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Designs for Diallel Cross Experiments with Specific Combining Abilities

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SUMMARY

For the data collected via a diallel cross experiment, a model that incorporates both general and specific combining ability effects is postulated. Under such a model, conditions are derived for a block design to be orthogonal in the sense that contrasts among the general combining ability effects, after eliminating the block effects, are estimated free from the specific combining ability effects. Conditions are also derived for such a design to be universally optimal. Some remarks are made on the existence of universally optimal designs.

1. *Introduction and Preliminaries*

The diallel cross is a type of mating design used in plant breeding to study the genetic properties of a set of inbred lines. A common diallel cross experiment involves $v = p(p-1)/2$ crosses of the type $(i \times j),\ i < j,\ i, j = 1, 2, \ldots, p$, where p is the number of inbred lines under consideration. Henceforth, a cross $(i \times j)$ will be denoted simply by (i, j). The problem of finding optimal block designs for diallel cross experiments has received considerable attention recently and for a brief review and references, see Dey (2002). It may be noted that with a few exceptions (e.g., Chai and Mukerjee (1999), Choi *et al.* (2002)), most of the results on optimal block designs for diallel crosses have been derived under a model that includes the general combining ability (g.c.a.) effects, apart from the block effects, but no specific combining ability (s.c.a.) effects. A model where s.c.a. effects are ignored cannot always be justified from practical considerations and thus, it is often necessary to consider a model that includes both the g.c.a. and s.c.a. effects, even if interest centres around the estimation of g.c.a. effects contrasts only. In this paper, we work under a model that includes both the g.c.a. and s.c.a. effects, though our primary interest is in the estimation of contrasts among g.c.a. effects.

In a diallel cross experiment the v crosses are regarded as treatments. If the (fixed) effect of cross (i, j) is denoted by τ_{ij}, then we have the representation

$$\tau_{ij} = \bar{\tau} + g_i + g_j + s_{ij}, \tag{1}$$

where $\bar{\tau}$ is the mean effect of the treatments, the $\{g_i\}$ stand for the general combining ability (g.c.a.) effects, $\{s_{ij}\}$ denote the specific combining ability (s.c.a.) effects, and

$$g_1 + \cdots + g_p = 0, \tag{2}$$

$$s_{1i} + \cdots + s_{(i-1)i} + s_{i(i+1)} + \cdots + s_{ip} = 0, \; 1 \le i \le p. \tag{3}$$

In what follows, we arrange the crosses in the order $(1,2),(1,3),\ldots,(1,p)$, $(2,3),\ldots,(2,p),\ldots,(p-1,p)$. Let $\boldsymbol{g} = (g_1,\ldots,g_p)'$ and let $\boldsymbol{\tau}$ and $\boldsymbol{s}$ be $v \times 1$ vectors with elements $\{\tau_{ij}\}$ and $\{s_{ij}\}$ respectively. We follow Chai and Mukerjee (1999) to express the general and specific combining abilities, i.e., $\boldsymbol{g}$ and $\boldsymbol{s}$ in terms of $\boldsymbol{\tau}$. Define Q to be a $p \times v$ matrix with rows indexed by $1,\ldots,p$ and columns by the pairs (i,j), $1 \le i < j \le p$, such that the $\{u,(i,j)\}$th entry of Q is 1 if $u \in (i,j)$ and zero, otherwise. We then have

$$QQ' = (p-2)I_p + J_{pp}, \; (QQ')^{-1} = (p-2)^{-1}\{I_p - (2(p-1))^{-1}J_{pp}\}, \tag{4}$$

$$Q\mathbf{1}_v = (p-1)\mathbf{1}_p, \; Q'\mathbf{1}_p = 2\mathbf{1}_v, \tag{5}$$

where, for positive integers c, d, I_c is the cth order identity matrix, $\mathbf{1}_c$ is the $c \times 1$ vector of all ones and $J_{cd} = \mathbf{1}_c\mathbf{1}_d'$. In view of this, (1) can be expressed as

$$\boldsymbol{\tau} = \bar{\tau}\mathbf{1}_v + Q'\boldsymbol{g} + \boldsymbol{s}, \tag{6}$$

