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Dependence, Dispersiveness and Multivariate Hazard Rate Ordering

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Abstract

To compare two multivariate random vectors of the same dimension, we define a new stochastic order called *upper orthant dispersive ordering* and study its properties. We study its relationship with positive dependence and multivariate hazard rate ordering as defined by Hu, Khaledi and Shaked (*J. Mult. Anal.*, 2002). It is shown that if two random vectors have a common copula and if their marginal distributions are ordered according to dispersive ordering in the same direction, then the two random vectors are ordered according to this new upper orthant dispersive ordering. Also it is shown that the marginal distributions of two upper orthant dispersive ordered random vectors are also dispersive ordered. Examples and applications are given.

Keywords and phrases : Copula, hazard gradient, multivariate hazard rate ordering, right tail increasing, positive quadrant dependence.

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1 Introduction

It is of interest to compare two random variables in terms of their variability. While this topic has been studied extensively in the univariate case, several attempts have been made to extend it to the multivariate case. Important contributions in this case have been made by Giovagnoli and Wynn (1995) Shaked and Shathikumar (1998) and Fernadez-Ponse and Suarez-Llorens (2003), among others.

Let X and Y be two univariate random variables with distribution functions F and G; and with survival functions $\overline{F}$ and $\overline{G}$, respectively. A basic concept for comparing variability in distributions is that of dispersive ordering. X is said to be less dispersed than Y (denoted by $X \leq_{disp} Y$) if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha) \text{ whenever } \quad 0 < \alpha \leq \beta < 1, \tag{1.1}$$

where F^{-1} and G^{-1} are the right continuous inverses of the distribution functions F and G, respectively. This means that the difference between any two quantiles of X is smaller than the difference between the corresponding quantiles of Y. In case the random variables X and Y are of continuous type with hazard rates r_F and r_G, respectively, then $X \leq_{disp} Y$ if and only if

$$r_G(G^{-1}(p)) \leq r_F(F^{-1}(p)), \ \forall p \in [0,1]. \tag{1.2}$$

For more details on dispersive ordering, see Section 2B of Shaked and Shanthikumar (1994).

In analogy with the characterization (1.2) of the univariate dispersive ordering, we introduce a new order in the multivariate case which we call *upper orthant dispersive ordering* and study its properties. According to (1.2), $X \leq_{disp} Y$ if and only if the hazard rates of X and Y at the quantiles of the same order p are ordered for all values of $p \in [0,1]$. To this end, we first recall the definition of hazard rate (or hazard gradient) in the multivariate case. Consider a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ with a partially differentiable survival function $\overline{F}(\mathbf{x}) = P\{\mathbf{X} > \mathbf{x}\}$. The function $R = -\log \overline{F}$ is called the hazard function of $\mathbf{X}$, and the vector $\mathbf{r_X}$ of partial derivatives, defined by

$$\mathbf{r_X}(\mathbf{x}) = (r_{\mathbf{X}}^{(1)}(\mathbf{x}), \ldots, r_{\mathbf{X}}^{(n)}(\mathbf{x})) = (\frac{\partial}{\partial x_1}R(\mathbf{x}), \ldots, \frac{\partial}{\partial x_n}R(\mathbf{x})),$$

for all $\mathbf{x} \in \{\mathbf{x} : \overline{F}(\mathbf{x}) > 0\}$, is called the *hazard gradient* of $\mathbf{X}$ (see Johnson and Kotz, 1975 and Marshall, 1975). Note that $r_{\mathbf{X}}^{(i)}(\mathbf{x})$ can be interpreted as the conditional hazard rate of X_i evaluated at x_i, given that $X_j > x_j$ for all $j \neq i$. That is,

$$r_{\mathbf{X}}^{(i)}(\mathbf{x}) = \frac{f_i(x_i \mid \bigcap\limits_{j \neq i}\{X_j > x_j\})}{\overline{F}_i(x_i \mid \bigcap\limits_{j \neq i}\{X_j > x_j\})},$$

where $f_i(\cdot \mid \bigcap_{j \neq i}\{X_j > x_j\})$ and $\overline{F}_i(\cdot \mid \bigcap_{j \neq i}\{X_j > x_j\})$ are, respectively, the conditional density and the conditional survival functions of X_i, given that $X_j > x_j$ for all $j \neq i$. For convenience, here and below we set $r^{(i)}_{\mathbf{X}}(\mathbf{x}) = \infty$ for all $\mathbf{x} \in \{\mathbf{x} : \overline{F}(\mathbf{x}) = 0\}$. Now we define *upper orthant dispersive ordering.*

Definition 1.1. *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be two random vectors with respective survival functions* $\overline{F}$ *and* $\overline{G}$. *We say that* $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *according to upper orthant dispersive ordering (denoted by* $\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}$*) if for all* $u_j \in [0,1]$, $j = 1, \ldots, n$, $j \neq i$,

$$\left(X_i \mid \bigcap_{j \neq i} \{X_j > F_j^{-1}(u_j)\} \right) \leq_{disp} \left(Y_i \mid \bigcap_{j \neq i} \{Y_j > G_j^{-1}(u_j)\} \right), \tag{1.3}$$

for $i = 1, \ldots, n$.

In case the distributions under consideration are absolutely continuous, the upper orthant dispersive ordering can be equivalently expressed in terms of the hazard gradients at the quantiles of the same orders of the conditional distributions. If we denote by $x_i(\beta; \mathbf{u})$ and $y_i(\beta; \mathbf{u})$, the βth quantiles of the conditional distributions $\left(X_i \mid \bigcap_{j \neq i} \{X_j > F_j^{-1}(u_j)\} \right)$ and $\left(Y_i \mid \bigcap_{j \neq i} \{Y_j > G_j^{-1}(u_j)\} \right)$, respectively, then

$$\begin{aligned} \mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y} \;\Leftrightarrow\; & r^{(i)}_{\mathbf{X}} \left(F_1^{-1}(u_1), \ldots, F_{i-1}^{-1}(u_{i-1}), x_i(\beta; \mathbf{u}), \ldots, F_n^{-1}(u_n) \right) \\ \geq\; & r^{(i)}_{\mathbf{Y}} \left(G_1^{-1}(u_1), \ldots, G_{i-1}^{-1}(u_{i-1}), y_i(\beta; \mathbf{u}), \ldots, G_n^{-1}(u_n) \right), \end{aligned} \tag{1.4}$$

for every $\beta \in [0,1]$, $\mathbf{u} \in [0,1]^{n-1}$ and $i = 1, \ldots, n$, where $r^{(i)}_{\mathbf{X}}$ and $r^{(i)}_{\mathbf{Y}}$ stand for the ith components of the hazard gradients of $\mathbf{X}$ and $\mathbf{Y}$, respectively.

The following slightly modified version of a theorem of Saunders and Moran (1978) provides a useful tool for establishing dispersive ordering among members of a parametric family of distributions.

THEOREM **1.1.** *Let* X_a *be a random variable with distribution function* F_a *for each* $a \in R$ *such that*

(i) F_a *is supported on some interval* $(x_-^{(a)}, x_+^{(a)}) \subseteq (-\infty, \infty)$ *and has density* f_a *which does not vanish on any subinterval of* $(x_-^{(a)}, x_+^{(a)})$,

(ii) the derivative of F_a *with respect to* a *exists and denoted by* F'_a.

Then,

$$X_a \geq_{disp} X_{a^*} \quad \textit{for} \;\; a,\; a^* \in R \;\; \textit{and}\; a > a^*, \tag{1.5}$$

if and only if,

$$F'_a(x)/f_a(x) \quad \textit{is decreasing in} \;\; x. \tag{1.6}$$

In the next example we identify conditions under which two bivariate normal random vectors are ordered according to upper orthant dispersive ordering. More examples are discussed in the last section.

Example 1.1 (Bivariate Normal Distribution).

