_0,R_\alpha} \cup B^{0,\beta}_{R_0,R_\beta}(u)|\},$$

and write $\triangle(\mathbf{x}_k, \mathbf{y}_\ell)$ for $\triangle(\mathbf{x}_k, \mathbf{y}_\ell | \mathbf{0})$. The following two lemmas are important to show the main theorem. Their proofs are given in the appendix.

Lemma 4.3 *Let $\mathbf{x}_k \in (\mathbb{R}^2)^k$ with $x_k = \mathbf{0}$ and $\mathbf{y}_\ell \in (\mathbb{R}^2)^\ell$ with $y_\ell = \mathbf{0}$.*
(i) *Suppose that $2H_0 < H_\alpha, H_\beta$. If*

$$\begin{aligned} M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) &< H_\alpha - 2H_0 \quad \textit{and} \\ M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) &< H_\beta - 2H_0 \end{aligned} \tag{4.5}$$

hold, then we have

$$\triangle(\mathbf{x}_k, \mathbf{y}_\ell) \leq \frac{1}{C_{\alpha,\beta}} \{|B^{0,\alpha}_{R_0, R_\alpha - R^\alpha_0}(\mathbf{x}_k) \setminus B^{0,\alpha}_{R_0, R_\alpha - R^\alpha_0}| + |B^{0,\beta}_{R_0, R_\beta - R^\beta_0}(\mathbf{y}_\ell) \setminus B^{0,\beta}_{R_0, R_\beta - R^\beta_0}|\},$$

$$\begin{aligned} \triangle(\mathbf{x}_k, \mathbf{y}_\ell) \quad &\geq \quad \frac{1}{C_{\alpha,\beta}} \{|B^{0,\alpha}_{R_0, R_\alpha - R^\alpha_0}(\mathbf{x}_k) \setminus B^{0,\alpha}_{R_0, R_\alpha - R^\alpha_0}| + |B^{0,\beta}_{R_0, R_\beta - R^\beta_0}(\mathbf{y}_\ell) \setminus B^{0,\beta}_{R_0, R_\beta - R^\beta_0}|\} \\ &- \quad M(h_\alpha(\mathbf{y}_\ell)) M(h_\beta(\mathbf{x}_k)). \end{aligned}$$

(ii) *Suppose that $2H_0 \geq \min\{H_\alpha, H_\beta\}$ and $H_\alpha > H_\beta$. If $M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) < H_\alpha - H_\beta$ and $M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) < H_\beta$ hold, then we have*

$$\begin{aligned} \triangle(\mathbf{x}_k, \mathbf{y}_\ell) \quad &\leq \quad \frac{1}{C_{\alpha,\beta}} \{|B^{0,\alpha}_{R_0, R_\alpha - \frac{1}{2} R^\alpha_\beta}(\mathbf{x}_k) \setminus B^{0,\alpha}_{R_0, R_\alpha - \frac{1}{2} R^\alpha_\beta}| + |B^{0,\beta}_{\frac{1}{2} R^0_\beta, \frac{1}{2} R_\beta}(\mathbf{y}_\ell) \setminus B^{0,\beta}_{\frac{1}{2} R^0_\beta, \frac{1}{2} R_\beta}|\} \\ &+ \quad \frac{1}{2} M(h_\beta(\mathbf{x}_k))^2 + \frac{1}{2} M(h_\alpha(\mathbf{y}_\ell))^2, \end{aligned} \tag{4.6}$$

$$
\begin{aligned}
\triangle(\mathbf{x}_k,\mathbf{y}_\ell) &\geq \frac{1}{C_{\alpha,\beta}}\{|B^{0,\alpha}_{R_0,R_\alpha-\frac{1}{2}R^\alpha_\beta}(\mathbf{x}_k)\setminus B^{0,\alpha}_{R_0,R_\alpha-\frac{1}{2}R^\alpha_\beta}|+|B^{0,\beta}_{\frac{1}{2}R^0_\beta,\frac{1}{2}R_\beta}(\mathbf{y}_\ell)\setminus B^{0,\beta}_{\frac{1}{2}R^0_\beta,\frac{1}{2}R_\beta}|\}\\
&- M(h_\beta(x_k))M(h_\beta(y_\ell))-M(h_\beta(x_k))M(h_\alpha(y_\ell))\\
&- (M(h_\beta(x_k)))^2-(M(h_\alpha(y_\ell)))^2.
\end{aligned}
$$

(iii) *Suppose that* $2H_0 \geq H_\alpha = H_\beta$. *If* $M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) < H_\alpha$ *and* $M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) < H_\beta$ *hold, then we have*

$$
\begin{aligned}
\triangle(\mathbf{x}_k,\mathbf{y}_\ell) &\leq \frac{1}{C_{\alpha,\beta}}\{|B^{0,\alpha}_{\frac{1}{2}R^0_\alpha,\frac{1}{2}R_\alpha}(\mathbf{x}_k)\setminus B^{0,\alpha}_{\frac{1}{2}R^0_\alpha,\frac{1}{2}R_\alpha}|+|B^{0,\beta}_{\frac{1}{2}R^0_\beta,\frac{1}{2}R_\beta}(\mathbf{y}_\ell)\setminus B^{0,\beta}_{\frac{1}{2}R^0_\beta,\frac{1}{2}R_\beta}|\}\\
&+ (2H_0-H_\beta)M(\overline{h}_0(\mathbf{x}_k\cdot\mathbf{y}_\ell))+\frac{1}{2}M(h_\beta(\mathbf{x}_k))^2+\frac{1}{2}M(h_\alpha(\mathbf{y}_\ell))^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\triangle(\mathbf{x}_k,\mathbf{y}_\ell) &\geq \frac{1}{C_{\alpha,\beta}}\{|B^{0,\alpha}_{\frac{1}{2}R^0_\alpha,\frac{1}{2}R_\alpha}(\mathbf{x}_k)\setminus B^{0,\alpha}_{\frac{1}{2}R^0_\alpha,\frac{1}{2}R_\alpha}|+|B^{0,\beta}_{\frac{1}{2}R^0_\beta,\frac{1}{2}R_\beta}(\mathbf{y}_\ell)\setminus B^{0,\beta}_{\frac{1}{2}R^0_\beta,\frac{1}{2}R_\beta}|\}\\
&+ (2H_0-H_\beta)M(\overline{h}_0(\mathbf{x}_k\cdot\mathbf{y}_\ell))-\frac{1}{2}M(\overline{h}_0(\mathbf{x}_k\cdot\mathbf{y}_\ell))^2\\
&- \min\{M(\overline{h}_0(\mathbf{x}_k)),M(\overline{h}_0(\mathbf{y}_\ell))\}\{M(h_\beta(\mathbf{x}_k))+M(h_\alpha(\mathbf{y}_\ell))\}.
\end{aligned}
\tag{4.7}
$$

Lemma 4.4 *Let* $\mathbf{x}_k \in (\mathbb{R}^2)^k$ *with* $x_k = \mathbf{0}$, $\mathbf{y}_\ell \in (\mathbb{R}^2)^\ell$ *with* $y_\ell = \mathbf{0}$ *and* $u \in \mathbb{R}^2$.
(i) *Suppose that* $2H_0 < H_\alpha, H_\beta$. *If* $M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) + |h_\alpha(u)| < H_\alpha - 2H_0$ *and* $M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) + |h_\beta(u)| < H_\beta - 2H_0$ *hold, then we have*

$$
\triangle(\mathbf{x}_k,\mathbf{y}_\ell|u)=\triangle(\mathbf{x}_k,\mathbf{y}_\ell).
$$

(ii) *Suppose that* $2H_0 \geq \min\{H_\alpha, H_\beta\}$ *and* $H_\alpha > H_\beta$. *If* $M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) + |h_\alpha(u)| < H_\alpha - H_\beta$ *and* $M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) + |h_\beta(u)| < H_\beta$ *hold, then we have*

$$
|\triangle(\mathbf{x}_k,\mathbf{y}_\ell|u)-\triangle(\mathbf{x}_k,\mathbf{y}_\ell)|\leq h_\beta(u)^2.
$$