where, from (2) and (3), we have

$$\mathbf{1}_p'\boldsymbol{g} = 0, \; Q\boldsymbol{s} = 0. \tag{7}$$

Premultiplying (6) by Q and using (4), (5) and (7), one has

$$\boldsymbol{g} = H_1\boldsymbol{\tau}, \; \boldsymbol{s} = \boldsymbol{\tau} - \bar{\tau}\mathbf{1}_v - Q'\boldsymbol{g} = H_2\boldsymbol{\tau}, \tag{8}$$

where

$$H_1 = (QQ')^{-1}Q - (2v)^{-1}J_{pv} = (p-2)^{-1}(Q - 2p^{-1}J_{pv}), \tag{9}$$

and

$$H_2 = I_v - Q'(QQ')^{-1}Q = I_v - (p-2)^{-1}\{Q'Q - 2(p-1)^{-1}J_{vv}\}. \tag{10}$$

Since

$$H_1\mathbf{1}_v = 0, \; H_2\mathbf{1}_v = 0, \; H_1H_2' = 0, \; \mathrm{Rank}(H_1) = p-1, \; \mathrm{Rank}(H_2) = v-p, \tag{11}$$

it is clear that $\boldsymbol{g}$ and $\boldsymbol{s}$ represent treatment contrasts carrying $p-1$ and $v-p$ degrees of freedom respectively and the contrasts representing $\boldsymbol{g}$ are orthogonal to those representing $\boldsymbol{s}$. It may be noted that for $p = 3$ lines, $\boldsymbol{s} = 0$ and hence, in the present paper, we take $p \ge 4$ throughout.

Consider now an arrangement of v treatments (crosses) in a block design with b blocks each of size $k \ge 2$. The usual fixed effects model incorporating both g.c.a and s.c.a. effects and with uncorrelated homoscedastic errors is postulated. The main interest is in the estimation of contrasts among the g.c.a. effects and, the s.c.a. effects, alongwith the block effects are treated as nuisance parameters. Under this set up, we derive conditions on the block design such that the g.c.a. effects, after eliminating the block effects, are estimated free from the s.c.a. effects. Designs with this property will be called *orthogonal* designs. We also derive sufficient conditions for orthogonal designs to be universally optimal.

2. *Orthogonal Designs*

To characterize orthogonal designs, it will be convenient to consider the problem via complete sets of orthonormal contrasts, say $L_1\boldsymbol{\tau}$ and $L_2\boldsymbol{\tau}$, representing $\boldsymbol{g}$ and $\boldsymbol{s}$ respectively. Then, by (8) and (11), we have

$$L_1L_1' = I_{p-1},\ L_2L_2' = I_{v-p},\ L_1L_2' = 0,\ \ \mathcal{R}(L_1) = \mathcal{R}(H_1), \mathcal{R}(L_2) = \mathcal{R}(H_2), \tag{12}$$

where $\mathcal{R}(\cdot)$ denotes the row span of a matrix. Note that the subsequent results do not depend on the specific choice of L_1 and L_2. Let P_1 be a $(p-1)\times p$ matrix such that $\begin{pmatrix} p^{-\frac{1}{2}}\mathbf{1}_p' \\ P_1 \end{pmatrix}$ is an orthogonal matrix and P_2 be a $(v-p)\times v$ matrix satisfying

$$P_2P_2' = I_{v-p} \ \text{ and } \ P_2Q' = 0. \tag{13}$$

It follows then that

$$P_1\mathbf{1}_p = 0,\ \ P_1P_1' = I_{p-1},\ \ P_2\mathbf{1}_v = 0, \tag{14}$$

where the last identity follows from (13) and (5). It is then easy to see that L_1 and L_2 satisfying (12) can be expressed as

$$L_1 = (p-2)^{-\frac{1}{2}}P_1Q \ \text{ and } \ L_2 = P_2 \tag{15}$$

for any P_1 and P_2 satisfying (13) and (14). Thus,

$$L_1'L_1 = (p-2)^{-1}(Q'Q - 4p^{-1}J_{vv}). \tag{16}$$

Under the stated model and a block design d, the joint information matrix for $\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}\boldsymbol{\tau}$ is given by

$$\mathcal{I}_d = \begin{bmatrix} L_1C_dL_1' & L_1C_dL_2' \\ L_2C_dL_1' & L_2C_dL_2' \end{bmatrix}, \tag{17}$$