Let $\mathbf{X}$ and $\mathbf{Y}$ follow bivariate normal distributions with each with mean vector $(0,0)$ and with dispersion matrices

$$\mathbf{\Sigma}^X = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \text{ and } \mathbf{\Sigma}^Y = \begin{pmatrix} \sigma_1^2 & \rho'\sigma_1\sigma_2 \\ \rho'\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix},$$

respectively, with $\sigma_i > 0$ for $i = 1, 2$. For the time being we are assuming that the marginal distributions of $\mathbf{X}$ and $\mathbf{Y}$ are identical. The general case is considered later. We use Theorem 1.1 to prove that in case ρ and ρ' are of the same sign, then

$$|\rho'| \le |\rho| \Rightarrow \mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}.$$

Let us denote by G_ρ, the distribution function of $\{X_1|X_2 > y\}$ (we are suppressing its dependence on y for the clarity of notation). Then

$$\begin{aligned} G_\rho(x) &= \frac{1}{P(X_2 > y)} \int_{-\infty}^{x} \int_{y}^{+\infty} f_{X_1,X_2}(u,v)\,du\,dv \\ &= \frac{1}{P(X_2 > y)} \int_{y}^{+\infty} P(X_1 \le x | X_2 = v) f_{X_2}(v)\,dv \\ &= \frac{1}{P(X_2 > y)} \int_{y}^{+\infty} \Phi\left(\frac{x - \frac{\rho\sigma_1}{\sigma_2}v}{\sigma_1(1-\rho^2)^{1/2}}\right) \frac{1}{\sigma_2}\phi\left(\frac{v}{\sigma_2}\right) dv \end{aligned}$$

and the corresponding conditional density function is

$$g_\rho(x) = \frac{1}{P(X_2 > y)\sigma_1(1-\rho^2)^{1/2}} \int_{y}^{+\infty} \phi\left(\frac{x - \frac{\rho\sigma_1}{\sigma_2}v}{\sigma_1(1-\rho^2)^{1/2}}\right) \frac{1}{\sigma_2}\phi\left(\frac{v}{\sigma_2}\right) dv$$

Now we compute

$$\begin{aligned} G'_\rho(x) &= \frac{\partial}{\partial\rho} G_\rho(x) \\ &= \frac{1}{P(X_2 > y)} \int_{y}^{+\infty} \left\{ \frac{-\frac{\sigma_1}{\sigma_2}v(\sigma_1(1-\rho^2)^{1/2}) + \frac{\sigma_1\rho}{(1-\rho^2)^{1/2}}(x - \frac{\rho\sigma_1}{\sigma_2}v)}{\sigma_1^2(1-\rho^2)} \right\} \\ &\quad \times \phi(\frac{x - \frac{\rho\sigma_1}{\sigma_2}v}{\sigma_1(1-\rho^2)^{1/2}}) \frac{1}{\sigma_2}\phi(\frac{v}{\sigma_2})dv, \end{aligned}$$

which simplifies to

$$\begin{aligned} G'_\rho(x) &= \frac{e^{-\frac{x^2}{2\sigma_1^2}}}{P(X_2 > y)\sqrt{2\pi}} \int_{y}^{+\infty} \frac{(\frac{\rho\,\sigma_2}{\sigma_1} - v)}{\sqrt{2\pi}\sigma_2{}^2(1-\rho^2)^{3/2}} e^{-\frac{1}{2\sigma_2{}^2(1-\rho^2)}(v - \frac{\rho x\sigma_2}{\sigma_1})^2} dv \\ &= -\frac{e^{-\frac{x^2}{2\sigma_1^2}}}{P(X_2 > y)(2\pi)(1-\rho^2)^{1/2}} e^{-\frac{1}{2\sigma_2{}^2(1-\rho^2)}(y - \frac{\rho x\sigma_2}{\sigma_1})^2}. \end{aligned} \tag{1.7}$$

Similarly, the conditional density function $g_\rho(x)$ can be written as

$$g_\rho(x) = \frac{e^{-\frac{x^2}{2\sigma_1^2}}}{P(X_2 > y)\,\sqrt{2\pi}\sigma_1} \int_y^{+\infty} \frac{1}{\sigma_2(1-\rho^2)^{1/2}(2\pi)^{1/2}}\; e^{-\frac{1}{2\sigma_2{}^2(1-\rho^2)}(v-\frac{\rho x\sigma_2}{\sigma_1})^2} dv. \qquad (1.8)$$

This gives

$$h(x) = \frac{G'_\rho(x)}{g_{\rho(x)}} = -\sigma_2\,\sigma_1\, r_{W_x}(y),$$

where, $r_{W_x}(.)$ denotes the hazard rate of W_x, a normal random variable with mean $\rho x\sigma_2/\sigma_1$ and variance $\sigma_2^2(1-\rho^2)$. It is known that the family of normal random variables with a fixed variance but with different means, is ordered according to hazard rate order and the one with smaller mean having greater hazard rate. Using this fact, it follows that if $\rho > 0$, then $h(x)$ is increasing in x. Hence by Theorem 1.1, $0 \le \rho' \le \rho \Rightarrow \mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}$. If $\rho < 0$, then $h(x)$ is decreasing in x, hence (X_1, X_2) is increasing in ρ in the sense of upper orthant dispersive ordering. This proves the required result.

■

The organization of the paper is as follows. In Section 2 we study some properties of the upper orthant dispersive ordering as defined above. It is proved that if two random vectors have the same dependence structure (copula), then they are ordered according to upper orthant dispersive ordering if and only if their corresponding marginals are ordered according to univariate dispersive ordering. In Section 3 we consider the special case of nonnegative random variables (more generally if the conditional distributions have common left end points of their supports). It is shown that if two random vectors have the same marginal distributions and they are ordered according to upper orthant dispersive ordering, then their bivariate copulas are ordered implying that one random vector is dependent in the sense of positive quadrant dependence (PQD) than the other. We also study the connection between upper orthant dispersive ordering and multivariate hazard rate ordering as introduced by Hu, Khaledi and Shaked (2002). The last section is devoted to some examples and applications. It is shown that if two univariate distributions are ordered according to dispersive ordering, then the corresponding vectors of order statistics from them are ordered according to upper orthant dispersive ordering.

2 Properties of upper orthant dispersive Ordering

In this section we establish an interesting property of the upper orthant dispersive ordering that if two n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$ have the same dependence structure in the sense that they have the same copula, then dispersive ordering among the marginal distributions implies upper orthant dispersive ordering and vice versa. The notion of copula has been introduced by Sklar (1959), and studied, among others by Kimeldrof and Sampson (1975), under the name of uniform representation, and by Deheuvels (1978), under the name of dependence function. A copula C is a cumulative distribution function with uniform margins on

$[0,1]$. Given a copula C, if one defines

$$F(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \ldots F_n(x_n)), \quad \mathbf{x} \in I\!R^n, \tag{2.1}$$

then F is a multivariate distribution function with margins as $F_1, F_2, \ldots, F_n$. For any multivariate distribution function F with margins as $F_1, F_2, \ldots, F_n$, there exists a copula C such that (2.1) holds. If F is continuous, then C is unique and can be constructed as follows

$$C(\mathbf{u}) = F[F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_n^{-1}(u_n)], \quad \mathbf{u} \in [0,1]^n. \tag{2.2}$$

It follows that if $\mathbf{X}$ and $\mathbf{Y}$ are two n- dimensional random vectors with margins as $(F_1, F_2, \ldots, F_n)$ and $(G_1, G_2, \ldots, G_n)$, respectively and if they have the same copula, then

$$(F_1(X_1), F_2(X_2), \ldots F_n(X_n)) \stackrel{st}{=} (G_1(Y_1), G_2(Y_2), \ldots G_n(Y_n)). \tag{2.3}$$

For $i = 1, \ldots, n$, let us denote by $H_{i,\mathbf{u}}^X$ the cdf of the conditional distribution $\left(X_i \mid \bigcap_{j \neq i} \{X_j > F_j^{-1}(u_j)\}\right)$ and by $H_{i,\mathbf{u}}^Y$ that of $\left(Y_i \mid \bigcap_{j \neq i} \{Y_j > G_j^{-1}(u_j)\}\right)$. To prove the next theorem, we first prove the following lemma, which may be of independent interest.