(iii) *Suppose that* $2H_0 \geq H_\alpha = H_\beta$. *If* $M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) + |h_\alpha(u)| < H_\alpha$ *and* $M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) + |h_\beta(u)| < H_\beta$ *hold, then we have*

$$
\begin{aligned}
&|\triangle(\mathbf{x}_k,\mathbf{y}_\ell|u)-\triangle(\mathbf{x}_k,\mathbf{y}_\ell)\\
&\qquad -(2H_0-H_\beta)\{M(\overline{h}_0(\mathbf{x}_k\cdot(\mathbf{y}_\ell+u)))-|\overline{h}_0(u)|-M(\overline{h}_0(\mathbf{x}_k\cdot\mathbf{y}_\ell))\}\,|\\
&\leq h_\alpha(u)^2+h_\beta(u)^2+|M(\overline{h}_0(\mathbf{x}_k\cdot(\mathbf{y}_\ell+u)))-|\overline{h}_0(u)|-M(\overline{h}_0(\mathbf{x}_k\cdot\mathbf{y}_\ell))|\\
&\times\{M(h_\alpha(\mathbf{x}_k))+M(h_\alpha(\mathbf{y}_\ell))+|h_\alpha(u)|+M(h_\beta(\mathbf{x}_k))+M(h_\beta(\mathbf{y}_\ell))+|h_\beta(u)|\},
\end{aligned}
$$

if $M(h_\alpha(\mathbf{x}_k)) + M(h_\alpha(\mathbf{y}_\ell)) + |h_\alpha(u)| < H_\alpha$, $M(h_\beta(\mathbf{x}_k)) + M(h_\beta(\mathbf{y}_\ell)) + |h_\beta(u)| < H_\beta$.

4.2 The asymptotic shape

First, we examine the behaviour of the function $\mu_{\lambda\rho}(C_\mathbf{0} \in \Lambda(\mathbf{k})|\Gamma_0)$ as $\lambda \to \infty$ when $\mathbf{k} = (0, k_\alpha, k_\beta)$. When $\mathbf{k} = (k_0, k_\alpha, 0)$ or $\mathbf{k} = (k_0, 0, k_\beta)$, we can estimate similarly. From (3.1) we have

$$
\mu_{\lambda\rho}(C_\mathbf{0}\in\Lambda(0,k_\alpha,k_\beta)\mid\Gamma_0)=\lambda^{|\mathbf{k}|-1}|\mathbf{k}|\frac{p_\alpha^{k_\alpha}p_\beta^{k_\beta}}{(k_\alpha)!k_\beta!}F_\lambda(0,k_\alpha,k_\beta),
\tag{4.8}
$$

where

$$\begin{aligned} F_\lambda(0,k_\alpha,k_\beta) &= \int\limits_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{y}_{k_\alpha-1} \int\limits_{(\mathbb{R}^2)^{k_\beta}} d\mathbf{z}_{k_\beta} 1_{\Lambda(0,k_\alpha,k_\beta)}(C_\mathbf{0}(\mathbf{y}_{k_\alpha},\mathbf{z}_{k_\beta})) \\ &\times \quad e^{-\lambda\{p_0|B^{\alpha,0}_{R_\alpha,R_0}(\mathbf{y}_{k_\alpha})\cup B^{\beta,0}_{R_\beta,R_0}(\mathbf{z}_{k_\beta})|+p_\alpha|B^{\beta,\alpha}_{R_\beta,R_\alpha}(\mathbf{z}_{k_\beta})|+p_\beta|B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{y}_{k_\alpha})|\}}. \end{aligned}$$

We put

$$\begin{aligned} &\Phi(\mathbf{p}) = p_0|B^{\alpha,0}_{R_\alpha,R_0}\cup B^{\beta,0}_{R_\beta,R_0}| + p_\alpha|B^{\beta,\alpha}_{R_\beta,R_\alpha}| + p_\beta|B^{\alpha,\beta}_{R_\alpha,R_\beta}|, \\ &f_\lambda(0,k_\alpha,k_\beta) = F_\lambda(0,k_\alpha,k_\beta)e^{\lambda\Phi(\mathbf{p})}. \end{aligned} \tag{4.9}$$

To examine the function $f_\lambda(\mathbf{k})$, we introduce the following function

$$\chi_c^{\theta_1,\theta_2,\theta_3}(\mathbf{x},\mathbf{y}|z) = e^{-c\{|B^{\theta_1,\theta_2}_{R_{\theta_1},R_{\theta_2}}(\mathbf{x})\cup B^{\theta_1,\theta_3}_{R_{\theta_1},R_{\theta_3}}(\mathbf{y}+z)|-|B^{\theta_1,\theta_2}_{R_{\theta_1},R_{\theta_2}}\cup B^{\theta_1,\theta_3}_{R_{\theta_1},R_{\theta_3}}(z)|\}}, \tag{4.10}$$

for $\theta_1,\theta_2,\theta_3\in[0,\pi)$, $c>0$, $\mathbf{x}\in(\mathbb{R}^2)^k$ $\mathbf{y}\in(\mathbb{R}^2)^{k'}$, $k,k'\in\mathbb{N}$ and $z\in\mathbb{R}^2$. We write $\chi_c^{\theta_1,\theta_2,\theta_3}(\mathbf{x},\mathbf{y})$ for $\chi_c^{\theta_1,\theta_2,\theta_3}(\mathbf{x},\mathbf{y}|\mathbf{0})$. By using these functions we obtain

$$\begin{aligned} f_\lambda(0,k_\alpha,k_\beta) &= \int\limits_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{y}_{k_\alpha-1} \int\limits_{(\mathbb{R}^2)^{k_\beta}} d\mathbf{z}_{k_\beta} 1_{\Lambda(0,k_\alpha,k_\beta)}(C_\mathbf{0}(\mathbf{y}_{k_\alpha},\mathbf{z}_{k_\beta})) \\ &\qquad \times\chi^{0,\alpha,\beta}_{\lambda p_0}(\mathbf{y}_{k_\alpha},\mathbf{z}_{k_\beta})\chi^{\alpha,\beta}_{\lambda p_\alpha}(\mathbf{z}_{k_\beta})\chi^{\alpha,\beta}_{\lambda p_\beta}(\mathbf{y}_{k_\alpha}). \end{aligned}$$

Putting $\mathbf{u}_{k_\alpha}=\mathbf{y}_{k_\alpha}-y_{k_\alpha}$, $\mathbf{v}_{k_\beta}=\mathbf{z}_{k_\beta}-z_{k_\beta}$ and $z_\beta=z$, we have

$$f_\lambda(0,k_\alpha,k_\beta) = \int\limits_{\mathbb{R}^2} dz g_\lambda(0,k_\alpha,k_\beta,z)\chi^{0,\alpha,\beta}_{\lambda p_0}(\mathbf{0},z),$$

where

$$\begin{aligned} g_\lambda(0,k_\alpha,k_\beta,z) &= \int\limits_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{u}_{k_\alpha-1} \int\limits_{(\mathbb{R}^2)^{k_\beta-1}} d\mathbf{v}_{k_\beta-1} 1_{\Lambda(0,k_\alpha,k_\beta)}(C_\mathbf{0}(\mathbf{u}_{k_\alpha},\mathbf{v}_{k_\beta}+z)) \\ &\times \quad \chi^{0,\alpha,\beta}_{\lambda p_0}(\mathbf{u}_{k_\alpha},\mathbf{v}_{k_\beta}|z)\chi^{\alpha,\beta}_{\lambda p_\alpha}(\mathbf{v}_{k_\beta})\chi^{\alpha,\beta}_{\lambda p_\beta}(\mathbf{u}_{k_\alpha}). \end{aligned} \tag{4.11}$$