where $C_d = R_d - k^{-1}M_dM_d'$, R_d is the diagonal matrix of the replications of the crosses under d and M_d is the $v\times b$ incidence matrix of crosses versus blocks. Note that C_d is the usual C-matrix of d with crosses as treatments and hence, $C_d\mathbf{1}_v = 0$. As shown in Lemma 1 below, in order that the design d is orthogonal in the sense of Section 1, we must have

$$L_2C_dL_1' = 0. \tag{18}$$

By (15), this is equivalent to

$$P_1QC_dP_2' = 0. \tag{19}$$

Let $\mathcal{I}_{gd}$ be the information matrix for $L_1\boldsymbol{\tau}$. Then, we have the following result.

Lemma 1. *(a) $L_1C_dL_1' - \mathcal{I}_{gd}$ is a nonnegative definite (n.n.d.) matrix.*
(b) Furthermore, $\mathcal{I}_{gd} = L_1C_dL_1'$ if and only if $L_2C_dL_1' = 0$.

Proof. Recall that the rows of $[L_1', L_2']'$ form an orthonormal basis of the orthocomplement of $\mathcal{R}(\mathbf{1}_v')$ in the v-dimensional Euclidean space. It follows from (17) that $\mathcal{I}_{gd}$ is $L_1C_dL_1' - L_1C_dL_2'(L_2C_dL_2')^- L_2C_dL_1'$, where A^- denotes a generalized inverse of a matrix A. Thus,

$$L_1C_dL_1' - \mathcal{I}_{gd} = L_1C_dL_2'(L_2C_dL_2')^- L_2C_dL_1'. \tag{20}$$

Suppose $\text{Rank}(C_d) = t$. Since C_d is n.n.d., there exists a $v \times t$ matrix H of full column rank, such that $C_d = H'H$. Then, by (20),

$$L_1C_dL_1' - \mathcal{I}_{gd} = L_1H'HL_2'(L_2H'HL_2')^- L_2H'HL_1' = L_1H'\text{pr}(HL_2')HL_1', \tag{21}$$

is n.n.d., as $\text{pr}(HL_2')$ is n.n.d., where for a matrix X, $\text{pr}(X)$ denotes the projection on to the column span of X. This proves (a).

By (21), $L_1C_dL_1' = \mathcal{I}_{gd}$ if and only if $\text{pr}(HL_2')HL_1' = 0$, i.e., if and only if $L_2H'HL_1' = L_2C_dL_1' = 0$. This proves (b). □

Lemma 2. *The following conditions are equivalent:*
(i) $L_2C_dL_1' = 0$.
(ii) $L_1'L_1C_d = C_dL_1'L_1$.
(iii) $Q'QC_d = C_dQ'Q$.

Proof. Suppose $L_1'L_1C_d = C_dL_1'L_1$. Then, $L_2C_dL_1' = L_2C_dL_1'L_1L_1' = L_2L_1'L_1C_dL_1' = 0$. Conversely, suppose $L_2C_dL_1' = 0$. Then, $L_2'L_2C_dL_1' = 0$ and since $L_1'L_1 + L_2'L_2 = I_v - v^{-1}J_{vv}$, this implies that $(I_v - v^{-1}J_{vv} - L_1'L_1)C_dL_1' = 0 \Rightarrow C_dL_1' = L_1'L_1C_dL_1' \Rightarrow C_dL_1'L_1 = L_1'L_1C_dL_1'L_1$. Now, since $L_1'L_1C_dL_1'L_1$ is symmetric, $C_dL_1'L_1 = (C_dL_1'L_1)' = L_1'L_1C_d$. Finally, from (14) and (15), it follows that $L_1'L_1C_d = C_dL_1'L_1 \Leftrightarrow Q'QC_d = C_dQ'Q$. □

