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Existence of Optimal Markov Solutions for Ergodic Control of Markov Processes

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EXISTENCE OF OPTIMAL MARKOV SOLUTIONS FOR ERGODIC CONTROL OF MARKOV PROCESSES

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SUMMARY. For the ergodic control problem for a large class of controlled Markov processes in continuous time, existence of an optimal ergodic solution and an optimal, possibly time-inhomogeneous Markov solution, in both cases corresponding to a stationary Markov relaxed control policy, are known separately under suitable conditions (Bhatt and Borkar, 1996). The aim of this article is to refine this result to establish the existence of an optimal *time-homogeneous* Markov solution. The proof is based upon Krylov's Markov selection procedure.

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1 Introduction

In Bhatt and Borkar (1996), the ergodic control problem for a broad class of controlled Markov processes was analyzed and existence of an optimal ergodic solution and an optimal Markov, though possibly time-inhomogeneous solution were separately established under reasonable hypotheses, both corresponding to a stationary Markov relaxed control policy. Experience with simpler situations (countable Markov chains, nondegenerate diffusions in $\mathcal{R}^d$, $\cdots$) suggests that one would have an optimal ergodic time-homogeneous Markov control. The aim of this article is to provide such a result by adapting Krylov's Markov selection procedure (Chapter 12, Stroock and Varadhan (1979)). Originally intended for extracting a Markov family of probability measures satisfying a 'martingale problem' in presence of nonuniqueness, this procedure was adapted to extract an optimal Markov solution to degenerate controlled diffusions in Haussmann (1986) and El Kaoui et al (1987), following a suggestion of Varadhan. A compact treatment appears in Chapter IV (page 84) of Borkar (1989), which we use as our starting point. As it stands, the procedure in the above references is geared for 'integral costs'

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such as finite horizon cost, infinite horizon discounted cost, cost up to an exit time, etc., and its application to ergodic control calls for an additional limiting procedure, viz., the 'vanishing discount limit', which is the main contribution of this work.

The next section recalls the problem set-up from Bhatt and Borkar (1996). Section 3 proves some preliminary results for the discounted cost problem which pave way for the 'vanishing discount' argument of section 4. Section 5 proves the main result. Section 6 concludes with some remarks.

Notation:

1. E: a Polish space $\approx$ the state space of the controlled Markov process $X(\cdot)$
2. U: a compact metric 'control' space
3. $\mathcal{U}$: the space of measurable maps $[0,\infty) \to V \stackrel{def}{=} \mathcal{P}(U)$ with the coarsest topology that renders continuous each of the maps
$$\mu(\cdot) = \mu(\cdot, du) \in \mathcal{U} \mapsto \int_0^T g(t) \int_U h(u)\mu(t, du)dt,$$
for all $T > 0, g \in L_2[0,T], h \in C_b(U)$. This is compact metrizable (see, e.g., Borkar (1989)).
4. For a Polish space S:
    - $\mathcal{B}(S)$ is its Borel $\sigma-$field and $\mathcal{P}(S)$ the space of probability measures on $(S, \mathcal{B}(S))$ with the Prohorov topology,
    - $B(S)$ (resp., $C_b(S)$) is the space of bounded measurable (resp., continuous) maps from $S \to \mathcal{R}$.
5. $\mathcal{L}(\cdots)$ stands for '*the law of*' $\cdots$.
6. For $\{f_k\}, f \in B(S)$, $f_k \stackrel{bp}{\to} f$ (where '*bp*' stands for '*bounded pointwise*') if $sup_{x,k}|f_k(x)| < \infty$ and $f_k(x) \to f(x)\ \forall x$. $Q \subset B(S)$ is *bp-closed* if $f_k \in Q\ \forall k$ and $f_k \stackrel{bp}{\to} f$ together imply $f \in Q$. For $Q \subset B(S)$, define $bp-closure(Q) \stackrel{def}{=}$ the smallest bp-closed subset of $B(S)$ containing Q.

2 The control problem

Let A be an operator with domain $\mathcal{D}(A) \subset C_b(E)$ and range $\mathcal{R}(A) \subset C_b(E \times U)$. Let $\nu \in \mathcal{P}(E)$.