LEMMA 2.1. *If two n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$ have the same copula, then for $i = 1, \ldots n$,*

$$F_i \circ H_{i,\mathbf{u}}^{X\ -1}(\beta) = G_i \circ H_{i,\mathbf{u}}^{Y\ -1}(\beta), \quad \beta \in [0,1], \quad \mathbf{u} \in [0,1]^{n-1}. \tag{2.4}$$

Proof: Proving (2.4) is equivalent to proving that for $i = 1, \ldots n$ and $\mathbf{u} \in [0,1]^{n-1}$,

$$\begin{aligned}
& \overline{H}_{i,\mathbf{u}}^X \circ F_i^{-1}(v) = \overline{H}_{i,\mathbf{u}}^Y \circ G_i^{-1}(v), \quad \forall v \in [0,1] \\
\Leftrightarrow \quad & P[X_i > F_i^{-1}(v) \mid \bigcap_{j \neq i} \{X_j > F_j^{-1}(u_j)\}] \\
& = P[Y_i > G_i^{-1}(v) \mid \bigcap_{j \neq i} \{Y_j > G_j^{-1}(u_j)\}], \quad \forall v \in [0,1] \\
\Leftrightarrow \quad & P[F_i(X_i) > v \mid \bigcap_{j \neq i} \{F_j(X_j) > u_j\}] \\
& = P[G_i(Y_i) > v \mid \bigcap_{j \neq i} \{G_j(Y_j) > u_j\}], \quad \forall v \in [0,1].
\end{aligned}$$

which is true because of (2.3) since $\mathbf{X}$ and $\mathbf{Y}$ have the same copula.

∎

THEOREM **2.1.** *Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors with the same copula. Then $\mathbf{X} \stackrel{uo-disp}{\preceq} \mathbf{Y}$ if and only if $X_i \leq_{disp} Y_i$, $i = 1, \ldots, n$.*

Proof : By definition

$$\mathbf{X} \stackrel{uo-disp}{\preceq} \mathbf{Y} \Leftrightarrow H_{i,\mathbf{u}}^{Y\ -1}(\beta) - H_{i,\mathbf{u}}^{X\ -1}(\beta) \text{ is increasing in } \beta \in [0,1],\ \mathbf{u} \in [0,1]^{n-1}, \text{ for } i = 1, \ldots, n. \tag{2.5}$$

It follows from (2.4) that if $\mathbf{X}$ and $\mathbf{Y}$ have the same copula, then for $i = 1, \ldots n$ and $\mathbf{u} \in [0,1]^{n-1}$,

$$H_{i,\mathbf{u}}^{Y\ -1}(\beta) - H_{i,\mathbf{u}}^{X\ -1}(\beta) = G_i^{-1} \circ F_i(H_{i,\mathbf{u}}^{X\ -1}(\beta)) - H_{i,\mathbf{u}}^{X\ -1}(\beta). \tag{2.6}$$

for $i = 1, \ldots n$ and for every $\beta \in [0,1]$.

It is easy to see that the right hand side of (2.6) is increasing in β if and only if $G_i^{-1}F_i(x) - x$ is increasing in x, that is, if and only if $X_i \leq_{disp} Y_i, \quad i = 1, \ldots, n$. This proves the desired result. ■

Recently Müller and Scarsini (2001) have investigated some other multivariate stochastic orders for which results parallel to Theorem 2.1 hold for those orders. The following interesting property of the upper orthant dispersive ordering immediately follows from the above theorem.

Corollary 2.1. *Let* $\mathbf{Y} = (a_1X_1 + b_1, a_2X_2 + b_2, \ldots, a_nX_n + b_n)$. *Then for* $a_i \geq 1,\ b_i \in I\!R$, $\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}$.

Proof : Since $\mathbf{X}$ and $\mathbf{Y}$ have the same copula, the required result follows immediately from Theorem 2.1 ■

Example 2.1 (Multivariate Normal Distributions).

Let $\mathbf{X}$ follow the p-variate multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and dispersion matrix $\mathbf{\Sigma} = ((\sigma_{ij}))$, with $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, $\rho_{ii} = 1$, $\sigma_i > 0$, $i,j = 1, \ldots, p$. Let $\mathbf{Y}$ follow the p-variate multivariate normal distribution with mean vector $\boldsymbol{\mu}'$ and dispersion matrix $\mathbf{\Sigma}' = ((\sigma'_{ij}))$, with $\sigma'_{ij} = \rho_{ij}\sigma'_i\sigma'_j$, $\sigma'_i > 0$, $i,j = 1, \ldots, p$. It is known that $\mathbf{X}$ and $\mathbf{Y}$ have the same copula. It follows from Theorem 2.1 that $\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}$ if and only if $\sigma_i \leq \sigma'_i$ for $i = 1, \ldots, p$. This result in conjunction with Example 1.1 leads us to the following result for comparing two bivariate normal distributions.

Let $\mathbf{X}$ and $\mathbf{Y}$ follow bivariate normal distributions with dispersion matrices

$$\mathbf{\Sigma} = \begin{pmatrix} {\sigma_1}^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & {\sigma_2}^2 \end{pmatrix},\ \mathbf{\Sigma}' = \begin{pmatrix} {\sigma'_1}^2 & \rho'\sigma'_1\sigma'_2 \\ \rho'\sigma'_1\sigma'_2 & {\sigma'_2}^2 \end{pmatrix},$$

respectively. If $0 < \sigma_i \leq \sigma'_i$ for $i = 1,2$ and $|\rho'| \leq |\rho| < 1$, then $\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}$.

It will be interesting to find necessary and sufficient conditions under which two multivariate normal random vectors will be ordered according to upper orthant dispersive ordering in the general case.

It follows immediately that if two random vectors are ordered according to upper orthant dispersive ordering, then so are their corresponding subsets. In particular, their marginal distributions will be then ordered according to univariate dispersive ordering.

Theorem **2.2.** *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two n-dimensional random vectors such that* $\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}$. *Then*

$$\mathbf{X}_I \overset{uo-disp}{\preceq} \mathbf{Y}_I,$$

where $I = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, $\mathbf{X}_I = (X_{i_1}, \ldots, X_{i_k})$, $\mathbf{Y}_I = (Y_{i_1}, \ldots, Y_{i_k})$ *and* $k = 1, \ldots, n$.

The proof of the next result is also immediate.

THEOREM **2.3.** *Let* $\mathbf{X}_1, \ldots, \mathbf{X}_m$ *be a set of independent random vectors where the dimension of* $\mathbf{X}_i$ *is* k_i, $i = 1, \ldots, m$. *Let* $\mathbf{Y}_1, \ldots, \mathbf{Y}_m$ *be another set of independent random variables where the dimension of* $\mathbf{Y}_i$ *is* k_i, $i = 1, \ldots, m$. *Then*

$$\left(\mathbf{X}_i \overset{uo-disp}{\preceq} \mathbf{Y}_i,\ i = 1, \ldots, m\right) \Rightarrow (\mathbf{X}_1, \ldots, \mathbf{X}_m) \overset{uo-disp}{\preceq} (\mathbf{Y}_1, \ldots, \mathbf{Y}_m). \tag{2.7}$$

Remark 2.1. : *A consequence of (2.7) is that if* $X_1, \ldots, X_n$ *is a collection of independent univariate random variables and* $Y_1, \ldots, Y_n$ *is another set of independent random variables, then* $X_i \leq_{disp} Y_i$, $i = 1, \ldots, n$ *implies* $\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}$.

In general, there does not seem to be any direct connection between upper orthant dispersive ordering and the multivariate dispersive ordering as introduced by Fernadez-Ponse and Suarez-Llorens (2003). According to their definition $\mathbf{X} \leq_{Disp} \mathbf{Y}$ may not imply that $X_i \leq_{Disp} Y_i$ for $i = 1, \ldots, n$. Also the multivariate dispersive ordering as defined by them may not be preserved under permutations of the variables. On the other hand the upper orthant dispersive ordering is invariant under same permutation of the two vectors and their marginals are also ordered according to univariate dispersive ordering. Obviously if $\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}$, then $tr\, \Sigma_X \leq tr\, \Sigma_Y$, where Σ_X and Σ_Y denote the dispersion matrices of $\mathbf{X}$ and $\mathbf{Y}$, respectively.