Writing $g_\lambda(\mathbf{k})$ for $g_\lambda(\mathbf{k},\mathbf{0})$, we have

$$\begin{aligned} &\mu_{\lambda\rho}(C_\mathbf{0}\in\Lambda(0,k_\alpha,k_\beta)\mid\ \Gamma_\mathbf{0}) \\ =\quad & e^{-\lambda\Phi(\mathbf{p})}\lambda^{|\mathbf{k}|-1}|\mathbf{k}|\frac{p_\alpha^{k_\alpha}p_\beta^{k_\beta}}{k_\alpha!k_\beta!}\int\limits_{\mathbb{R}^2} dz g_\lambda(0,k_\alpha,k_\beta,z)\chi^{0,\alpha,\beta}_{\lambda p_0}(\mathbf{0},z). \end{aligned} \tag{4.12}$$

Remark 4.2. The function $\chi^{0,\alpha,\beta}_{\lambda p_0}$ determines the structure of finite clusters. From Remark 4.1 we see that $\chi^{0,\alpha,\beta}_{\lambda p_0}(0,z)=\exp[-\lambda p_0 C_{\alpha,\beta}\triangle(z)]=1$ if and only if

$$\begin{aligned} &z\in B^{\alpha,\beta}_{R_\alpha-2R_0^\alpha,R_\beta-2R_0^\beta}, \qquad \text{when } H_\alpha,H_\beta>2H_0, \\ &z\in B^{\alpha,\beta}_{[R_\alpha-R_\beta^\alpha]_+,[R_\beta-R_\alpha^\beta]_+}, \qquad \text{when } \min\{H_\alpha,H_\beta\}\le 2H_0. \end{aligned}$$

We divide into four cases and obtain estimates.

Case (1) $2H_0 < H_\alpha, H_\beta$. In this case we will show that

$$\begin{aligned}
&\mu_{\lambda\rho}(C_{\mathbf{0}} \in \Lambda(0, k_\alpha, k_\beta)|\Gamma_0) \\
&\quad \sim \exp[-4C_{\alpha,\beta}\lambda\{p_0 H_0(H_\alpha + H_\beta - H_0) + (1-p_0)H_\alpha H_\beta\}] \\
&\quad \times \left(\frac{1}{4C_{\alpha,\beta}\lambda}\right)^{|\mathbf{k}|-3} |\mathbf{k}| H_\alpha H_\beta (H_\alpha - 2H_0)(H_\beta - 2H_0) \\
&\quad \times p_\alpha^{k_\alpha} k_\alpha! G^{k_\alpha}(p_0 H_0 + p_\beta H_\beta, p_\beta H_\alpha, p_0(H_\alpha - H_0)) \\
&\quad \times p_\beta^{k_\beta} k_\beta! G^{k_\beta}(p_\alpha H_\beta, p_0 H_0 + p_\alpha H_\alpha, p_0(H_\beta - H_0)), \qquad (4.13)
\end{aligned}$$

where for $c_1, c_2, c_3 > 0$

$$G^k(c_1, c_2, c_3) = (\frac{1}{k!})^2 \int_{(\mathbb{R}^2)^{k-1}} d\mathbf{u}_{k-1} \gamma^k(c_1, c_2, c_3)(\mathbf{u}_k), \qquad (4.14)$$

$$\gamma^k(c_1, c_2, c_3)(\mathbf{u}_k) = \exp[-\{c_1 M(\mathbf{u}_k^1) + c_2 M(\mathbf{u}_k^2) + c_3 M(\mathbf{u}_k^1 + \mathbf{u}_k^2)\}]. \qquad (4.15)$$

From Remark 4.2 we see that the asymptotic shape of the cluster is given by

$$\{x \in \mathbb{R}^2 : |h_\alpha(x)| \le H_\alpha - 2H_0,\ |h_\beta(x)| \le H_\beta - 2H_0\}.$$

By Lemma 4.2 (i) and Lemma 4.4 (i) we have

$$f_\lambda(0, k_\alpha, k_\beta) \sim |B^{\alpha,\beta}_{R_\alpha - 2R_0^\alpha, R_\beta - 2R_0^\beta}| g_\lambda(0, k_\alpha, k_\beta), \quad \text{as } \lambda \to \infty. \qquad (4.16)$$

By Lemma 4.3 (i) we have

$$\begin{aligned}
&g_\lambda(0, k_\alpha, k_\beta) \\
&\sim \int_{(\mathbb{R}^2)^{k_\alpha - 1}} d\mathbf{u}_{k_\alpha - 1} e^{-\lambda\{p_0|B^{0,\alpha}_{R_0, R_\alpha - R_0^\alpha}(\mathbf{u}_{k_\alpha}) \setminus B^{0,\alpha}_{R_0, R_\alpha - R_0^\alpha}| + p_\beta |B^{\alpha,\beta}_{R_\alpha, R_\beta}(\mathbf{u}_{k_\alpha}) \setminus B^{\alpha,\beta}_{R_\alpha, R_\beta}|\}} \\
&\times \int_{(\mathbb{R}^2)^{k_\beta - 1}} d\mathbf{v}_{k_\beta - 1} e^{-\lambda\{p_0|B^{0,\beta}_{R_0, R_\beta - R_0^\beta}(\mathbf{v}_{k_\beta}) \setminus B^{0,\beta}_{R_0, R_\beta - R_0^\beta}| + p_\alpha |B^{\alpha,\beta}_{R_\alpha, R_\beta}(\mathbf{v}_{k_\beta}) \setminus B^{\alpha,\beta}_{R_\alpha, R_\beta}|\}}
\end{aligned}$$

Using Lemma 3.1 and putting $\hat{\mathbf{u}} = A^{\alpha,\beta}_{2\lambda \sin\beta, 2\lambda\sin\alpha}\mathbf{u}$, by a simple calculation we have

$$\begin{aligned}
&\int_{(\mathbb{R}^2)^{k_\alpha - 1}} d\mathbf{u}_{k_\alpha - 1} e^{-\lambda\{p_0|B^{0,\alpha}_{R_0, R_\alpha - R_0^\alpha}(\mathbf{u}_{k_\alpha}) \setminus B^{0,\alpha}_{R_0, R_\alpha - R_0^\alpha}| + p_\beta |B^{\alpha,\beta}_{R_\alpha, R_\beta}(\mathbf{u}_{k_\alpha}) \setminus B^{\alpha,\beta}_{R_\alpha, R_\beta}|\}} \\
&\sim \int_{(\mathbb{R}^2)^{k_\alpha - 1}} d\mathbf{u}_{k_\alpha - 1} e^{-2C_{\alpha,\beta}\lambda[(p_0 H_0 + p_\beta H_\beta) M(h_\alpha(\mathbf{u}_{k_\alpha})) + p_0(H_\alpha - H_0) M(\overline{h}_0(\mathbf{u}_{k_\alpha})) + p_\beta H_\beta M(h_\alpha(\mathbf{u}_{k_\alpha}))]} \\
&= \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{k_\alpha - 1} G^{k_\alpha}(p_0 H_0 + p_\beta H_\beta, p_\beta H_\alpha, p_0(H_\alpha - H_0)).
\end{aligned}$$

Similarly, we have

$$\int_{(\mathbb{R}^2)^{k_\beta-1}} d\mathbf{v}_{k_\beta-1} e^{-\lambda\{p_0|B^{0,\beta}_{R_0,R_\beta-R_0^\beta}(\mathbf{v}_{k_\beta})\backslash B^{0,\beta}_{R_0,R_\beta-R_0^\beta}|+p_\alpha|B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{v}_{k_\beta})\backslash B^{\alpha,\beta}_{R_\alpha,R_\beta}|\}}$$

$$\sim \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{k_\beta-1} G^{k_\beta}(p_\alpha H_\beta, p_0 H_0 + p_\alpha H_\alpha, p_0(H_\beta - H_0))$$

Since by Lemma 4.1 (i)

$$\Phi(\mathbf{p}) = 4C_{\alpha,\beta}\{p_0 H_0(H_\alpha + H_\beta - H_0) + (1-p_0)H_\alpha H_\beta\}, \tag{4.17}$$

we have (4.13) from (4.12) and the above estimates.