We now find an upper bound to $\text{tr}(\mathcal{I}_{gd})$, the trace of $\mathcal{I}_{gd}$. Let $\mathcal{D}$ be the collection of all designs that keep $L_1\boldsymbol{\tau}$ estimable. Since by Lemma 1(a), for any $d \in \mathcal{D}$, $L_1C_dL_1' - \mathcal{I}_{gd}$ is n.n.d., we have

$$\begin{aligned}
\text{tr}(\mathcal{I}_{gd}) &\leq \text{tr}(L_1C_dL_1') \\
&= \text{tr}(C_dL_1'L_1) \\
&= \text{tr}(C_d\{(p-2)^{-1}Q'Q - 4p^{-1}J_{vv}\}) \\
&= (p-2)^{-1}\text{tr}(C_dQ'Q) \\
&= (p-2)^{-1}\text{tr}(R_dQ'Q - k^{-1}M_dM_d'Q'Q).
\end{aligned}$$

Now, $\text{tr}(M_dM_d'Q'Q) = \text{tr}(QM_dM_d'Q') = \text{tr}(N_dN_d') = \sum_{i=1}^{p}\sum_{j=1}^{b} n_{dij}^2$, where $N_d = (n_{dij}) = QM_d$ is the $p \times b$ lines versus blocks incidence matrix. Since $\{n_{dij}\}$ are integers and $\sum_{i=1}^{p}\sum_{j=1}^{b} n_{dij} = 2bk$, $\sum_{i=1}^{p}\sum_{j=1}^{b} n_{dij}^2 \geq b\{2k(2x+1) - px(x+1)\}$, where $x = [2kp^{-1}]$ and $[\cdot]$ is the greatest integer function. Also, $\text{tr}(R_dQ'Q) = 2\text{tr}(R_d) = 2bk$, since each diagonal element of $Q'Q$ equals 2. Thus, $\text{tr}(\mathcal{I}_{gd}) \leq (p-2)^{-1}\{2bk - bk^{-1}(2k(2x+1) - px(x+1))\}$ or,

$$\text{tr}(\mathcal{I}_{gd}) \leq (k(p-2))^{-1}b\{2k(k-2x-1) + px(x+1)\} = \omega \text{ (say)}. \tag{22}$$

Remark. The expression in (22) is actually obvious from Das, Dey and Dean (1998,Theorem 2.1) if one notes that $\operatorname{tr}(C_d Q'Q) = \operatorname{tr}(QC_d Q')$ and that $QC_d Q'$ is the C-matrix of the block design with lines as treatments.

We next have the following result. **Theorem 1**. *Suppose there exists a design $d_0 \in \mathcal{D}$ such that*
(i) $Q'QC_{d_0} = C_{d_0}Q'Q$ and,
(ii) $QC_{d_0}Q' = (p-1)^{-1}(p-2)\omega(I_p - p^{-1}J_{pp})$, where $\omega > 0$ is as in (22).
Then, d_0 is universally optimal in $\mathcal{D}$ for inference on $L_1\boldsymbol{\tau}$.

Proof. Let $\mathcal{I}_{gd_0}$ be the information matrix for $L_1\boldsymbol{\tau}$ under d_0. Then, by (i), part (b) of Lemma 1 and Lemma 2, $\mathcal{I}_{gd_0} = L_1 C_{d_0} L_1'$. Also, by (ii), $L_1 C_{d_0} L_1' = (p-2)^{-1}P_1 Q C_{d_0} Q' P_1' = (p-1)^{-1}\omega P_1(I_p - p^{-1}J_{pp})P_1' = (p-1)^{-1}\omega I_{p-1}$. Thus,

$$\mathcal{I}_{gd_0} = (p-1)^{-1}\omega I_{p-1}. \tag{23}$$

Also, from (22), for every $d \in \mathcal{D}$,

$$\operatorname{tr}(\mathcal{I}_{gd}) \le \omega, \tag{24}$$

and from (23),

$$\operatorname{tr}(\mathcal{I}_{gd_0}) = \operatorname{tr}((p-1)^{-1}\omega I_{p-1}) = \omega. \tag{25}$$

Since by (23), $\mathcal{I}_{gd_0}$ is a constant times the identity matrix, in view of (24) and (25), the claimed universal optimality of d_0 now follows from Kiefer (1975) and Sinha and Mukerjee (1982). □

We now derive a condition, which is equivalent to the conditions of Theorem 1. The conditions in Theorem 1 are

$$Q'QC_{d_0} = C_{d_0}Q'Q, \text{ and } QC_{d_0}Q' = (p-1)^{-1}(p-2)\omega(I_p - p^{-1}J_{pp}). \tag{26}$$