3 The case of nonnegative random variables

In this section we will restrict our attention to the case when the random vectors under consideration are nonnegative or more generally, they have a finite common left endpoint of their supports. We will see that certain results hold in this case which may not hold in the general case. The following assumptions will be made at some places in this paper :

Assumption A : The random variables $\{X_i \mid \bigcap_{j \neq i} \{X_j > F_j^{-1}(u_j)\}\}$ and $\{Y_i \mid \bigcap_{j \neq i} \{Y_j > G_j^{-1}(u_j)\}\}$ have a *finite* common left endpoint of their supports for all $\boldsymbol{u}$ and for $i = 1, \ldots, n$.

In the univariate case, for nonnegative random variables, there is an intimate connection between hazard rate ordering and dispersive ordering and which is made more explicit in the following result of Bagai and Kochar (1986). We use this theorem to prove some of the results of this section.

THEOREM **3.1.** *Let* X *and* Y *be two univariate random variables with distribution functions* F *and* G, *respectively such that* $F(0) = G(0) = 0$. *Then*

(a) *if* $Y \leq_{hr} X$ *and either* F *or* G *is DFR (decreasing failure rate), then* $Y \leq_{disp} X$;

(b) if $Y \leq_{disp} X$ and either F or G is IFR (increasing failure rate), then $Y \leq_{hr} X$.

For a bivariate random vector (S,T), we say that T is *right tail increasing* in S if $P[T > t|S > s]$ is increasing in s for all t, and denote this relationship by RTI$(T|S)$. If S and T are continuous lifetimes, then T is right tail increasing in S if and only if $r(s|T > t) \leq r(s|T > 0) = r_S(s)$ for all $s > 0$ and for each fixed t. The RTI property is weaker than the RCSI property, but stronger than PQD (positive quadrant dependence). In the next theorem we study the effect of positive dependence on upper orthant dispersive ordering for *nonnegative* random vectors.

THEOREM **3.2.** *Let $\mathbf{X} = (X_1, X_2)$ be a bivariate random vector such that the left end point of the support of $\left\{X_i|X_j > F_j^{-1}(u)\right\}$ is finite and independent of $u \in [0,1]$ for $i,j = 1,2$. Let $\mathbf{X^I} = (X_1^I, X_2^I)$ be a random vector of independent random variables such that $X_i \overset{st}{=} X_i^I$, i=1,2.*
(a) If X_i is RTI in X_j, $i \neq j$, and X_i is DFR for i,j=1,2. Then

$$(X_1, X_2) \overset{uo-disp}{\succeq} (X_1^I, X_2^I). \tag{3.1}$$

(b) If $(X_1, X_2) \overset{uo-disp}{\succeq} (X_1^I, X_2^I)$ and X_i is IFR for $i = 1,2$, then X_i is RTI in X_j, $i \neq j$, $i,j = 1,2$.

Proof : (a) Note that $RTI(X_i|X_j)$ if and only if for all $u \geq 0$,

$$\left\{X_i|X_j > F_j^{-1}(u)\right\} \geq_{hr} X_i. \tag{3.2}$$

It follows from Theorem 3.1(a) that if, in addition, X_i is DFR, then

$$\left\{X_i|X_j > F_j^{-1}(u)\right\} \geq_{disp} X_i \overset{st}{=} X_i^I. \tag{3.3}$$

Since this holds for $i,j = 1,2$, the required result follows.

(b) $(X_1, X_2) \overset{uo-disp}{\succeq} (X_1^I, X_2^I)$ implies for all $u \geq 0$,

$$\left\{X_i|X_j > F_j^{-1}(u)\right\} \geq_{disp} X_i^I \overset{st}{=} X_i, \quad i \neq j,\ i,j = 1,2.$$

This together with the assumption that X_1 and X_2 are IFR implies (3.2) by Theorem 3.1 (b). This proves that $RTI(X_i|X_j)$, $i \neq j$, $i,j = 1,2$.

■

THEOREM **3.3.** *Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors satisfying the Assumption A and such that $X_i \overset{st}{=} Y_i$, $i = 1, \ldots, n$. Then $\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}$ implies that*

$$C_{i,j}^X \leq C_{i,j}^Y, \quad \textit{for } i,j = 1, \ldots, n,\ i \neq j, \tag{3.4}$$

where $C_{i,j}^X$ $(C_{i,j}^Y)$ denotes the copula of $(X_i, X_j)((Y_i, Y_j))$.

Proof :
$$\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y} \Rightarrow (X_i, X_j) \overset{uo-disp}{\preceq} (Y_i, Y_j),\ i \neq j,\ i,j \in \{1, \ldots, n\}, \tag{3.5}$$

from which it follows that

$$\left\{X_i \mid X_j > F_j^{-1}(u_j)\right\} \leq_{disp} \left\{Y_i \mid Y_j > G_j^{-1}(u_j)\right\},\ u_j \in [0,1],$$

and which in turn implies

$$\left\{X_i \mid X_j > F_j^{-1}(u_j)\right\} \leq_{st} \left\{Y_i \mid Y_j > G_j^{-1}(u_j)\right\},\ u_j \in [0,1], \tag{3.6}$$

under the Assumption A since dispersive ordering implies stochastic ordering when the random variables have a finite common left end point of their supports. If we denote by $\overline{F}_{i,j}$ ($\overline{G}_{i,j}$) the joint survival function of (X_1, X_j) ((Y_i, Y_j)), then (3.6) can be written as,

$$\begin{array}{rc}
& \overline{F}_{i,j}(x, F_j^{-1}(u_j)) \leq \overline{G}_{i.j}(x, G_j^{-1}(u_j)),\ \text{for } x \geq 0, u_j \in [0,1] \\
\Leftrightarrow & \overline{F}_{i,j}(F_i^{-1}(u_i), F_j^{-1}(u_j)) \leq \overline{G}_{i.j}(F_i^{-1}(u_i), G_j^{-1}(u_j)) \ \text{ for all } u_i, u_j \in [0,1] \\
\Leftrightarrow & \overline{F}_{i,j}(F_i^{-1}(u_i), F_j^{-1}(u_j)) \leq \overline{G}_{i.j}(G_i^{-1}(u_i), G_j^{-1}(u_j)) \ \text{ for all } u_i, u_j \in [0,1] \\
& (\text{ since } X_i \overset{st}{=} Y_i,\ i = 1, \ldots, n) \\
\Leftrightarrow & \overline{C}_{i,j}^X(u_i, u_j) \leq \overline{C}_{i,j}^Y(u_i, u_j) \ \text{ for all } u_i, u_j \in [0,1] \\
\Leftrightarrow & C_{i,j}^X(u_i, u_j) \leq C_{i,j}^Y(u_i, u_j) \ \text{ for all } u_i, u_j \in [0,1]
\end{array}$$

where $\overline{C}(u, v) = 1 - u - v + C(u.v)$.

■

If (3.4) holds and the margins of (X_i, X_j) and (Y_i, Y_j) are equal, then we say that (Y_i, Y_j) is *more PQD* (positive quadrant dependent) than (X_i, X_j) (cf. Joe, 1997, p.36). Note that $C_{i,j}^X(u_i, u_j) \equiv u_i u_j$ in case X_i and X_j are independent and $C_{i,j}^X(u_i, u_j) \geq u_i u_j$ for all $u_i, u_j \in [0,1]$ in case X_i and X_j are PQD. Thus according to Theorem 3.3, if the Assumption A holds and if $\mathbf{X}$ and $\mathbf{Y}$ have the same margins and $\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}$, then the Y_i's are more dependent than the X_i's according to PQD ordering. We get the following result as a special case.

COROLLARY 3.1. *Let $\mathbf{X} = (X_1, X_2)$ be a bivariate random vector such that the left end point of the support of $\left\{X_i | X_j > F_j^{-1}(u)\right\}$ is finite and independent of $u \in [0,1]$, for $i \neq j$ and $i,j = 1,2$. Let $\mathbf{X^I} = (X_1^I, X_2^I)$ be a random vector of independent random variables such that $X_i \overset{st}{=} X_i^I$, i=1,2. Then*

$$(X_1, X_2) \overset{uo-disp}{\succeq} (X_1^I, X_2^I) \Rightarrow \mathbf{X} \text{ is } PQD. \tag{3.7}$$

Contrast this result with Theorem 3.2 (b) which is a stronger one since RTI implies PQD. However, here no assumption on the monotonicity of the hazard rates is made in the second case.