DEFINITION. *An $E \times V-$valued process $(X(\cdot), \pi(\cdot) = \pi(\cdot, du))$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is said to be a solution to the relaxed controlled martingale problem for (A, ν) with respect to a filtration $\{\mathcal{F}_t, t \geq 0\}$ if:*

- $(X(\cdot), \pi(\cdot))$ *is* $\{\mathcal{F}_t\}$*−progressive;*
- $\mathcal{L}(X(0)) = \nu$*;*
- *for* $f \in \mathcal{D}(A)$*,*
$$f(X(t)) - \int_0^t \int_U Af(X(s), u)\pi(s, du)ds, \ t \geq 0, \tag{2.1}$$
is an $\{\mathcal{F}_t\}$*−martingale.*

We omit explicit mention of $\{\mathcal{F}_t\}$ or ν when they are apparent from the context. Further, we may simplify notation by setting

$$\bar{A}f(x, \mu) \stackrel{def}{=} \int_U Af(x, u)\mu(du), \ f \in \mathcal{D}(A), x \in E, \mu \in V,$$

and rewrite (2.1) as

$$f(X(t)) - \int_0^t \bar{A}f(X(s), \pi(s))ds, \ t \geq 0. \tag{2.2}$$

The operator A is assumed to satisfy the following conditions:

1. (C1) There exists a countable subset $\{g_k\} \subset \mathcal{D}(A)$ such that
$$\{(g, Ag) : g \in \mathcal{D}(A)\} \subset bp - closure(\{(g_k, Ag_k) : k \geq 1\}).$$
2. (C2) $\mathcal{D}(A)$ is an algebra that separates points in E and contains constant functions. Also, $A\mathbf{1} = 0$, where $\mathbf{1}$ is the constant function identically equal to 1.
3. (C3) For each $u \in U$, let $A^u f(\cdot) \stackrel{def}{=} Af(\cdot, u)$. Then there exists an r.c.l.l. solution to the martingale problem for (A^u, δ_x) for all $u \in U, x \in E$.

We also make the following assumption: Let $(X(\cdot), \pi(\cdot))$ be a solution to the relaxed controlled martingale problem above.

(†) For fixed initial law ν of X_0, $\mathcal{L}((X(\cdot), \pi(\cdot)))$ form a tight set $\bar{\mathcal{M}}(\nu)$ of $\mathcal{P}(C([0, \infty); E) \times \mathcal{U})$.

Simple sufficient conditions for this can be given in specific cases mentioned above. An immediate corollary is the following:

LEMMA **2.1** $\bar{\mathcal{M}}(\nu)$ *is a compact convex set.*

PROOF. Let $\mathcal{U}_s$ denote the set of restrictions of $\pi(\cdot) \in \mathcal{U}$ to $[0, s]$ with the induced topology and let $C([0, s]; E)$ be the space of continuous maps from $[0, s]$ to E. A process $(X(\cdot), \pi(\cdot))$ is a solution to the relaxed control martingale problem for A if and only if for all $t > s \geq 0, f \in \mathcal{D}(A)$ and $G \in C_b(C([0,s];E) \times \mathcal{U}_s)$,

$$E\left[\left(f(X(t)) - f(X(s)) - \int_s^t \bar{A}f(X(y),\pi(y))dy\right) G\left(X(\cdot)|_{[0,s]}, \pi(\cdot)|_{[0,s]}\right)\right] = 0.$$

This relation is retained under convex combinations of laws and also under convergence in $\mathcal{P}(C([0,\infty);E) \times \mathcal{U})$. The claim follows. □

Let $k : E \times U \to [0,\infty]$ be a *running cost* function. The ergodic control problem is to minimize the *ergodic cost*

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t E\left[\int_U k(X(s),u)\pi(s,du))\right] ds. \tag{2.3}$$

We assume that the set of $\mathcal{L}((X(\cdot),\pi(\cdot)))$ for which this is finite is nonempty.

3 The discounted cost problem

The main argument will be based on combining the Markov selection procedure applied to the infinite horizon discounted cost problem with the 'vanishing discount limit'. With this goal, introduce the discounted cost

$$J(\Phi, x) \stackrel{def}{=} E\left[\int_0^\infty e^{-\alpha t} \int_U k(X(t),u)\pi(t,du)dt \middle| X(0) = x\right], \tag{3.1}$$

where $\Phi \stackrel{def}{=} \mathcal{L}((X(\cdot),\pi(\cdot)))$ and $\alpha > 0$ is the discount factor. Let

$$J(\Phi,\nu) \stackrel{def}{=} \int_E J(\Phi,x)\nu(dx),$$

$$\Psi_0^\alpha(\nu) \stackrel{def}{=} \inf_{\Phi \in \bar{\mathcal{M}}(\nu)} J(\Phi,\nu),$$

and

$$\mathcal{M}_0^\alpha(\nu) \stackrel{def}{=} \left\{\Phi \in \bar{\mathcal{M}}(\nu) : J(\Phi,\nu) = \Psi_0^\alpha(\nu)\right\}.$$