Lemma 3. *The conditions in Theorem 1 are equivalent to*

$$QC_{d_0} = (p-1)^{-1}\omega(Q - 2p^{-1}J_{pv}). \tag{27}$$

Proof. Suppose (27) holds. Then, $QC_{d_0}Q' = (p-1)^{-1}\omega(QQ' - 2(p-1)p^{-1}J_{pp}) = (p-1)^{-1}(p-2)\omega(I_p - p^{-1}J_{pp})$, and $Q'QC_{d_0} = (p-1)^{-1}\omega(Q'Q - 4p^{-1}J_{vv})$, which is symmetric, i.e., $Q'QC_{d_0} = C_{d_0}Q'Q$. Thus (27) $\Rightarrow$ (26).

Conversely, suppose (26) holds. Then, $QC_{d_0}Q'Q = (p-1)^{-1}(p-2)\omega(I_p - p^{-1}J_{pp})Q$, by the second condition of (26). This implies that $QQ'QC_{d_0} = (p-1)^{-1}(p-2)\omega(I_p - p^{-1}J_{pp})Q$, by the first condition in (26). Now, using (4), we have

$$QC_{d_0} = (p-1)^{-1}\omega(I_p - (2(p-1))^{-1}J_{pp})(I_p - p^{-1}J_{pp})Q$$

or, $QC_{d_0} = (p-1)^{-1}\omega(Q - 2p^{-1}J_{pv})$ and thus, (26) $\Rightarrow$ (27). $\square$

3. *Construction of Designs*

In this section, we consider the issue of determining designs that satisfy the conditions of Theorem 1. We first have the following result, which is in the spirit of the discussion in Dey and Mukerjee (1999; Section 2.3, page 12) in a different context.

Lemma 4. *In order to keep $L_1\boldsymbol{\tau}$ estimable under a design d (blocked or unblocked), it is necessary that every cross appears at least once in the design.*

Proof. Suppose $L_1\boldsymbol{\tau}$ is estimable under d and if possible, suppose some cross, say $(1,2)$, never appears in d. Then, $\mathcal{R}(L_1) \subset \mathcal{R}(C_d)$ and the first column of C_d is null. Hence the first column of L_1 is also null. We shall now show that this is impossible. Suppose the first column of L_1 is null. Then, the first column of $L_1'L_1$ is null, i.e., by (16), the first column of $Q'Q - 4p^{-1}J_{vv}$ is null. This implies that the first element of the first column of $Q'Q$ must equal $4p^{-1}$. But, by the definition of Q, the first element in the first column of $Q'Q$ equals 2, which leads to a contradiction, since $p > 2$. $\square$

In view of Lemma 4, *the smallest design that keeps $L_1\boldsymbol{\tau}$ estimable under the stated model is one in which each cross is replicated just once.*

For a single replicate design d_1, $C_{d_1} = I_v - k^{-1}M_{d_1}M_{d_1}'$. Hence, (27) $\Leftrightarrow$ $Q(I_v - k^{-1}M_{d_1}M_{d_1}') = (p-1)^{-1}\omega(Q - 2p^{-1}J_{pv})$, which is equivalent to

$$(p-1-\omega)Q + 2\omega p^{-1}J_{pv} = k^{-1}(p-1)QM_{d_1}M_{d_1}'. \tag{28}$$

Also, for d_1, $M_{d_1}'M_{d_1} = kI_b$. Hence,

$$(28) \Rightarrow QM_{d_1} = N_{d_1} = 2kp^{-1}J_{pb}, \tag{29}$$

where N_{d_1} is the $p \times b$ lines versus blocks incidence matrix. That is, each line occurs $2kp^{-1}$ times in each block.

We next check whether (29) $\Rightarrow$ (28). Suppose (29) holds. Then,

$$k^{-1}QM_{d_1}M_{d_1}' = k^{-1}2kp^{-1}J_{pb}M_{d_1}' = 2p^{-1}\mathbf{1}_p\mathbf{1}_b'M_{d_1}' = 2p^{-1}J_{pv}. \tag{30}$$

Furthermore, if (29) holds, then $2kp^{-1}$ is an integer, i.e., $x = 2kp^{-1}$ and, in such a case, one can show that $\omega = p-1$. Hence, under (29), the right hand side of (28) equals $2(p-1)p^{-1}J_{pv}$ and by (30), it follows that (29) $\Rightarrow$ (28). Hence, we get the following result.