Remark 3.1. *The Assumption A is very crucial for Theorem 3.3 and Corollary 3.1 to hold. As a counter example, let Y_1 and Y_2 be two independent $U(0,1)$ random variables. Let $X_1 = X_2$ be also uniformly distributed over $(0,1)$. Note that X_1 and X_2 are strongly positively dependent as they satisfy the Frechet upper bound. Let us compare (X_1, X_2) with (Y_1, Y_2) according to upper orthant dispersive ordering. The relevant conditional distributions to compare are*

$$[X_1|X_1 > u] \quad \text{and} \quad [Y_1|Y_2 > u],\ 0 < u < 1.$$

The left-hand conditional distribution is $U(u,1)$ and the right-hand conditional distribution is $U(0,1)$. Hence $(X_1, X_2) \overset{uo-disp}{\preceq} (Y_1, Y_2)$, but $C^X_{1,2} \geq C^Y_{1,2}$, contradictory to (3.4). The reason for this contradiction is that unless the Assumption A is satisfied, dispersive ordering may not imply stochastic ordering.

THEOREM **3.4.** *Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors satisfying the Assumption A and such that $\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}$. Then*

$$uv \leq C^X_{i,j} \leq C^Y_{i,j},\ i,j = 1, \ldots, n,\ i \neq j. \tag{3.8}$$

implies

$$Cov(h_1(X_i), h_2(X_j)) \leq Cov(h_1(Y_i), h_2(Y_j)) \tag{3.9}$$

for all increasing convex functions h_1 and h_2 for which the above covariances exit.

Proof : Without loss of generality, let $i = 1$ and $j = 2$. The survival functions of $(h_1(X_1), h_2(X_2))$, $h_1(X_1)$, $h_2(X_2)$, respectively, are $\overline{H}(x_1, x_2) = \overline{F}(h_1^{-1}(x_1), h_2^{-1}(x_2))$, $\overline{H}_1(x_1) = \overline{F}_1(h_1^{-1}(x_1))$ and $\overline{H}_2(x_1) = \overline{F}_2(h_2^{-1}(x_2))$. Similarly, the survival functions of $(h_1(Y_1), h_2(Y_2))$, $h_1(Y_1)$, $h_2(Y_2)$, respectively, are $\overline{K}(x_1, x_2) = \overline{G}(h_1^{-1}(x_1), h_2^{-1}(x_2))$, $\overline{K}_1(x_1) = \overline{G}_1(h_1^{-1}(x_1))$ and $\overline{K}_2(x_1) = \overline{G}_2(h_2^{-1}(x_2))$. Covariance between $h_1(X_1)$ and $h_2(X_2)$, if it exists, can be expressed as

$$\begin{aligned}
Cov\,(h_1(X_1), h_2(X_2)) &= \int\!\!\int \left(\overline{H}(x_1,x_2) - \overline{H}_1(x_1)\overline{H}_2(x_2)\right) dx_1 dx_2 \\
&= \int\!\!\int \left(\overline{F}(h_1^{-1}(x_1), h_2^{-1}(x_2)) - \overline{F}_1(h_1^{-1}(x_1))\overline{F}_2(h_2^{-1}(x_2))\right) dx_1 dx_2 \\
&= \int\!\!\int \left(\overline{F}(F_1^{-1}(u), F_2^{-1}(v)) - (1-u)(1-v)\right) \\
&\times \left(\frac{h_1'(F_1^{-1}(u))}{f_1(F_1^{-1}(u)}\right)\left(\frac{h_2'(F_2^{-1}(v))}{f_2(F_2^{-1}(v)}\right) dudv \\
&= \int_0^1\!\!\int_0^1 \left(\overline{C}^{\mathbf{X}}_{1,2}(u,v) - (1-u)(1-v)\right) \\
&\times \left(\frac{h_1'(F_1^{-1}(u))}{f_1(F_1^{-1}(u)}\right)\left(\frac{h_2'(F_2^{-1}(v))}{f_2(F_2^{-1}(v)}\right) dudv
\end{aligned} \tag{3.10}$$

where $h_1^{-1}(x_1) = F_1^{-1}(u)$ and $h_2^{-1}(x_2) = F_2^{-1}(v)$. The assumption $\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y}$ implies that $X_i \leq_{disp} Y_i$ from which it follows that $f_i(F_i^{-1}(u)) \geq g_i(G_i^{-1}(u))$ and $F_i^{-1}(u) \leq G_i^{-1}(u)$, $i = 1, 2$, under the Assumption A. Now $h_i'(x)$ is increasing in x, since $h(x)$ is convex. Combining these facts, the required result follows from assumption (3.8).

■

THEOREM **3.5.** *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two* n*-dimensional random vectors satisfying the Assumption A and let* $\phi_1, \ldots, \phi_n$ *be increasing convex functions on* $\mathbb{R}^+$*. Then*

$$\mathbf{X} \overset{uo-disp}{\preceq} \mathbf{Y} \Rightarrow (\phi_1(X_1), \ldots, \phi_n(X_n)) \overset{uo-disp}{\preceq} (\phi_1(Y_1), \ldots, \phi_n(Y_n))$$

Proof : Note that the cdf of $\phi_j(X_j)$ is $F_{\phi_j}(x) = F_j(\phi_j^{-1}(x))$ with its inverse as $F_{\phi_j}^{-1}(u) = \phi_j(F_j^{-1}(u))$. We have to prove that for $i = 1, \ldots, n$, $u_j \in (0,1)$, $\quad j = 1, \ldots, n$, $j \neq i$,

$$\left(\phi_i(X_i) \mid \bigcap_{j\neq i}\{\phi_j(X_j) > F_{\phi_j}^{-1}(u_j)\}\right) \leq_{disp} \left(\phi_i(Y_i) \mid \bigcap_{j\neq i}\{\phi_j(Y_j) > G_{\phi_j}^{-1}(u_j)\}\right).$$

That is,

$$\left(\phi_i(X_i) \mid \bigcap_{j\neq i}\{\phi_j(X_j) > \phi_j(F_j^{-1}(u_j))\}\right) \leq_{disp} \left(\phi_i(Y_i) \mid \bigcap_{j\neq i}\{\phi_j(Y_j) > \phi_j(G_j^{-1}(u_j))\}\right),$$

which is equivalent to

$$\left(\phi_i(X_i) \mid \bigcap_{j\neq i}\{X_j > F_j^{-1}(u_j)\}\right) \leq_{disp} \left(\phi_i(Y_i) \mid \bigcap_{j\neq i}\{Y_j > G_j^{-1}(u_j)\}\right).$$

Using Assumption A, for $i = 1, \ldots, n$, $u_j \in (0,1)$, $\quad j = 1, \ldots, n$, $j \neq i$,

$$\left(X_i \mid \bigcap_{j\neq i}\{X_j > F_j^{-1}(u_j)\}\right) \leq_{disp} \left(Y_i \mid \bigcap_{j\neq i}\{Y_j > G_j^{-1}(u_j)\}\right)$$

implies

$$\left(X_i \mid \{X_j > F_j^{-1}(u_j)\}\right) \leq_{st} \left(Y_i \mid \{Y_j > G_j^{-1}(u_j)\}\right).$$

Now the required result follows from Theorem 2.2 of Rojo and He (1991), since ϕ_i's are increasing convex functions.

■

Hu, Khaledi and Shaked (2002) gave the following definition of multivariate (weak) hazard rate ordering.

Definition 3.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be* n*-dimensional random vectors with hazard gradients* $r_{\mathbf{X}}$ *and* $r_{\mathbf{Y}}$*, respectively. We say that* $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *according to weak hazard rate ordering (written as* $\mathbf{X} \leq_{\mathrm{whr}} \mathbf{Y}$*) if*

$$\left(X_i \mid \bigcap_{j\neq i}\{X_j > x_j\}\right) \leq_{hr} \left(Y_i \mid \bigcap_{j\neq i}\{Y_j > x_j\}\right),$$

for $i=1,2,\ldots,n,\ \mathbf{x}\in I\!R^n$.

That is, if

$$r^{(i)}_{\mathbf{X}}(\mathbf{x})\geq r^{(i)}_{\mathbf{Y}}(\mathbf{x}),\quad i=1,2,\ldots,n,\ \mathbf{x}\in I\!R^n.$$

In the next theorem, we establish results analogous to Theorem 3.1 between upper orthant dispersive ordering and multivariate weak hazard rate ordering under the Assumption A.