Let $\{f_i\} \subset C_b(E)$ be a countable separating class for $\mathcal{P}(E)$ and $\{\beta_j\}$ be a countable dense set in $(0,\infty)$. For $i,j \geq 1$, define $F_{ij} : \mathcal{P}\left(C_b([0,\infty);E) \times \mathcal{U}\right) \mapsto \mathcal{R}$ as follows. Let $\Phi \in \mathcal{P}\left(C_b([0,\infty);E) \times \mathcal{U}\right)$ denote the law of a process $(X(\cdot),\pi(\cdot))$. Then

$$F_{ij}(\Phi) \stackrel{def}{=} \int_0^\infty E\left[e^{-\beta_i t} f_j(X(t))\right] dt. \tag{3.2}$$

Enumerate F_{ij}'s as $F_1, F_2, \cdots$, by a suitable relabelling. For $i \geq 1$, define inductively

$$\Psi_i^\alpha(\nu) \stackrel{def}{=} \inf_{\Phi \in \mathcal{M}_{i-1}^\alpha(\nu)} F_i(\Phi),$$

$$\mathcal{M}_i^\alpha(\nu) \stackrel{def}{=} \left\{\Phi \in \mathcal{M}_{i-1}^\alpha(\nu) : F_i(\Phi) = \Psi_i^\alpha(\nu)\right\}.$$

Since $\mathcal{M}_i^\alpha(\nu)$ have been obtained as *Argmins* of lower semi-continuous linear functionals on compact convex sets of measures, we have the following immediate corollary to Lemma 2.1:

COROLLARY **3.1** *For fixed α, ν, $\{\mathcal{M}_i^\alpha(\nu), i \geq 0\}$ is a nested, decreasing family of compact convex nonempty sets.*

In particular, $\mathcal{M}_\infty^\alpha(\nu) \stackrel{def}{=} \cap_i \mathcal{M}_i^\alpha(\nu)$ is nonempty compact and convex. For sake of simplicity, we shall denote $\Psi_i^\alpha(\delta_x), \mathcal{M}_i^\alpha(\delta_x)$, where $\delta_x \stackrel{def}{=}$ the Dirac measure at x, by $\Psi_i^\alpha([x]), \mathcal{M}_i^\alpha([x])$ resp. The following lemma mimicks Lemma 1.2, p. 86, of Borkar (1989).

LEMMA **3.1** *For $\Phi \approx \mathcal{L}((X(\cdot), \pi(\cdot))) \in \mathcal{M}_i^\alpha(\nu)$, the regular conditional law of $(X(\cdot), \pi(\cdot))$ given $X(0) = x$ is in $\mathcal{M}_i^\alpha([x])$ for $\nu-$a.s. x. (Equivalently, $\Psi_i^\alpha(\nu) = \int \Psi_i^\alpha([x])\nu(dx)$.)*

PROOF. For $i = 0$, the claim follows by a standard argument of dynamic programming (see, e.g., Lemma 1.2, pp. 55-56, of Borkar (1989), in fact this is the same argument as for $i \geq 1$ below from (3.4) onwards, except that the steps preceding (3.4) are not needed for $i = 0$). Suppose the claim is true for all $j \leq i$. Thus

$$\Psi_j^\alpha(\nu) = \int \Psi_j^\alpha([x])\nu(dx), \; j \leq i.$$

Let $\epsilon > 0$ and $\epsilon_n \stackrel{def}{=} \epsilon/2^n, n \geq 1$. For $n \geq 1$, let $\Phi^n = \mathcal{L}((X^n(\cdot), \pi^n(\cdot))) \in \mathcal{M}_i^\alpha(\nu)$ be such that

$$F_{i+1}(\Phi^n) \leq \Psi_{i+1}^\alpha(\nu) + \epsilon_n. \tag{3.3}$$

Let $\Phi^n(x)$ denote the regular conditional law of $(X^n(\cdot), \pi^n(\cdot))$ given $X^n(0) = x$. By the induction hypothesis, we can find a set $N \subset E$ with $\nu(N) = 0$ such that the $\Phi^n(x)$ is in $\mathcal{M}_j^\alpha([x])$ for all $x \notin N$ and for all $j \leq i$. Let

$$B_n = \left\{x \in E : F_{i+1}(\Phi^n(x)) < \Psi_{i+1}^\alpha([x]) + \epsilon\right\}, \; n \geq 1.$$