Theorem 2. *For a single replicate design d_1, (27) holds if and only if*

$$N_{d_1} = 2kp^{-1}J_{pb}.$$

We next consider a general equireplicate design, say d_2. For such a design, $C_{d_2} = rI_v - k^{-1}M_{d_2}M'_{d_2}$, where r is the common replication of the crosses and M_{d_2} is the crosses versus blocks incidence matrix of d_2. It follows then that

$$(27) \Leftrightarrow (r(p-1)-\omega)Q + 2\omega p^{-1}J_{pv} = k^{-1}(p-1)QM_{d_2}M'_{d_2}. \tag{31}$$

We need the following result in the sequel.

Lemma 5. *The $v \times v$ matrix $W = \begin{bmatrix} Q \\ P_2 \end{bmatrix}$ is nonsingular.*

Proof. With W as above, we have

$$WW' = \begin{bmatrix} QQ' & QP'_2 \\ P_2Q' & P_2P'_2 \end{bmatrix}.$$

The result then follows, since $QQ' = (p-2)I_p + J_{pp}$, $P_2P'_2 = I_{v-p}$, and $P_2Q' = 0$.

□

Now, since W is nonsingular matrix of order v and M_{d_2} is a $v \times b$ matrix, the column span of M_{d_2} is a subspace of the column span of W', i.e., there exist matrices A_1 and A_2 of orders $p \times b$ and $(v-p) \times b$ respectively, such that

$$M_{d_2} = Q'A_1 + P'_2A_2. \tag{32}$$

In view of (32), we have $k\mathbf{1}'_b = \mathbf{1}'_vM_{d_2} = \mathbf{1}'_vQ'A_1 + \mathbf{1}'_vP'_2A_2 = \mathbf{1}'_vQ'A_1 = (p-1)\mathbf{1}'_pA_1$, so that

$$\mathbf{1}'_pA_1 = (p-1)^{-1}k\mathbf{1}'_b.$$

Also, $r\mathbf{1}_v = M_{d_2}\mathbf{1}_b = Q'A_1\mathbf{1}_b + P'_2A_2\mathbf{1}_b$. This implies that $rQ\mathbf{1}_v = QQ'A_1\mathbf{1}_b + QP'_2A_2\mathbf{1}_b = QQ'A_1\mathbf{1}_b$. Thus, using (4),

$$A_1\mathbf{1}_b = r(QQ')^{-1}Q\mathbf{1}_v = \frac{r}{2}\mathbf{1}_p.$$

Finally, $r\mathbf{1}_v = M_{d_2}\mathbf{1}_b = Q'A_1\mathbf{1}_b + P'_2A_2\mathbf{1}_b = Q'(\frac{r}{2}\mathbf{1}_p) + P'_2A_2\mathbf{1}_b = r\mathbf{1}_v + P'_2A_2\mathbf{1}_b$. Premultiplying by P_2, this implies that

$$A_2\mathbf{1}_b = 0.$$

We now have the following result.

Lemma 6. *The condition (31) is equivalent to*

$$A_1A'_2 = 0 \tag{33}$$

and

$$A_1A'_1 = k(p-1)^{-1}(p-2)^{-1}\{(r(p-1)-\omega)I_p - (2p)^{-1}(rp-2\omega)J_{pp}\}. \tag{34}$$

Proof. By (32), $QM_{d_2} = QQ'A_1 + QP_2'A_2 = QQ'A_1$, which implies that

$$QM_{d_2}M_{d_2}' = QQ'A_1(A_1'Q + A_2'P_2). \tag{35}$$

Now suppose (31) holds. Then

$$(r(p-1)-\omega)Q + 2\omega p^{-1}J_{pv} = k^{-1}(p-1)QM_{d_2}M_{d_2}' = k^{-1}(p-1)QQ'A_1(A_1'Q + A_2'P_2). \tag{36}$$

Postmultiplying both sides of (36) by P_2' and recalling that $QP_2' = 0,\ J_{pv}P_2' = 0,\ P_2P_2' = I_{v-p}$, we have $0 = k^{-1}(p-1)QQ'A_1A_2' \Rightarrow A_1A_2' = 0$, as QQ' is nonsingular. This proves (33).