Case (2) $2H_0 \geq H_\beta$, $H_\alpha > H_\beta$. In this case we will show that

$$\begin{aligned}
&\mu_{\lambda\rho}(C_{\mathbf{0}} \in \Lambda(0,k_\alpha,k_\beta)|\Gamma_0) \\
&\quad \sim \exp[-4C_{\alpha,\beta}\lambda\{p_0 H_0 H_\alpha + \frac{p_0}{4}H_\beta^2 + (1-p_0)H_\alpha H_\beta\}] \\
&\quad \times \left(\frac{1}{4C_{\alpha,\beta}\lambda}\right)^{|\mathbf{k}|-\frac{5}{2}} |\mathbf{k}||H_\alpha - H_\beta|(\frac{\pi}{p_0})^{\frac{1}{2}} \\
&\quad \times p_\alpha^{k_\alpha} k_\alpha! G^{k_\alpha}(p_0 H_0 + p_\beta H_\beta, p_\beta H_\alpha, p_0(H_\alpha - \frac{1}{2}H_\beta)) \\
&\quad \times p_\beta^{k_\beta} k_\beta! G^{k_\beta}(p_\alpha H_\beta, \frac{1}{2}p_0 H_\beta + p_\alpha H_\alpha, \frac{1}{2}p_0 H_\beta).
\end{aligned} \tag{4.18}$$

From Remark 4.2 we see that the asymptotic shape of the cluster is given by

$$\{x \in \mathbb{R}^2 : |h_\alpha(x)| \leq H_\alpha - H_\beta,\ |h_\beta(x)| = 0\}.$$

By Lemma 4.4 (ii) and a simple calculation we have

$$g_\lambda(0,k_\alpha,k_\beta,z) \sim g_\lambda(0,k_\alpha,k_\beta) \quad \text{as } \lambda \to \infty,$$

when $|h_\alpha(z)| < H_\alpha - H_\beta$ $|h_\beta(z)| = o(1)$. ¿From Lemma 4.2 (ii) we have

$$\chi^{0,\alpha.\beta}_{\lambda p_0}(\mathbf{0},z) = e^{-p_0 C_{\alpha,\beta}\lambda h_\beta(z)^2},$$

if $|\overline{h}_0(z)| \leq H_\alpha - H_\beta$, $|h_\beta(z)| \leq 2H_0 - H_\beta$. Then we have

$$\begin{aligned}
f_\lambda(0,k_\alpha,k_\beta) &\sim g_\lambda(0,k_\alpha,k_\beta)\int_{\mathbb{R}^2} dz\chi^{0,\alpha,\beta}_{\lambda p_0}(\mathbf{0},z) \\
&\sim 2|H_\alpha - H_\beta|(\frac{C_{\alpha,\beta}\pi}{p_0\lambda})^{1/2} g_\lambda(0,k_\alpha,k_\beta) \quad \text{as } \lambda \to \infty.
\end{aligned} \tag{4.19}$$

By Lemma 3.1 and Lemma 4.3 (ii) and similar calculations as above, we have

$$\begin{aligned}
g_\lambda(0,k_\alpha,k_\beta) &\sim \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{k_\alpha-1} G^{k_\alpha}(p_0 H_0 + p_\beta H_\beta, p_\beta H_\alpha, p_0(H_\alpha - \frac{1}{2}H_\beta)) \\
&\times \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{k_\beta-1} G^{k_\beta}(p_\alpha H_\beta, \frac{1}{2}p_0 H_\beta + p_\alpha H_\alpha, \frac{1}{2}p_0 H_\beta).
\end{aligned}$$

Since by Lemma 3.1 (ii)

$$\Phi(\mathbf{p}) = 4C_{\alpha,\beta}\{p_0H_0H_\alpha + \frac{p_0}{4}H_\beta^2 + (1-p_0)H_\alpha H_\beta\}, \tag{4.20}$$

we have (4.18) from (4.12) and the above estimates

Case (3) $2H_0 = H_\alpha = H_\beta$. In this case we will show that

$$\begin{aligned}
&\mu_{\lambda\rho}(C_{\mathbf{0}} \in \Lambda(0,k_\alpha,k_\beta)|\Gamma_0) \\
&\quad \sim \exp[-4C_{\alpha,\beta}\lambda(4-p_0)H_0^2] \\
&\quad \times \left(\frac{1}{4C_{\alpha,\beta}\lambda}\right)^{|\mathbf{k}|-2}|\mathbf{k}|\frac{3\pi+4}{2p_0} \\
&\quad \times p_\alpha^{k_\alpha}k_\alpha!G^{k_\alpha}((p_0+2p_\beta)H_0, 2p_\beta H_0, p_0H_0) \\
&\quad \times p_\beta^{k_\beta}k_\beta!G^{k_\beta}(2p_\alpha H_0, (p_0+2p_\alpha)H_0, p_0H_0).
\end{aligned} \tag{4.21}$$

From Remark 4.2 we see that the asymptotic shape of the cluster is given by

$$\{x \in \mathbb{R}^2 : |h_\alpha(x)| \le 0,\ |h_\beta(x)| \le 0\} = \{\mathbf{0}\}.$$

By Lemma 4.4 (iii) and a simple calculation we have

$$g_\lambda(0,k_\alpha,k_\beta,z) \sim g_\lambda(0,k_\alpha,k_\beta) \quad \text{as } \lambda \to \infty,$$

when $|h_\alpha(z)| = o(1)$, $|h_\beta(z)| = o(1)$. ¿From Lemma 4.2 (ii) we have

$$\chi_{\lambda p_0}^{0,\alpha,\beta}(\mathbf{0},z) = \begin{cases} \exp[-\frac{1}{2}C_{\alpha,\beta}p_0\lambda(h_\alpha(z)^2 + h_\beta(z)^2)], & \overline{h}_0(z)h_\beta(z) > 0, \\ \exp[-\frac{1}{2}C_{\alpha,\beta}p_0\lambda h_\alpha(z)^2], & \overline{h}_0(z)h_\beta(z) > 0, \end{cases} \tag{4.22}$$

if $|h_\alpha(z)| \le H_\alpha$, $|h_\beta(z)| \le H_\beta$. Then we have

$$\begin{aligned}
f_\lambda(0,k_\alpha,k_\beta) &\sim g_\lambda(0,k_\alpha,k_\beta)\int_{\mathbb{R}^2} dz\chi_{\lambda p_\beta}^{0,\alpha,\beta}(\mathbf{0},z) \\
&\sim (\frac{3\pi+4}{2p_0\lambda})g_\lambda(0,k_\alpha,k_\beta) \quad \text{as } \lambda \to \infty.
\end{aligned} \tag{4.23}$$

By Lemma 3.1 and Lemma 4.3 (iii) and similar calculations as above, we have

$$\begin{aligned}
g_\lambda(0,k_\alpha,k_\beta) &\sim \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{k_\alpha-1}G^{k_\alpha}(\frac{1}{2}p_0H_\alpha + p_\beta H_\beta, p_\beta H_\alpha, \frac{1}{2}p_0H_\alpha) \\
&\times \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{k_\beta-1}G^{k_\beta}(p_\alpha H_\beta, \frac{1}{2}p_0H_\beta + p_\alpha H_\alpha, \frac{1}{2}p_0H_\beta).
\end{aligned}$$

Since by Lemma 3.1 (ii), $\Phi(\mathbf{p}) = 4C_{\alpha,\beta}(4-p_0)H_0^2)$, we have (4.21) from (4.12) and the above estimates