Then, (3.3) implies that $\nu(B_n^c) < 2^{-n}$. Define $C_n \subset E, n \geq 1$, successively by

$$C_1 = B_1, C_n = B_n \cap (\cup_{m<n} B_m)^c, n > 1.$$

Then clearly C_n's are disjoint and $\nu(\cup_n C_n) = \nu(\cup_n B_n) = 1$. Define $\bar{\Phi}$ by

$$\bar{\Phi} = \sum_n \int_{C_n} \Phi^n(x)\nu(dx).$$

Then $\bar{\Phi}$ is the law of a process $(\bar{X}(\cdot), \bar{\pi}(\cdot))$ such that $\mathcal{L}(\bar{X}(0)) = \nu$ and for $x \in C_n, n \geq 1$, the regular conditional law of $(\bar{X}(\cdot), \bar{\pi}(\cdot))$ given $\bar{X}(0) = x$ is $\Phi^n(x)$. Using the fact that $\Phi^n(x) \in \mathcal{M}_j^\alpha([x])$ for all $n \geq 1$ and $0 < j \leq i$, we get

$$
\begin{aligned}
F_j(\bar{\Phi}) &= \sum_n \int_{C_n} F_j(\Phi^n(x))\nu(dx) \\
&= \sum_n \int_{C_n} \Psi_j^\alpha([x])\nu(dx) \\
&= \Psi_j^\alpha(\nu),
\end{aligned}
$$

where the last equality above follows from the induction hypothesis. Thus $\bar{\Phi} \in \mathcal{M}_j^\alpha(\nu)$, $0 < j \leq i$. A similar argument works for $j = 0$ where $F_0 = J$ is the discounted cost defined in (3.1). Furthermore, another similar argument using (3.3) leads to

$$
F_{i+1}(\bar{\Phi}) \leq \int_E \Psi_{i+1}^\alpha([x])\nu(dx) + \epsilon. \tag{3.4}
$$

Since $\epsilon > 0$ was arbitrary, we get

$$
\Psi_{i+1}^\alpha(\nu) \leq \int_E \Psi_{i+1}^\alpha([x])\nu(dx). \tag{3.5}
$$

Now consider an arbitrary $\Phi^* = \mathcal{L}^*((X(\cdot), \pi^*(\cdot))) \in \mathcal{M}_i^\alpha(\nu)$. Let $\Phi^*(x)$ denote the regular conditional law of $(X^*(\cdot), \pi^*(\cdot))$ given $X^*(0) = x$. Then we have

$$
\begin{aligned}
F_{i+1}(\Phi^*) &= \int_E F_{i+1}(\Phi^*(x))\nu(dx) \\
&\geq \int_E \Psi_{i+1}^\alpha([x])\nu(dx).
\end{aligned}
$$

Thus, taking the infimum over all $\Phi^* \in \mathcal{M}_i^\alpha(\nu)$ on the left hand side, we get

$$
\Psi_{i+1}^\alpha(\nu) \geq \int_E \Psi_{i+1}^\alpha([x])\nu(dx). \tag{3.6}
$$

Now (3.5) and (3.6) together complete the induction step. $\square$

For $\Phi \approx \mathcal{L}((X(\cdot), \pi(\cdot)))$, let $\nu' \stackrel{def}{=} \mathcal{L}(X(t))$ for some $t \geq 0$. Suppose $\Phi' \approx \mathcal{L}((X'(\cdot), \pi'(\cdot)))$ is such that $\mathcal{L}(X'(0)) = \nu'$. We define a $t-$concatenation $\tilde{\Phi}$ of Φ with Φ' to be $\mathcal{L}((\tilde{X}(\cdot), \tilde{\pi}(\cdot)))$ such that

$$
\mathcal{L}((\tilde{X}(\cdot), \tilde{\pi}(\cdot))|_{[0,t]}) = \mathcal{L}((X(\cdot), \pi(\cdot))|_{[0,t]}),
$$