Again, postmultiplying both sides of (36) by Q' and simplifying, we get A_1A_1' as in (34).

Conversely, let (33) and (34) hold. Then, by (35), we have $k^{-1}(p-1)QM_{d_2}M_{d_2}' = k^{-1}(p-1)(QQ'A_1A_1'Q + QQ'A_1A_2'P_2) = k^{-1}(p-1)QQ'A_1A_1'Q$, by (33). Substituting for QQ' and A_1A_1' from (4) and (34) respectively, the right hand side yields on simplification $(r(p-1)-\omega)Q + 2\omega p^{-1}J_{pv}$, which is (31). □

To summarize, in order that Theorem 1 holds in the case of an equireplicate design, M_{d_2} must be of the form

$$M_{d_2} = Q'A_1 + P_2'A_2$$

where (i) A_1 and A_2 satisfy (33) and (34) and, (ii) $A_1\mathbf{1}_b = \frac{r}{2}\mathbf{1}_p$, $\mathbf{1}_p'A_1 = k(p-1)^{-1}\mathbf{1}_b'$ and $A_2\mathbf{1}_b = 0$.

From (31) it is seen that d_2 is orthogonal and balanced for g.c.a. effects if

$$(r(p-1)-\delta)Q + 2\delta p^{-1}J_{pv} = k^{-1}(p-1)QM_{d_2}M_{d_2}', \tag{36}$$

where δ is any positive constant. Furthermore, if $\delta = \omega$, the design is universally optimal as well.

If $M_{d_2}M_{d_2}' = \alpha Q'Q + \beta J_{vv}$ for some scalars α, β satisfying $(p-1)(4\alpha + p\beta) = 2rk$, then (36) holds. It can be verified that equireplicate designs for diallel crosses derived from triangular incomplete block designs with usual parameters $v = p(p-1)/2, b, r, k, \lambda_1, \lambda_2$ (see e.g., Dey and Midha (1996)) satisfy (36) and are thus, orthogonal and balanced for g.c.a. effects. Furthermore, if

$$p(p-1)(p-2)\lambda_1 = bx\{4k - p(x+1)\},$$

where $x = [2kp^{-1}]$ then $\delta = \omega$ and in such a case, the design is universally optimal for estimating g.c.a. effects in the presence of s.c.a. effects. This supports the findings in Chai and Mukerjee (1999).

Now, consider an equireplicate design d_2, such that $N_{d_2} = QM_{d_2} = 2kp^{-1}J_{pb}$. Then, $\omega = r(p-1)$ and thus, (31) holds. Hence, such a design is universally optimal for g.c.a. effects under the stated model. Families of such designs have been given by Choi *et al.* (2002).

The existence of the above two classes of designs does not preclude the existence of other designs which are also universally optimal under the present set up. Further work needs to be done for determining such designs.

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REFERENCES

[1] Chai, F.-S. and Mukerjee, R. (1999). Optimal designs for diallel crosses with specific combining abilities. *Biometrika* **86**, 453-458.

[2] Choi, K. C., Chatterjee, K., Das, A. and Gupta, S. (2002). Optimality of orthogonally blocked diallels with specific combining abiities. *Statist. Probab. Lett.* **57**, 145-150.

[3] Das, A., Dey, A. and Dean, A. M. (1998). Optimal designs for diallel cross experiments. *Statist. Probab. Lett.* **36**, 427-436.

[4] Dey, A. (2002). Optimal designs for diallel crosses. *J. Indian Soc. Agric. Statist.* **55**, 1-16.

[5] Dey, A. and Midha, C. K. (1996). Optimal block designs for diallel crosses. *Biometrika* **83**, 484-489.

[6] Dey, A. and Mukerjee, R. (1999). *Fractional Factorial Plans.* New York : Wiley.

[7] Kiefer, J. (1975). Construction and optimality of generalized Youden designs. In : *A Survey of Statistical Design and Linear Models*, Ed. J. N. Srivastava, pp. 333-353. Amsterdam : North-Holland.

[8] Sinha, B. K. and Mukerjee, R. (1982). A note on the universal optimality criterion for full rank models. *J. Statist. Plann. Infer.* **7**, 97-100.