Case (4) $2H_0 > H_\alpha = H_\beta$. In this case we will show that

$$\begin{aligned}
&\mu_{\lambda\rho}(C_{\mathbf{0}} \in \Lambda(0,k_\alpha,k_\beta)|\Gamma_0) \\
&\quad \sim \exp[-4C_{\alpha,\beta}\lambda\{p_0 H_0 H_\alpha + (1-\frac{3}{4}p_0)H_\alpha^2\}] \\
&\quad \times \left(\frac{1}{4C_{\alpha,\beta}\lambda}\right)^{|\mathbf{k}|-\frac{3}{2}} |\mathbf{k}|(\frac{2\pi}{p_0})^{\frac{1}{2}} p_\alpha^{k_\alpha} k_\alpha! p_\beta^{k_\beta} k_\beta! \\
&\quad \times G^{k_\alpha,k_\beta}_{\frac{1}{2}(2H_0-H_\alpha)}((\frac{p_0}{2}+p_\beta)H_\alpha, p_\beta H_\alpha, \frac{p_0}{2}H_\alpha, p_\alpha H_\alpha, (\frac{p_0}{2}+p_\alpha)H_\alpha, \frac{p_0}{2}H_\alpha).
\end{aligned} \tag{4.24}$$

where

$$\begin{aligned}
G_z^{k,\ell}(c_1,c_2,c_3,c_4,c_5,c_6) &= (\frac{1}{k!})^2(\frac{1}{\ell!})^2 \int\limits_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{u}_{k_\alpha-1} \int\limits_{(\mathbb{R}^2)^{k_\beta-1}} d\mathbf{v}_{k_\beta-1} \\
&\quad \times J_z(\mathbf{u}_{k_\alpha},\mathbf{v}_{k_\beta})\gamma(c_1,c_2,c_3)(\mathbf{u}_{k_\alpha})\gamma(c_4,c_5,c_6)(\mathbf{v}_{k_\beta}),
\end{aligned}$$

From Remark 4.2 we see that the asymptotic shape of the cluster is given by

$$\{x \in \mathbb{R}^2 : |h_\alpha(x)| \le 0,\ |h_\beta(x)| \le 0\} = \{\mathbf{0}\}.$$

By Lemma 4.3 (iii), Lemma 4.4 (iii) and a simple calculation we have

$$\begin{aligned}
\triangle(\mathbf{x}_k,\mathbf{y}_\ell|z) &\sim \frac{1}{C_{\alpha,\beta}}\{|B^{0,\alpha}_{\frac{1}{2}R^0_\alpha,\frac{1}{2}R_\alpha}(\mathbf{x}_k) \setminus B^{0,\alpha}_{\frac{1}{2}R^0_\alpha,\frac{1}{2}R_\alpha}| + |B^{0,\beta}_{\frac{1}{2}R^0_\beta,\frac{1}{2}R_\beta}(\mathbf{y}_\ell) \setminus B^{0,\beta}_{\frac{1}{2}R^0_\beta,\frac{1}{2}R_\beta}|\} \\
&+ (2H_0 - H_\beta)\{M(\overline{h}_0(\mathbf{x}_k \times (\mathbf{y}_\ell + z))) - \overline{h}_0(z)\}
\end{aligned}$$

when $|h_\alpha(z)| = o(1)$, $|h_\beta(z)| = o(1)$. ¿From Lemma 4.2 (ii)

$$\triangle(z) = \frac{1}{2}(h_\alpha(z)^2 + h_\beta(z)^2) + (2H_0 - H_\beta)|\overline{h}_0(z)|,$$

if $|h_\alpha(z)| \le H_\alpha$, $|h_\beta(z)| \le H_\beta$. Then

$$\begin{aligned}
f_\lambda(0,k_\alpha,k_\beta) &\sim \int\limits_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{u}_{k_\alpha-1} \int\limits_{(\mathbb{R}^2)^{k_\beta-1}} d\mathbf{v}_{k_\beta-1} K_\lambda(\mathbf{u}_{k_\alpha},\mathbf{v}_{k_\beta}) \\
&\quad \times \chi^{0,\alpha,\beta}_{\lambda p_0}(\mathbf{u}_{k_\alpha},\mathbf{v}_{k_\beta})\chi^{\alpha,\beta}_{\lambda p_\alpha}(\mathbf{v}_{k_\beta})\chi^{\alpha,\beta}_{\lambda p_\beta}(\mathbf{u}_{k_\alpha}),
\end{aligned}$$

where

$$\begin{aligned}
K_\lambda(\mathbf{u}_{k_\alpha},\mathbf{v}_{k_\beta}) &= \int_{\mathbb{R}^2} dz \exp[-\frac{1}{2}C_{\alpha,\beta}p_0\lambda(h_\alpha(z)^2 + h_\beta(z)^2)] \\
&\quad \times \exp[-\lambda C_{\alpha,\beta}p_0(2H_0 - H_\beta)M(\overline{h}_0(\mathbf{u}_{k_\alpha}\cdot(\mathbf{v}_{k_\beta}+z)))].
\end{aligned}$$

By Lemma 3.1 and Lemma 4.3 (iii) and similar calculations as above, we have

$$\begin{aligned}
f_\lambda(0,k_\alpha,k_\beta) &\sim \left(\frac{1}{4C_{\alpha,\beta}\lambda^2}\right)^{|k|-1}\left(\frac{8\pi C_{\alpha,\beta}\lambda}{p_0}\right)^{\frac{1}{2}} \\
&\times \int\limits_{(\mathbb{R}^2)^{k_\alpha-1}} d\mathbf{u}_{k_\alpha-1} \int\limits_{(\mathbb{R}^2)^{k_\beta-1}} d\mathbf{v}_{k_\beta-1} J_{\frac{p_0}{2}(2H_0-H_\beta)}(\mathbf{u}_{k_\alpha},\mathbf{v}_{k_\beta}) \\
&\times \gamma^{k_\alpha}((\frac{1}{2}p_0 + p_\beta)H_\alpha, p_\beta H_\alpha, \frac{1}{2}p_0 H_\alpha)\gamma^{k_\beta}(p_\alpha H_\alpha, (\frac{1}{2}p_0 + p_\alpha)H_\alpha, \frac{1}{2}p_0 H_\alpha).
\end{aligned}$$

Since by Lemma 4.1 (ii), $\Phi(\mathbf{p}) = 4C_{\alpha,\beta}\{p_0 H_0 H_\alpha(1-\frac{3}{4}p_0)H_\alpha^2\}$, we have (4.24) from (4.12) and the above estimates.

Proof of Theorem 2.2 First we examine the behaviour of the function $\mu_{\lambda\rho}(C_{\mathbf{0}} \in \Lambda(\mathbf{k})|\Gamma_0)$ as $\lambda \to \infty$ when $\mathbf{k} = (k_0, k_\alpha, k_\beta)$, with $k_0, k_\alpha, k_\beta \in \mathbb{N}$. From (1.3) and an argument similar to that needed to obtain (4.1) we have

$$\mu_{\lambda\rho}(C_{\mathbf{0}} \in \Lambda(\mathbf{k}) \mid \ \Gamma_{\mathbf{0}}) = \lambda^{|\mathbf{k}|-1}|\mathbf{k}|\frac{p_0^{k_0}p_\alpha^{k_\alpha}p_\beta^{k_\beta}}{k_0!k_\alpha!k_\beta!}F_\lambda(\mathbf{k}), \tag{4.25}$$