and the regular conditional law of $(\tilde{X}(t + \cdot), \tilde{\pi}(t + \cdot))$ given $(\tilde{X}(\cdot), \tilde{\pi}(\cdot))|_{[0,t]}$ is the same as the regular conditional law thereof given $\tilde{X}(t)$, which is in turn the same as the regular conditional law of $(X'(\cdot), \pi'(\cdot))$ given $X'(0)$, $\nu'-$a.s. The next two lemmas are in the spirit of Lemmas 1.3-1.4, pp. 87-88, of Borkar (1989). They are stated separately, but their induction steps are interlinked: The $i-$th induction step of Lemma 3.2 invokes the claim of the $i-$th induction step of Lemma 3.3, whereas the $i-$th induction step of Lemma 3.3 invokes the claim of the $(i-1)-$th induction step of Lemma 3.2. Let $\Phi, \Phi', \tilde{\Phi}, (X(\cdot), \pi(\cdot)), (X'(\cdot), \pi'(\cdot)), (\tilde{X}(\cdot), \tilde{\pi}(\cdot))$ be as above, with $\Phi \in \mathcal{M}_i^\alpha(\nu)$ and $\Phi' \in \mathcal{M}_i^\alpha(\nu')$ for some $i \geq 0$.

LEMMA **3.2** $\tilde{\Phi} \in \mathcal{M}_i^\alpha(\nu)$.

PROOF. For $i = 0$, this is a standard part of the dynamic programming argument (see section III.1 of Borkar (1989)). Let $i \geq 1$ and suppose that the claim is true for all $\ell < i$. Now, $\Phi \in \mathcal{M}_j^\alpha(\nu)$ and $\Phi' \in \mathcal{M}_j^\alpha(\nu')$ for $0 \leq j \leq i$.

Fix $j, 0 < j \leq i$. Let m, n be such that for $\Phi^* = \mathcal{L}((X^*(\cdot), \pi^*(\cdot)))$ (see (3.2))

$$F_j(\Phi^*) = \int_0^\infty E\left[e^{-\beta_m s} f_n(X^*(s))\right] ds.$$

For any $\Phi^* \in \mathcal{P}\left(C_b([0,\infty); E) \times \mathcal{U}\right)$ as above let $\Phi^{*t} \in \mathcal{P}\left(C_b([0,\infty); E) \times \mathcal{U}\right)$ be defined by $\Phi^{*t} = \mathcal{L}((X^*(t + \cdot), \pi^*(t + \cdot)))$.

We have

$$F_j(\Phi^t) = \Psi_j^\alpha(\nu') = F_j(\Phi') = F_j(\tilde{\Phi}^t) \qquad 0 < j \leq i,$$

where the first equality follows from Lemma 3.3 below, the second follows because $\Phi' \in \mathcal{M}_j^\alpha(\nu')$, and the third follows from our definition of $\tilde{\Phi}$. Multiply both extremes of the above equality by $e^{-\beta_m t}$ and add

$$E\left[\int_0^t e^{-\beta_m s} f_n(X(s)) ds\right]$$

to both sides. But since $\mathcal{L}(X(s) : 0 \leq s \leq t) = \mathcal{L}(\tilde{X}(s) : 0 \leq s \leq t)$ we get

$$\Psi_j^\alpha(\nu) = F_j(\Phi) = F_j(\tilde{\Phi}),$$

implying $\tilde{\Phi} \in \mathcal{M}_j^\alpha(\nu)$ for $0 \leq j \leq i$. □

COROLLARY **3.2** *The following 'dynamic programming principle' holds: For $i \geq 1$ and m, n as above and $\Phi \stackrel{def}{=} \mathcal{L}((X(\cdot), \pi(\cdot))) \in \mathcal{M}_i^\alpha(\nu)$,,*

$$\Psi_i^\alpha(\nu) = E\left[\int_0^t e^{-\beta_m s} f_n(X(s)) ds + e^{-\beta_m t} \Psi_i^\alpha([X(t)])\right]. \tag{3.7}$$