THEOREM **3.6.** *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two* n*-dimensional random vectors satisfying the Assumption A.*

(a) If $\mathbf{Y}\leq_{\rm whr}\mathbf{X}$ *and either* $r^{(i)}_{\mathbf{X}}(\mathbf{x})$ *or* $r^{(i)}_{\mathbf{Y}}(\mathbf{x})$ *is decreasing in* $\mathbf{x}$*, for* $i=1,\ldots,n$*, then* $\mathbf{Y}\overset{uo-disp}{\preceq}\mathbf{X}$.

(b) If $\mathbf{Y}\overset{uo-disp}{\preceq}\mathbf{X}$ *and either* $r^{(i)}_{\mathbf{X}}(\mathbf{x})$ *or* $r^{(i)}_{\mathbf{Y}}(\mathbf{x})$ *is increasing in* $\mathbf{x}$*, for* $i=1,\ldots,n$*, then* $\mathbf{Y}\leq_{\rm whr}\mathbf{X}$.

Proof : (a) Under the Assumption A, $\mathbf{Y}\leq_{whr}\mathbf{X}$, implies that $Y_i\leq_{st}X_i$, $i=1,\ldots,n,$ which is equivalent to $G_i^{-1}(u)\leq F_i^{-1}(u)$, for $u\in[0,1]$, $i=1,\ldots,n$. Again, $\mathbf{Y}\leq_{whr}\mathbf{X}$ implies that for $i=1,\ldots,n$, $(Y_i\mid\bigcap_{j\neq i}\{Y_j>x_j\})\leq_{st}(X_i\mid\bigcap_{j\neq i}\{X_j>x_j\})$. Taking $x_j=F_j^{-1}(u_j)$, $j=1,\ldots,n,j\neq i$, we find that this implies $y_i'(\beta;\mathbf{u})\leq x_i(\beta;\mathbf{u})$, where $y_i'(\beta;\mathbf{u})$ is the βth quantiles of the conditional distribution $\left(Y_i\mid\bigcap_{j\neq i}\{Y_j>F_j^{-1}(u_j)\}\right)$ and $x_i(\beta;\mathbf{u})$ is as defined earlier. On the other hand, $r^{(i)}_{\mathbf{Y}}(\mathbf{x})$ decreasing in $\mathbf{x}$ implies that

$$\left(Y_i\mid\bigcap_{j\neq i}\{Y_j>x_j\}\right)\leq_{st}\left(Y_i\mid\bigcap_{j\neq i}\{Y_j>x_j'\}\right),$$

for $x_j\leq x_j'$, $j=1\ldots,n$, $j\neq i$. This along with $G_i^{-1}(u_i)\leq F_i^{-1}(u_i)$, implies that

$$y_i(\beta;\mathbf{u})\leq y_i'(\beta;\mathbf{u})\leq x_i(\beta;\mathbf{u}),\ i=1,\ldots,n.$$

Using these observations we get,
$r^{(i)}_{\mathbf{X}}\left(F_1^{-1}(u_1),\ldots,F_{i-1}^{-1}(u_{i-1}),x_i(\beta;\mathbf{u}),\ldots,F_n^{-1}(u_n)\right)\leq$

$$r^{(i)}_{\mathbf{Y}}\left(F_1^{-1}(u_1),\ldots,F_{i-1}^{-1}(u_{i-1}),x_i(\beta;\mathbf{u}),\ldots,F_n^{-1}(u_n)\right),$$

since $\mathbf{Y}\leq_{\rm whr}\mathbf{X}$. The right hand side of this inequality is less than or equal to

$$r^{(i)}_{\mathbf{Y}}\left(G_1^{-1}(u_1),\ldots,G_{i-1}^{-1}(u_{i-1}),y_i(\beta;\mathbf{u}),\ldots,G_n^{-1}(u_n)\right),$$

since $r^{(i)}_{\mathbf{Y}}(\mathbf{x})$ is decreasing in $\mathbf{x}$. This completes the proof of (a).

b) From the assumption $\mathbf{Y}\overset{uo-disp}{\preceq}\mathbf{X}$, it follows that

$$\begin{aligned}&r^{(i)}_{\mathbf{X}}\left(F_1^{-1}(u_1),\ldots,F_{i-1}^{-1}(u_{i-1}),x_i(\beta;\mathbf{u}),\ldots,F_n^{-1}(u_n)\right)\\ \leq\ &r^{(i)}_{\mathbf{Y}}\left(G_1^{-1}(u_1),\ldots,G_{i-1}^{-1}(u_{i-1}),y_i(\beta;\mathbf{u}),\ldots,G_n^{-1}(u_n)\right),\end{aligned}\tag{3.11}$$

and

$$\left(X_i \mid \bigcap_{j\neq i}\{X_j > F_j^{-1}(u_j)\},\right) \geq_{st} \left(Y_i \mid \bigcap_{j\neq i}\{Y_j > G_j^{-1}(u_j)\}\right),$$

which implies that, $x_i(\beta;\mathbf{u}) \geq y_i(\beta;\mathbf{u})$ and $F_j^{-1}(u_j) \geq G_j^{-1}(u_j)$, $j = 1,\ldots,n$, $j \neq i$. Using these facts and the assumption that $r_{\mathbf{Y}}^{(i)}(\mathbf{x})$ increasing in $\mathbf{x}$, it follows that, for $i = 1,\ldots,n$, the R.H.S. of (3.11) is less than or equal to

$$r_{\mathbf{Y}}^{(i)}\left(F_1^{-1}(u_1),\ldots,F_{i-1}^{-1}(u_{i-1}),x_i(\beta;\mathbf{u}),\ldots,F_n^{-1}(u_n)\right) = r_{\mathbf{Y}}^{(i)}(\mathbf{x}).$$

That is, we have shown that for $i = 1,\ldots,n$,

$$r_{\mathbf{X}}^{(i)}(\mathbf{x}) \leq r_{\mathbf{Y}}^{(i)}(\mathbf{x})$$

and hence the required result.

■

Remark 3.2. : *If $r_{\mathbf{X}}^{(i)}(x_1,\ldots,x_i,\ldots,x_n)$ increases in x_i for $i = 1,\ldots,n$ then we say that the the random vector $\mathbf{X}$ has a multivariate increasing hazard rate distribution (cf. Johnson and Kotz, 1975). The condition $r_{\mathbf{X}}^{(i)}(x_1,\ldots,x_i,\ldots,x_n)$ increasing in x_j, $j = 1,\ldots,n$, $j \neq i$, $i = 1,\ldots,n$ describes a condition of positive dependence that is equivalent to saying that the random vector $\mathbf{X}$ has* **RCSI** *(right corner set increasing) dependence property. That is,*

$$P[X_1 > x_1,\ldots,X_n > x_n | X_1 > x_1',\ldots,X_n > x_n']$$

is increases in x_i', $i = 1,\ldots,n$.

We will now study some preservation properties of the upper orthant dispersive order under random compositions. Such results are often referred to as preservations under 'random mapping' (see Shaked and Wong (1995)), or preservations of 'stochastic convexity' (see Shaked and Shanthikumar (1994, Chapter 6) and Denuit, Lefèvre, and Utev (1999), and references therein).