where

$$\begin{aligned} F_\lambda(\mathbf{k}) \quad = \quad & e^{-\lambda\{p_0|B^{\alpha,0}_{R_\alpha,R_0}\cup B^{\beta,0}_{R_\beta,R_0}|+p_\alpha|B^{0,\alpha}_{R_0,R_\alpha}\cup B^{\beta,\alpha}_{R_\beta,R_\alpha}|+p_\beta|B^{0,\beta}_{R_0,R_\beta}\cup B^{\alpha,\beta}_{R_\alpha,R_\beta}|\}} \\ & \times \int\limits_{(\mathbb{R}^2)^{k_0-1}} d\mathbf{x}_{k_0-1} \int\limits_{(\mathbb{R}^2)^{k_\alpha}} d\mathbf{y}_{k_\alpha} \int\limits_{(\mathbb{R}^2)^{k_\beta}} d\mathbf{z}_{k_\beta} 1_{\Lambda(\mathbf{k})}(C_{\mathbf{0}}(\mathbf{x}_{k_0}, \mathbf{y}_{k_\alpha}, \mathbf{z}_{k_\beta})) \\ & \times\chi^{0,\alpha,\beta}_{\lambda p_0}(\mathbf{y}_{k_\alpha}, \mathbf{z}_{k_\beta})\chi^{\alpha,\beta,0}_{\lambda p_\alpha}(\mathbf{z}_{k_\beta}, \mathbf{x}_{k_0})\chi^{\beta,0,\alpha}_{\lambda p_\beta}(\mathbf{x}_{k_0}, \mathbf{y}_{k_\alpha}). \end{aligned}$$

From the above we see that the probability that the cluster contains sticks of three distinct orientations is much smaller than that of only two distinct orientations.

For **case (1)**, when $a, b \geq 2$, from (4.13), (4.21) and (4.18) we have

$$\begin{aligned} \lim_{\lambda\to\infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(0,k,\ell)|\Gamma_0) &= p_0(a+b-1) + (1-p_0)ab, \\ \lim_{\lambda\to\infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k,0,\ell)|\Gamma_0) &= p_\alpha ab + \frac{p_\alpha}{4} + (1-p_\alpha)b, \\ \lim_{\lambda\to\infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k,\ell,0)|\Gamma_0) &= p_\beta ab + \frac{p_\beta}{4} + (1-p_\beta)a. \end{aligned}$$

Since

$$p_0(a+b-1) + (1-p_0)ab > \min\{p_\alpha ab + \frac{p_\alpha}{4} + (1-p_\alpha)b, p_\beta ab + \frac{p_\beta}{4} + (1-p_\beta)a\},$$

we obtain Theorem 2.2 (1) (i) and (ii). From (4.18) we see that

$$\mu_{\lambda\rho}(C_0 \in \Lambda(k,0,\ell)|\Gamma_0)\exp\{\lambda\Phi(\mathbf{p})\} \sim c\lambda^{k+\ell-5/2},$$

and

$$\mu_{\lambda\rho}(C_0 \in \Lambda(k,\ell,0)|\Gamma_0)\exp\{\lambda\Phi(\mathbf{p})\} \sim c'\lambda^{k+\ell-5/2},$$

with positive constants c and c' independent of λ. Thus we have (iii).

For **case (2)**, when $1/2 < \min\{a,b\} < 2$, $a \neq b$, $a, b \neq 1$, from (4.18) we have

$$\begin{aligned} \lim_{\lambda\to\infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(0,k,\ell)|\Gamma_0) &= f(0,\alpha,\beta), \\ \lim_{\lambda\to\infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k,0,\ell)|\Gamma_0) &= f(\beta,0,\alpha) \\ \lim_{\lambda\to\infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k,\ell,0)|\Gamma_0) &= f(\alpha,\beta,0). \end{aligned}$$

Thus we obtain Theorem 2.2 (2).

For **case (3)**, when $0 < a = b < 1$, from (4.18) and (4.21) we have

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(0,k,\ell)|\Gamma_0) &= p_0 a + (1 - \frac{3}{4}p_0)a^2, \\
\lim_{\lambda\to\infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k,0,\ell)|\Gamma_0) &= \frac{1}{4}p_\alpha a^2 + a, \\
\lim_{\lambda\to\infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k,\ell,0)|\Gamma_0) &= \frac{1}{4}p_\beta a^2 + a.
\end{aligned}$$

If $p_\alpha \geq p_\beta$, $A(\alpha,\beta)$ occurs whenever

$$p_0 a + (1 - \frac{3}{4}p_0)a^2 < \frac{1}{4}p_\beta a^2 + a,$$

i.e., $a < \mathbf{l}_1(p_0, p_\alpha, p_\beta)$. Since $\mathbf{l}_1(p_0, p_\alpha, p_\beta) \geq 1$ for $p_0 \leq p_\beta$, we obtain Theorem 2.2 (3).

Finally for **case (4)**, when $1 < a = b < 2$, from (4.18) and (4.21) we have

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(0,k,\ell)|\Gamma_0) &= p_0 a + (1 - \frac{3}{4}p_0)a^2, \\
\lim_{\lambda\to\infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k,0,\ell)|\Gamma_0) &= p_\alpha a^2 + \frac{1}{4}p_\alpha + (1-p_\alpha)a, \\
\lim_{\lambda\to\infty} \frac{-1}{4C_{\alpha,\beta}\lambda} \log \mu_{\lambda\rho}(C_0 \in \Lambda(k,\ell,0)|\Gamma_0) &= p_\beta a^2 + \frac{1}{4}p_\beta + (1-p_\beta)a.
\end{aligned}$$

If $p_\alpha \geq p_\beta$, we see that $A(\alpha,\beta)$ occurs whenever

$$p_0 a + (1 - \frac{3}{4}p_0)a^2 < p_\beta a^2 + \frac{1}{4}p_\beta + (1-p_\beta)a,$$

i.e., $a < \mathbf{l}_2(p_0, p_\alpha, p_\beta)$. Since $\mathbf{l}_2(p_0, p_\alpha, p_\beta) \leq 1$ for $p_0 \geq p_\beta$, we obtain Theorem 2.2 (4).

Also for **case (4)** $a = b = 1$, from (4.18) and (4.21) we have Theorem 2.2 (5), easily.

5 Appendix

Proof of Lemma 3.1: We bound the volume of $B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{x}_k)$ by the volume of the smallest parallelogram containing it.

$$\begin{aligned}
|B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{x}_k)| &\leq (2R_\alpha + M(\mathbf{x}^\alpha_k))(2R_\beta + M(\mathbf{x}^\beta_k))\sin(\beta-\alpha) \\
&= 2R_\alpha 2R_\beta \sin(\beta-\alpha) + (2R_\alpha M(\mathbf{x}^\beta_k) + 2R_\beta M(\mathbf{x}^\alpha_k))\sin(\beta-\alpha) \\
&\qquad + M(\mathbf{x}^\alpha_k)M(\mathbf{x}^\beta_k)\sin(\beta-\alpha) \\
&= |B^{\alpha,\beta}_{R_\alpha,R_\beta}| + 2C_{\alpha,\beta}\{H_\alpha M(h_\beta(\mathbf{x}_k)) + H_\beta M(h_\alpha(\mathbf{x}_k))\} \\
&\qquad + C_{\alpha,\beta} M(h_\beta(\mathbf{x}_k))M(h_\alpha(\mathbf{x}_k))
\end{aligned}$$

which yields (3.5).

The inequality (3.6) follows on observing that

(i) $|B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{x}_k)|$ must include an area $2R_\alpha \max\{x^\beta_1, \ldots, x^\beta_k\}\sin(\beta-\alpha)$ along the 'length' of the

connected cluster,
(ii) $|B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{x}_k)|$ must must include an area $2R_\beta \max\{x^\alpha_1,\dots,x^\alpha_k\}\sin(\beta-\alpha)$ along the 'breadth' of the connected cluster.
Thus removing the double counting obtained when we consider the parallelograms along the breadth of the cluster we obtain (3.6).