PROOF. By Lemma 3.1, the r.h.s. above equals

$$E\left[\int_0^t e^{-\beta_m s} f_n(X(s)) ds\right] + e^{-\beta_m t} \Psi_i^\alpha(\nu')$$

for $\nu' \stackrel{def}{=} \mathcal{L}(X(t))$. Take $\Phi' \stackrel{def}{=} \mathcal{L}((X'(\cdot), \pi'(\cdot))) \in \mathcal{M}_i^\alpha(\nu')$ and let $\tilde{\Phi} = \mathcal{L}((\tilde{X}(\cdot), \tilde{\pi}(\cdot)))$ be the $t-$concatenation of Φ with Φ'. Then by the above lemma $\tilde{\Phi} \in \mathcal{M}_i^\alpha(\nu)$, leading to

$$\begin{aligned}
\Psi_i^\alpha(\nu) &= E\left[\int_0^\infty e^{-\beta_m s} f_n(\tilde{X}(s)) ds\right] \\
&= E\left[\int_0^t e^{-\beta_m s} f_n(\tilde{X}(s)) ds\right] + E\left[\int_t^\infty e^{-\beta_m s} f_n(\tilde{X}(s)) ds\right] \\
&= E\left[\int_0^t e^{-\beta_m s} f_n(X(s)) ds\right] + E\left[\int_t^\infty e^{-\beta_m s} f_n(X'(s-t)) ds\right] \\
&= E\left[\int_0^t e^{-\beta_m s} f_n(X(s)) ds\right] + e^{-\beta_m t} \Psi_i^\alpha(\nu').
\end{aligned}$$

This completes the proof. $\square$

LEMMA **3.3** *If $\Phi = \mathcal{L}((X(\cdot), \pi(\cdot))) \in \mathcal{M}_i^\alpha(\nu)$, then for each $t > 0$, $\Phi^t \in \mathcal{M}_i^\alpha(\mathcal{L}(X(t)))$.*

PROOF. For $i = 0$, the claim follows from dynamic programming as follows. If not, recalling (3.1), one would have

$$E\left[\int_t^\infty e^{-\alpha s} \int k(X(s), u)\pi(s, du)ds\right] > e^{-\alpha t}\Psi_0^\alpha(\mathcal{L}(X(t))). \tag{3.8}$$

Hence,

$$\begin{aligned}\Psi_0^\alpha(\nu) &= E\left[\int_0^t e^{-\alpha s} \int k(X(s), u)\pi(s, du)ds\right] \\ &\quad + E\left[\int_t^\infty e^{-\alpha s} \int k(X(s), u)\pi(s, du)ds\right] \\ &> E\left[\int_0^t e^{-\alpha s} \int k(X(s), u)\pi(s, du)ds + e^{-\alpha t}\Psi_0^\alpha([X(t)])\right] \\ &= \Psi_0^\alpha(\nu),\end{aligned} \tag{3.9}$$

a contradiction. (Here the strict inequality follows from (3.8) and Lemma 3.1, and the last equality follows from the dynamic programming principle.)

Suppose now that the claim is true for $0 \le j < i, i \ge 1$. Let $\Phi \in \mathcal{M}_i^\alpha(\nu)$. Then by the induction hypothesis,

$$\Phi^t = \mathcal{L}((X(t+\cdot), \pi(t+\cdot))) \in \mathcal{M}_{i-1}^\alpha(\mathcal{L}(X(t))). \tag{3.10}$$

Let m, n be such that

$$F_i(\Phi^*) = \int_0^\infty E\left[e^{-\beta_m s} f_n(X^*(s))\right] ds$$

for any $\Phi^* = \mathcal{L}(X^*(\cdot), \pi^*(\cdot))$. Suppose now that the claim is false for i.
Then we get

$$E\left[\int_t^\infty e^{-\beta_m s} f_n(X(s))ds\right] > e^{-\beta_m t}\Psi_i^\alpha(\mathcal{L}(X(t))).$$

This implies

$$\begin{aligned}\Psi_i^\alpha(\nu) &= E\left[\int_0^t e^{-\beta_m s} f_n(X(s))ds\right] + E\left[\int_t^\infty e^{-\beta_m s} f_n(X(s))ds\right] \\ &> E\left[\int_0^t e^{-\beta_m s} f_n(X(s))ds + e^{-\beta_m t}\Psi_i^\alpha([X(t)])\right] \\ &= E\left[\int_0^t e^{-\beta_m s} f_n(X(s))ds\right] + e^{-\beta_m t}\Psi_i^\alpha(\nu')\end{aligned} \tag{3.11}$$

where $\nu' = \mathcal{L}(X(t))$. Let $\Phi' = \mathcal{L}((X'(\cdot), \pi'(\cdot))) \in \mathcal{M}_i^\alpha(\nu')$. Hence, by definition,

$$\Psi_i^\alpha(\nu') = F_i(\Phi') = E\left[\int_0^\infty e^{-\beta_m s} f_n(X'(s))ds\right]. \tag{3.12}$$