Let $\{\overline{F}_\theta,\, \theta \in \mathcal{X}\}$ be a family of n-dimensional survival functions, where $\mathcal{X}$ is a subset of the real line. Let $\boldsymbol{X}(\theta)$ denote a random vector with survival function $\overline{F}_\theta$. For any random variable Θ with support in $\mathcal{X}$, and with distribution function H, let us denote by $\boldsymbol{X}(\Theta)$ a random vector with survival function $\overline{G}$ given by

$$\overline{G}(\boldsymbol{x}) = \int_{\mathcal{X}} \overline{F}_\theta(\boldsymbol{x})\, dH(\theta), \quad \boldsymbol{x} \in I\!R^n.$$

THEOREM **3.7.** *Consider a family of n-dimensional survival functions $\{\overline{F}_\theta,\, \theta \in \mathcal{X}\}$ as above. Let Θ_1 and Θ_2 be two random variables with supports in $\mathcal{X}$ and distribution functions H_1 and H_2, respectively. Let $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ be two random vectors such that $\boldsymbol{Y}_i =_{\rm st} \boldsymbol{X}(\Theta_i)$, $i = 1,2$; that is, suppose that the survival function of $\boldsymbol{Y}_i$ is given by*

$$\overline{G}_i(\boldsymbol{x}) = \int_{\mathcal{X}} \overline{F}_\theta(\boldsymbol{x})\, dH_i(\theta), \quad \boldsymbol{x} \in I\!R^n,\, i = 1,2.$$

If

(a)

$$\boldsymbol{X}(\theta) \leq_{\rm whr} \boldsymbol{X}(\theta') \quad \textit{whenever} \;\; \theta \leq \theta', \tag{3.12}$$

(b) Θ_1 *and* Θ_2 *are ordered in the univariate hazard rate order; that is,*

$$\Theta_1 \leq_{\rm hr} \Theta_2, \tag{3.13}$$

(c) $r^{(i)}_{\mathbf{X}(\theta)}(x_1, \ldots, x_n)$ *is decreasing in* x_j, $j = 1, \ldots, n$, $i = 1, \ldots, n$, *then*

$$\mathbf{Y_1} \overset{uo-disp}{\preceq} \mathbf{Y_2}. \tag{3.14}$$

Proof : Hu, Khaledi and Shaked (2002) proved that assumptions (a) and (b) imply that $\boldsymbol{Y}_1 \leq_{\rm whr} \boldsymbol{Y}_2$. Now we show that for $i = 1, \ldots, n$, $r^{(i)}_{\mathbf{Y_1}}(x_1, \ldots, x_n)$ is decreasing in x_j, $j = 1, \ldots, n$, Then the required result will follow from Theorem 3.6 (a). Assumption (a) is equivalent to $\overline{F}_\theta(\mathbf{x})$ being TP_2 in (θ, x_j), $j = 1, \ldots, n$. $r^{(i)}_{\mathbf{X}(\theta)}(x_1, \ldots, x_n)$ decreasing in x_j, $j = 1, \ldots, n$, $j \neq i$ is equivalent to $\overline{F}_\theta(x_1, \ldots, x_n)$ being TP_2 in (x_i, x_j), $i, j = 1, \ldots, n$, $j \neq i$. Using these observations, it follows that $\overline{G}_1(\mathbf{x})$ is TP_2 in (x_i, x_j), $i, j = 1, \ldots, n$ (cf. Karlin, 1968) which is equivalent to $r^{(i)}_{\mathbf{Y_1}}(x_1, \ldots, x_n)$ decreasing in x_j, $j = 1, \ldots, n$, $j \neq i$. It is worth noting that $r^{(i)}_{\mathbf{X}(\theta)}(x_1, \ldots, x_n)$ decreasing in x_i is equivalent to the fact that $\{X_i | \cap_{j \neq i} \{X_j > x_j\}\}$ is a DFR random variable, $i = 1, \ldots, n$. Now $\overline{G}_1(\mathbf{x})$ can be written as follows

$$\overline{G}_1(\mathbf{x}) = \int P_\theta(X_i > x_i \mid \bigcap_{j \neq i} \{X_j > x_j\})\, P_\theta(\bigcap_{j \neq i} \{X_j > x_j\}, j = 1, \ldots, n, j \neq i) dH_1(\theta).$$

That is $(Y_1, \ldots, Y_n)$ is a sort of mixture of DFR random variables, therefore $-\partial \log \overline{G}_j(\mathbf{x})/\partial x_i$ is decreasing in x_i, which is equivalent to $r^{(i)}_{\mathbf{Y}_1}(x_1, \ldots, x_n)$ decreasing in x_i. This completes the proof.

■

4 Examples and Applications

Example 4.1 (Multivariate Pareto distributions).

For $a > 0$, let $\mathbf{X}_a = (X_{a,1}, \ldots, X_{a,n})$ have the survival function $\overline{F}_a$ given by

$$\overline{F}_a(x_1, \ldots, x_n) = \Big(\sum_{i=1}^{n} x_i - n + 1\Big)^{-a}, \quad x_i > 1, \; i = 1, 2, \ldots, n;$$

see, for example, Kotz, Balakrishnan, and Johnson (2000, page 600). The corresponding density function is given by

$$f_a(x_1, \ldots, x_n) = a(a+1) \cdots (a+n-1) \Big(\sum_{i=1}^{n} x_i - n + 1\Big)^{-a-n}, \quad x_i > 1, \; i = 1, 2, \ldots, n.$$

Hu, Khaledi and Shaked (2002) showed that $\mathbf{X}_{a_1} \leq_{whr} \mathbf{X}_{a_2}$, whenever $a_1 \geq a_2$. On the other hand, $r^{(i)}_{\mathbf{X}}(\mathbf{x}) = \frac{a}{\sum_{j=1}^{n} x_j - n+1}$, is decreasing in x_j, $j = 1, \ldots, n$. Then from Theorem 3.6(a) it follows that $\mathbf{X_{a_1}} \overset{uo-disp}{\preceq} \mathbf{X_{a_2}}$, whenever $a_1 \geq a_2$.

Example 4.2 (Bivariate Farlie-Gumbel-Morgenstern distributions).

For $\alpha \in (-1, 1)$, let $\mathbf{X}_\alpha = (X_{\alpha,1}, X_{\alpha,2})$ have the survival function $\overline{F}_\alpha$ given by

$$\overline{F}_\alpha(x_1, x_2) = \overline{F}_1(x_1)\overline{F}_2(x_2)[1 + \alpha(1 - \overline{F}_1(x_1))(1 - \overline{F}_2(x_1))]$$

and $\mathbf{Y}_\alpha = (Y_{\alpha,1}, Y_{\alpha,2})$ have the survival function $\overline{G}_\alpha$ given by

$$\overline{G}_\alpha(x_1, x_2) = \overline{G}_1(x_1)\overline{G}_2(x_2)[1 + \alpha(1 - \overline{G}_1(x_1))(1 - \overline{G}_2(x_1))]$$

where $\overline{F}_1$, $\overline{F}_2$, $\overline{G}_1$ and $\overline{G}_2$ are arbitrary univariate survival functions (which happen to be the marginal survival functions of $X_{\alpha,1}$, $X_{\alpha,2}$, $Y_{\alpha,1}$ and $Y_{\alpha,2}$, respectively, independently of α). Assume that $X_{\alpha,i} \leq_{disp} Y_{\alpha,i}$, $i = 1, 2$. It is easy to see that $\overline{C}^{\mathbf{X}_\alpha}(u, v) = \overline{C}^{\mathbf{Y}_\alpha}(u, v)$. Then from Theorem 2.1 it follows that $\mathbf{X}_\alpha \overset{uo-disp}{\preceq} \mathbf{Y}_\alpha$.

Example 4.3 (Multivariate Gumbel Exponential distributions).

For positive parameters $\boldsymbol{\lambda} = \{\lambda_I : I \subseteq \{1, 2, \ldots, n\}, I \neq \varnothing\}$, let $\boldsymbol{X}_{\boldsymbol{\lambda}} = (X_1, X_2, \ldots, X_n)$ have the survival function $\overline{F}_{\boldsymbol{\lambda}}$ given by

$$\overline{F}_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_n) = \exp\Big\{ -\sum_I \lambda_I \textstyle\prod_{i \in I} x_i\Big\}, \quad (x_1, x_2, \ldots, x_n) \geq (0, 0, \ldots, 0);$$

see Kotz, Balakrishnan, and Johnson (2000, page 406). For another set of positive parameters $\boldsymbol{\lambda}^* = \{\lambda^*_I : I \subseteq \{1, 2, \ldots, n\}, I \neq \varnothing\}$, let $\boldsymbol{Y}_{\boldsymbol{\lambda}^*} = (Y_1, Y_2, \ldots, Y_n)$ have the survival function $\overline{G}_{\boldsymbol{\lambda}^*}$. Let $X_i =_{st} Y_i$, $i = 1, \ldots, n$, that is $\lambda_i = \lambda^*_i$. We show that if $\boldsymbol{\lambda} \geq \boldsymbol{\lambda}^*$, then for $i = 1, \ldots, n$,

$$\left(X_i \mid \bigcap_{j \neq i} \{X_j > F_j^{-1}(u_j)\} \right) \leq_{disp} \left(Y_i \mid \bigcap_{j \neq i} \{Y_j > G_j^{-1}(u_j)\} \right), \ u_j \in [0, 1]. \tag{4.1}$$