To show the last inequality we must estimate the double counting more precisely. Observe that the two halves of the parallelograms on the extremes (in either of the two directions α or β) of the region $B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{x}_k)$ constitute an area $|B^{\alpha,\beta}_{R_\alpha,R_\beta}|$. Also if $B^{\alpha,\beta}_{R_\alpha,R_\beta}(\mathbf{x}_k)$ is connected, then the area of this region between the lines $\{x\in\mathbb{R}^2 : x^\alpha = \min\{x^\alpha_1,\dots,x^\alpha_k\}\}$ and $\{x\in\mathbb{R}^2 : x^\alpha = \max\{x^\alpha_1,\dots,x^\alpha_k\}\}$ has an area at least $(2R_\alpha\max\{x^\beta_1,\dots,x^\beta_k\}+2R_\beta\max\{x^\alpha_1,\dots,x^\alpha_k\})\sin(\beta-\alpha)-\max\{x^\alpha_1,\dots,x^\alpha_k\}\max\{x^\beta_1,\dots,x^\beta_k\}\sin(\beta-\alpha)$. Since $x_k=\mathbf{0}$, (3.7) follows. ■

Proof of Lemma 4.1 If $2H_0\ge H_\beta$ and $H_\alpha\ge H_\beta$. Then

$$\begin{aligned}
|B^{0,\alpha}_{R_0,R_\alpha}\cup B^{0,\beta}_{R_0,R_\beta}| &= |B^{0,\alpha}_{R_0,R_\alpha}\setminus B^{0,\beta}_{R_0,R_\beta}|+|B^{0,\beta}_{R_0,R_\beta}\setminus B^{0,\alpha}_{R_0,R_\alpha}|+|B^{0,\alpha}_{R_0,R_\alpha}\cap B^{0,\beta}_{R_0,R_\beta}|\\
&= 2R_0.2R_\alpha\sin\alpha+R_\beta\sin(\pi-\beta)R_\beta\sin(\beta-\alpha)(\sin\alpha)^{-1}\\
&= C_{\alpha,\beta}(4H_0H_\alpha+H_\beta^2).
\end{aligned}$$

If $2H_0\ge H_\alpha$ and $H_\beta\ge H_\alpha$. Then, similarly, we have

$$\begin{aligned}
|B^{0,\alpha}_{R_0,R_\alpha}\cup B^{0,\beta}_{R_0,R_\beta}| &= 2R_0.2R_\beta\sin\beta+R_\alpha\sin(\pi-\alpha)R_\alpha\sin(\beta-\alpha)(\sin\beta)^{-1}\\
&= C_{\alpha,\beta}(4H_0H_\beta+H_\alpha^2).
\end{aligned}$$

Finally if $H_\alpha,H_\beta>2H_0$, then

$$\begin{aligned}
|B^{0,\alpha}_{R_0,R_\alpha}\cup B^{0,\beta}_{R_0,R_\beta}| &= |B^{0,\alpha}_{R_0,R_\alpha}|+|B^{0,\beta}_{R_0,R_\beta}|-|B^{0,\alpha}_{R_0,R_\alpha}\cap B^{0,\beta}_{R_0,R_\beta}|\\
&= 4R_0R_\alpha\sin\alpha+4R_0R_\beta\sin\beta-4R_0^2\sin\alpha\sin\beta(\sin(\beta-\alpha))^{-1}\\
&= 4C_{\alpha,\beta}H_0(H_\alpha+H_\beta-H_0).
\end{aligned}$$

This proves the lemma. ■

Proof of Lemma 4.2 Suppose that $2H_0\ge H_\beta$ and $H_\alpha\ge H_\beta$. Also assume that $|\overline{h}_0(x)|\le H_\alpha-H_\beta$ and $|h_\beta(x)|\le 2H_0-H_\beta$. In this case we have $B^{0,\alpha}_{R_0,R_\alpha}\cup B^{0,\beta}_{R_0,R_\beta}$ represented as the union of the two parallelograms $ABCD$ and $EFGH$ in Figure 5, while $B^{0,\alpha}_{R_0,R_\alpha}\cup B^{0,\beta}_{R_0,R_\beta}(x)$ is the union of $ABCD$ and $IJKL$. The difference between these two regions is thus the difference of the "dashed" triangles and the "solid" triangles outside the parallelogram $ABCD$. It is easily seen that the sum of the area of the "dashed" triangles is $\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}[\frac{R_\beta^2\sin^2(\beta-\alpha)}{\sin^2\alpha}+(x_1-\frac{x_2}{\tan\alpha})]$, while the sum of the areas of the solid triangles is $\frac{R_\beta^2\sin\beta\sin(\beta-\alpha)}{\sin\alpha}$. This proves the first case Lemma 4.2 (i). By considering similar figures, the other parts of the lemma follow. ■

Proof of Lemma 4.3 First we consider the situation when $y_1=\mathbf{0}$, $\ell=1$ and $k=2$ with $x_2=\mathbf{0}$ and x_1 such that

$$|x^\alpha_1|\le R_\alpha-2R^\alpha_0,\quad |x^\beta_1|\le R_\beta-2R^\beta_0. \tag{5.26}$$

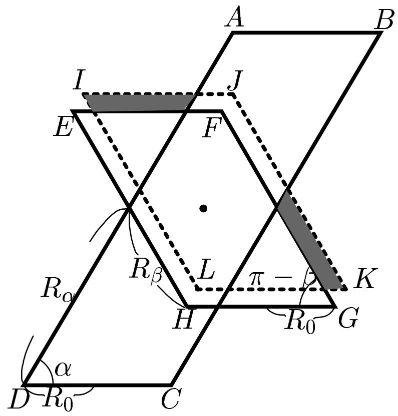

Figure 5: *Figure accompanying proof of Lemma 4.2.*

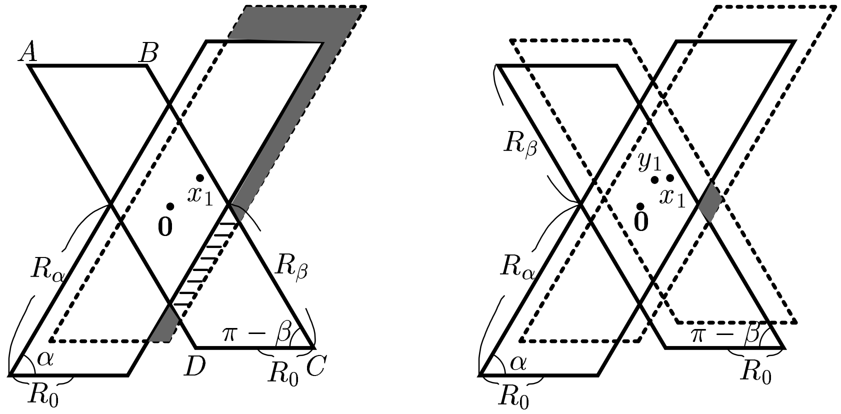

Figure 6: *The two shaded regions in the left figure combine on collapsing the lines AD and BC. The shaded parallelogram in the right figure is double counted.*

We note here that this choice of x_1 ensures the existence of the hatched region in Figure 6 which is isomorphic to a parallelogram with sides making angles α and β with the x-axis.

From Figure 6 we see that if we collapse the lines AD and BC into one and remove the parallelogram contained between these lines then each of the parallelograms $B^{0,\alpha}_{R_0,R_\alpha}$ and $B^{0,\alpha}_{R_0,R_\alpha}(x_1)$ become isomorphic to $B^{0,\alpha}_{R_0,R_\alpha-R^0_\alpha}$. Moreover the shaded area which represents $\left((B^{0,\alpha}_{R_0,R_\alpha}(x_1,x_2)\cup B^{0,\beta}_{R_0,R_\beta}(y_1))\setminus(B^{0,\alpha}_{R_0,R_\alpha}\cup B^{0,\beta}_{R_0,R_\beta})\right)$ is isomorphic to $\left(B^{0,\alpha}_{R_0,R_\alpha-R^0_\alpha}(x_1,x_2)\setminus B^{0,\alpha}_{R_0,R_\alpha-R^0_\alpha}\right)$.