Let $\tilde{\Phi}$ be the $t-$concatenation of Φ with Φ'. Since (3.10) holds, Lemma 3.2 can be used for $(i-1)$ and hence we get that $\tilde{\Phi} \in \mathcal{M}^{\alpha}_{i-1}(\nu)$. This implies that $F_i(\tilde{\Phi}) \geq \Psi^{\alpha}_i(\nu)$. Using this and the definition of $\tilde{\Phi}$, and substituting (3.12) in (3.11) we get

$$\begin{aligned}
\Psi^{\alpha}_i(\nu) &> E\left[\int_0^t e^{-\beta_m s} f_n(X(s))ds\right] + e^{-\beta_m t} E\left[\int_0^\infty e^{-\beta_m s} f_n(X'(s))ds\right] \\
&= E\left[\int_0^t e^{-\beta_m s} f_n(X(s))ds\right] + E\left[\int_t^\infty e^{-\beta_m s} f_n(X'(s-t))ds\right] \\
&= F_i(\tilde{\Phi}) \\
&\geq \Psi^{\alpha}_i(\nu),
\end{aligned} \tag{3.13}$$

a contradiction. This completes the induction step and hence the proof of the lemma. □

The following technical lemma is easily proved and is the same as Lemma 1.5, p. 89, of Borkar (1989).

LEMMA **3.4** *For bounded measurable* $f : [0,\infty) \to \mathcal{R}$,

$$\int_0^\infty e^{-\beta_j t} f(t)dt = 0 \ \forall j \Longrightarrow f(t) = 0 \ a.e.$$

COROLLARY **3.3** *For any two elements* $\mathcal{L}((X(\cdot),\pi(\cdot))), \mathcal{L}((X'(\cdot),\pi'(\cdot)))$ *of* $\mathcal{M}^{\alpha}_{\infty}(\nu)$, $X(\cdot), X'(\cdot)$ *have the same one dimensional marginals. Furthermore, there exists a* $\mathcal{L}((\hat{X}(\cdot),\hat{\pi}(\cdot))) \in \mathcal{M}^{\alpha}_{\infty}(\nu)$ *such that*

$$\hat{\pi}(t) \ (= \hat{\pi}(t,du)) = v(t,\hat{X}(t)) \ (= v(t,\hat{X}(t),du))$$

for some measurable $v : [0,\infty) \to V$ *(i.e.,* $\hat{\pi}(\cdot)$ *is a 'Markov control').*

PROOF. The first claim is immediate from Lemma 3.4 and our choice of $\{f_m\}$ as a separating class for $\mathcal{P}(E)$. The second is a consequence of Corollary 2.2, p. 1549, of Bhatt and Borkar (1996). □

Define $q^{\alpha}(x,t,B) \stackrel{def}{=} P(X(t) \in B)$ for $B \in \mathcal{B}(E)$ and any $\mathcal{L}((X(\cdot),\pi(\cdot))) \in \mathcal{M}^{\alpha}_{\infty}([x])$. The exact choice of the latter is immaterial by Corollary 3.3.

LEMMA **3.5** $\{q^{\alpha}(x,t,.), x \in E, t \geq 0\}$ *satisfy the Chapman-Kolmogorov equations.*

PROOF. This is immediate from Lemma 3.3. □

Consider a Markov process $X^{\alpha}(\cdot)$ with the transition kernel $q^{\alpha}(\cdot,\cdot,\cdot)$.

LEMMA **3.6** *For $X^\alpha(\cdot)$ constructed as above with $\mathcal{L}(X^\alpha(0)) = \nu$ and $v(\cdot,\cdot)$ as in Corollary 3.3, $(X^\alpha(\cdot), v(\cdot, X^\alpha(\cdot)))$ satisfies the relaxed controlled martingale problem for (A, ν).*

PROOF. By our construction $\mathcal{L}((X^\alpha(\cdot), v(\cdot, X^\alpha(\cdot))) \in \mathcal{M}_\infty^\alpha(\nu)$. This implies the claim. □

COROLLARY **3.4** *Without loss of generality, we may replace $v(t, X^\alpha(t))$ in Lemma 3.6 by $v^\alpha(X^\alpha(t))$ for a measurable $v^\alpha : E \to V$.*

PROOF. This follows as in Corollary 1.1, p. 13, of Borkar (1989). □

4 The vanishing discount limit

We recall the following notation. For $\Phi = \mathcal{L}((X(\cdot), \pi(\cdot)))$, $\Phi(x)$ denotes the regular conditional law of $(X(\cdot), \pi(\cdot))$ given $X(0) = x$. Also for any set D let $\overline{D}$ denote its closure.