Since $X_i =_{st} Y_i$, $i = 1, \ldots, n$, (4.1) is equivalent to

$$\left(X_i \mid \bigcap_{j \neq i} \{X_j > x_j\} \right) \leq_{disp} \left(Y_i \mid \bigcap_{j \neq i} \{Y_j > x_j\} \right), \ i = 1, \ldots, n. \tag{4.2}$$

Let $\mathbf{x_i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, n)$. The survival function of $\left(X_i \mid \cap_{j \neq i} \{X_j > x_j\} \right)$, denoted by $\overline{F}_i(x_i; \mathbf{x_i})$ is

$$\overline{F}_i(x_i; \mathbf{x_i}) = exp\left\{ -x_i \left\{ \lambda_i + \sum_j \lambda_{ij} x_j + \sum_{j \neq k} \lambda_{ijk} x_j x_k + \ldots + \lambda_{12 \ldots n} \prod_{j \neq i} x_j \right\}\right\}. \tag{4.3}$$

Similarly, the survival function of $\{Y_i \mid Y_j > x_j,\ j \neq i\}$, denoted by $\overline{G}_i(x_i; \mathbf{x_i})$ is

$$\overline{G}_i(x_i; \mathbf{x_i}) = exp\left\{ -x_i \left\{ \lambda_i + \sum_j \lambda^*_{ij} x_j + \sum_{j \neq k} \lambda^*_{ijk} x_j x_k + \ldots + \lambda^*_{12\ldots n} \prod_{j \neq i} x_j \right\}\right\}. \tag{4.4}$$

Now the ratio

$$\frac{\overline{G}_i(x_i; \mathbf{x_i})}{\overline{F}_i(x_i; \mathbf{x_i})} \tag{4.5}$$

is increasing in x_i. That is

$$\left(X_i \mid \bigcap_{j \neq i} \{X_j > x_j\} \right) \leq_{hr} \left(Y_i \mid \bigcap_{j \neq i} \{Y_j, > x_j\} \right),\ \ i = 1, \ldots, n. \tag{4.6}$$

On the other hand, the random variable $\{X_i \mid X_j > x_j,\ j \neq i\}$ has exponential distribution which is DFR. Combining this observation with (4.6), it follows form Theorem 3.1(a) that (4.2) holds. Now applying Theorem 3.3 to this example, we get that $C^{\lambda}_{i,j}$ is decreasing in λ, where $C^{\lambda}_{i,j}$ denotes the copula of (X_i, X_j)

Application 4.1. Order Statistics

Let $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$ be a random sample from a univariate distribution with strictly increasing distribution function F (G). Bartoszewicz (1986) has shown that $F \leq_{disp} G$ implies $X_{i:n} \leq_{disp} Y_{i:n}$, $i = 1, \ldots, n$ where $X_{i:n}$ $(Y_{i:n})$ is the ith order statistic of the X-sample (Y-sample). We shall strengthen this result to prove that $F \leq_{disp} G$ implies

$$(X_{1:n}, \ldots, X_{n:n}) \overset{uo-disp}{\preceq} (Y_{1:n}, \ldots, Y_{n:n}). \tag{4.7}$$

We first show that in the case of random samples from continuous distributions, the copulas of order statistics are independent of the parent distributions. Note that $Y_{i:n} \overset{st}{=} G^{-1}F(X_{i:n})$, $i = 1, \ldots n$. Since the function $G^{-1}F$ is strictly increasing, it follows from Theorem 2.4.3 of Nelsen (1998) that $C(X_{i:n}, X_{j:n}) = C(Y_{i:n}, Y_{j:n})$, for $i, j = 1, \ldots, n$. It now immediately follows from Theorem 2.1 that $F \leq_{disp} G$ implies (4.7).

Since the order statistics from a random sample are positively associated (cf. Boland et al., 1996) and since $(X_{1:n}, \ldots, X_{n:n})$ and $(Y_{1:n}, \ldots, Y_{n:n})$ have the same copula, the conditions of Theorem 3.4 are satisfied. Hence for $i, j \in \{1, \ldots, n\}$,

$$Cov(h_1(X_{i:n}), h_2(X_{j:n})) \leq Cov(h_1(Y_{i:n}), h_2(Y_{j:n})),$$

for all increasing convex functions h_1 and h_2 for which the above covariances exit. This result was originally proved by Bartoszewicz (1985) using a different method. A similar result can be established for record values.

Application 4.2. Record values

Let $X_1,\ldots,X_n,\ldots$ $(Y_1,\ldots,Y_n,\ldots)$ be a sequence of random variables from a univariate distribution F (G). It is known that $F \le_{disp} G$ implies $R^X_m \le_{disp} R^Y_m$, where R^X_m (R^Y_m) is the mth record value of the X-sequence (Y-sequence). We first show that in case of random sequences from continuous distributions, the copulas of record values are independent of the parent distributions. Then it will immediately follows from Theorem 2.1 that $F \le_{disp} G$ implies

$$(R^X_{m_1},\ldots,R^X_{m_n}) \overset{uo-disp}{\preceq} (R^Y_{m_1},\ldots,R^Y_{m_n}). \tag{4.8}$$

Let $M(x) = -\log \overline{F}(x)$, then the distribution function of R^X_m can be expressed as $F_{R^X_m}(x) = G_m(M(x))$, where $G_n(x)$ is the distribution function of a Gamma random variable with scale parameter one and shape parameter m. Similarly, let $N(x) = -\log \overline{G}(x)$, then the distribution function of R^Y_m is $F_{R^Y_m}(x) = G_m(N(x))$. Now $M(X_1),\ldots,M(X_n),\ldots$ and $N(Y_1),\ldots,N(Y_n),\ldots$ both are sequences of independent and identical exponential random variables with mean one. Using this observation, it follows that

$$(R^X_{m_1},\ldots,R^X_{m_n}) \overset{st}{=} (M^{-1}(R^*_{m_1}),\ldots,M^{-1}(R^*_{m_n})) \text{ and } (R^Y_{m_1},\ldots,R^Y_{m_n}) \overset{st}{=} (N^{-1}(R^*_{m_1}),\ldots,N^{-1}(R^*_{m_n})),$$

where R^*_m is the mth record value of a sequence of i.i.d exponential random variables with mean one.

$$\begin{aligned}
C^{\mathbf{R}^X}(u_1,\ldots,u_n) &= F_{\mathbf{R}^X}\left(F^{-1}_{R^X_{m_1}}(u_1),\ldots,F^{-1}_{R^X_{m_n}}(u_n)\right)\\
&= P\left(R^X_{m_1} \le F^{-1}_{R^X_{m_1}}(u_1),\ldots,R^X_{m_n} \le F^{-1}_{R^X_{m_n}}(u_n),\right)\\
&= P\left(M^{-1}(R^*_{m_1}) \le M^{-1}G^{-1}_{m_1}(u_1),\ldots,\right.\\
&\qquad \left. M^{-1}(R^*_{m_n}) \le M^{-1}G^{-1}_{m_n}(u_n)\right)\\
&= P\left(R^*_{m_1} \le G^{-1}_{m_1}(u_1),\ldots,R^*_{m_n} \le G^{-1}_{m_n}(u_n)\right)\\
&= P\left(N^{-1}(R^*_{m_1}) \le N^{-1}G^{-1}_{m_1}(u_1),\ldots,\right.\\
&\qquad \left. N^{-1}(R^*_{m_n}) \le N^{-1}G^{-1}_{m_n}(u_n)\right)\\
&= P\left(R^Y_{m_1} \le F^{-1}_{R^Y_{m_1}}(u_1),\ldots,R^Y_{m_n} \le F^{-1}_{R^Y_{m_n}}(u_n),\right)\\
&= C^{\mathbf{R}^Y}(u_1,\ldots,u_n).
\end{aligned}$$

This proves the desired result.

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