Since $(B^{0,\alpha}_{R_0,R_\alpha}\cup B^{0,\beta}_{R_0,R_\beta})\subseteq(B^{0,\alpha}_{R_0,R_\alpha}(x_1,x_2)\cup B^{0,\beta}_{R_0,R_\beta})$ and $B^{0,\alpha}_{R_0,R_\alpha-R^0_\alpha}(x_1,x_2)\supseteq B^{0,\alpha}_{R_0,R_\alpha-R^0_\alpha}$ we have

$$C_{\alpha,\beta}\Delta(\mathbf{x}_2,y_1)=|B^{0,\alpha}_{R_0,R_\alpha-R^0_\alpha}(\mathbf{x}_2)\setminus B^{0,\alpha}_{R_0,R_\alpha-R^0_\alpha}|. \tag{5.27}$$

Now observe that a similar result may be obtained when $x_1=\mathbf{0}$, $k=1$ and $\ell=2$, $y_2=\mathbf{0}$ and y_1 such that

$$|y^\alpha_1|\le R_\alpha-2R^\alpha_0,\quad |y^\beta_1|\le R_\beta-2R^\beta_0. \tag{5.28}$$

In this case we obtain

$$C_{\alpha,\beta}\Delta(x_1,\mathbf{y}_2) = |B^{0,\beta}_{R_0,R_\beta-R^0_\beta}(\mathbf{y}_2) \setminus B^{0,\beta}_{R_0,R_\beta-R^0_\beta}|. \tag{5.29}$$

In case both $k=2$ and $\ell=2$ with x_1 and y_1 satisfying (5.26) and (5.28) we see from Figure 6 that if we add the areas obtained in (5.27) and (5.29) there is double counting of the shaded parallelogram with sides of length $|x^\beta_1|$ and $|y^\alpha_1|$ and area $|x^\alpha_1||y^\beta_1|\sin(\beta-\alpha)$. Thus we have $C_{\alpha,\beta}\Delta(\mathbf{x}_2,\mathbf{y}_2) = |B^{0,\alpha}_{R_0,R_\alpha-R^0_\alpha}(\mathbf{x}_2)\setminus B^{0,\alpha}_{R_0,R_\alpha-R^0_\alpha}| + |B^{0,\beta}_{R_0,R_\beta-R^0_\beta}(\mathbf{y}_2)\setminus B^{0,\beta}_{R_0,R_\beta-R^0_\beta}| - |x^\beta_1||y^\alpha_1|\sin(\beta-\alpha)$.

In general, for any k and ℓ, we see that if

$$M(\mathbf{x}_k) \leq R_\alpha - 2R^0_\alpha, \text{ and } M(\mathbf{y}_\ell) \leq R_\beta - 2R^0_\beta \tag{5.30}$$

there will be many such shaded areas which will be double counted. These areas need not be all distinct and the total area of this double counted region is at most $M(\mathbf{x}^\beta_k)M(\mathbf{y}^\alpha_\ell)\sin(\beta-\alpha)$. Now note that the condition (4.5) guarantees that (5.30) holds. Hence Lemma 4.3 (i) follows.

The remaining parts of the lemmas follow from similar arguments and are explained through Figures 7 and 8. ∎

Lemma 4.4 follows similarly and its proof is omitted.

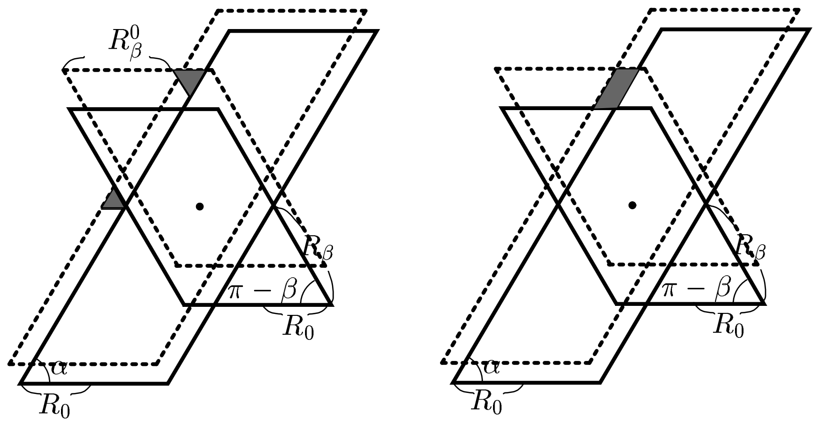

Figure 7: *The shaded triangles in the left figure give the last two terms in (4.6), while the shaded parallelogram in the right figure is double counted.*

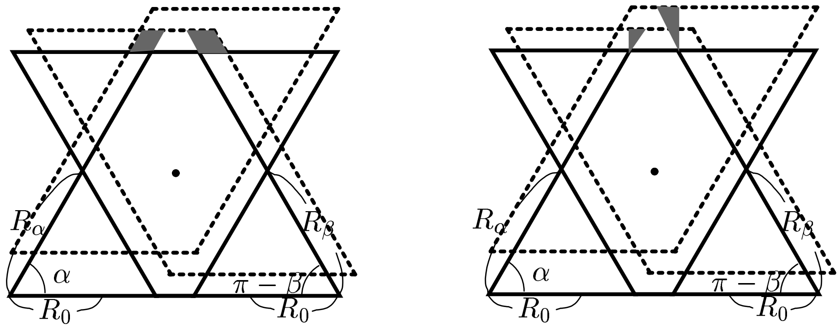

Figure 8: *The shaded areas are double counted and is deducted in (4.7).*

References

[1] Alexander, K., Finite clusters in high-density continuous percolation: compression and sphericality, *Probab. Th. Rel. Fields*, **97**, 35–63 (1993).

[2] Domany, E. and Kinzell, W., Equivalence of cellular automata to Ising models and directed percolation. *Phys. Rev.Letters*, **53**, 311–314 (1984)

[3] Ehrenfest, P., *Collected Scientific Papers.* (ed. M.J. Klein). North-Holland, Amsterdam.

[4] Grimmett, G., Stochastic pin-ball. *preprint*(1998).

[5] Harris, M., Nontrivial phase transition in a continuum mirror model. *preprint*(1996).

[6] Hall, P., *The theory of coverage processes.* John Wiley, New York (1988).

[7] Hall, P., On continuum percolation. *Annals Probab.*, **13**, 1250–1266 (1985).

[8] Menshikov, M., Coincidence of critical points in percolation problems. *Sov. Math. Dokl.*, **24**, 856–859 (1986).

[9] Roy, R., Percolation of Poisson sticks on the plane. *Probab. Th. Rel. Fields*, **89**, 503–517 (1991).

[10] Sarkar, A., Finite clusters in high density Boolean models with balls of varying sizes. *Adv. Appl. Probab. (SGSA)*, **30**, 929–947 (1998).

[11] Stoyan, D., Kendall, W. and J. Mecke, J., *Stochastic geometry and its applications.* 2nd. Ed. John Wiley, Chichester (1995).

[12] Weber, W., Zur methode der lokalisierung und charkterisierung tiefer bruchstrukturen für minerogenetische untersuchungen. In *Tiefenbruchstrukturen und post magmatische mineralization*, ed. W. Weber and V.A. Korcemagin. Freiberger Forschungshefte, C 329, Deutscher Verlag für Grund Stoffindustrie, Leipzig, 9–52 (1977).

Rahul Roy
Indian Statistical Institute,
7 SJS Sansanwal Marg,
New Delhi 110016, INDIA.
e-mail:- rahul@isid1.isid.ac.in

Hideki Tanemura
Department of Mathematics and Informatics,
Faculty of Science,
Chiba University,
1-33, Yayoi-cho, Inage-ku,
Chiba 263-8522, JAPAN.
e-mail:- tanemura@math.s.chiba-u.ac.jp