Now, for prescribed $i \geq 0, \nu \in \mathcal{P}(E)$, let $\mathcal{M}_i^0([x]) \stackrel{def}{=}$ the set of limit points of $\mathcal{M}_i^\alpha([y])$ as $\alpha \to 0$ and $y \to x$. In other words, letting ρ denote any compatible complete metric on E,

$$\mathcal{M}_i^0([x]) \stackrel{def}{=} \cap_{\alpha>0} \cap_{\epsilon>0} \overline{\cup_{0<\beta<\alpha} \cup_{\rho(y,x)<\epsilon} M_i^\alpha([y])}.$$

Further let

$$\begin{aligned} \mathcal{M}_i^0(\nu) \quad &\stackrel{def}{=} \quad \{\Phi = \mathcal{L}((X(\cdot), \pi(\cdot))) : \mathcal{L}(X(0)) = \nu, \\ &\qquad\qquad \Phi(x) \in \mathcal{M}_i^0([x]) \ \forall x \text{ a.s. } [\nu]\}. \end{aligned}$$

First we note down the following observation.

LEMMA **4.1** *$M_i^0(\nu)$ is compact and contains the set of limit points of $M_i^\alpha(\nu')$ as $(\alpha, \nu') \to (0, \nu)$ (in particular, is nonempty).*

PROOF. We first prove the second claim. Let $\nu_n \to \nu, \alpha_n \to 0$, and let $\mathcal{L}(X^n(\cdot), \pi^n(\cdot)) \in M_i^{\alpha_n}$. Then $\{\mathcal{L}(X^n(\cdot), \pi^n(\cdot))\}$ can be shown to be a tight sequence as in Lemma 3.6 (pp. 126) of Borkar (1989). By dropping to a subsequence if necessary we can assume that

$$\mathcal{L}(X^n(\cdot), \pi^n(\cdot)) \to \mathcal{L}(X(\cdot), \pi(\cdot)) \tag{4.1}$$

(say). Let $\psi_n : E \to \mathcal{P}(C([0, \infty); E) \times \mathcal{U})$ denote the regular conditional law of $(X^n(\cdot), \pi^n(\cdot))$ given $X^n(0)$. By dropping to a further subsequence if necessary, we may suppose that

$$(X^n(0), \psi_n(X^n(0))) \to (X'(0), \psi') \tag{4.2}$$

in law for an $E \times \mathcal{P}(C([0,\infty);E) \times \mathcal{U})-$ valued pair $(X'(0), \psi')$. Clearly, the law of both $X'(0)$ in (4.2) and $X(0)$ in (4.1) is ν. Let $f : E \times (C([0,\infty);E) \times \mathcal{U}) \to R$ be a bounded continuous function. Then by (4.2),

$$E[\int f(X^n(0), \omega)\psi^n(X^n(0), d\omega)] \to E[\int f(X'(0), \omega)\psi'(d\omega)]. \tag{4.3}$$

By Lemma 3.1, $\psi^n(X^n(0)) \in M_i^{\alpha_n}(X^n(0))$ $\forall n$, a.s. Thus

$$\psi' \in M_i^0(X'(0)) \ a.s. \tag{4.4}$$

Define $\psi : E \to \mathcal{P}(C([0,\infty);E) \times \mathcal{U})$ by:

$$\int h d\psi(X'(0)) = E[\int h d\psi' | X'(0)]$$

for h in any countable convergence determining class in $C_b(C([0,\infty);E) \times \mathcal{U})$. Clearly $M_i^0(\mu)$ is convex for all i, μ (being the set of limit points of a sequence of convex compact sets contained in a compact set). By (4.4), it then follows that $\psi(X'(0)) \in M_i^0(X'(0))$, a.s. By (4.3),

$$E[\int f(X^n(0), \omega)\psi^n(X^n(0), d\omega)] \to E[\int f(X'(0), \omega)\psi(d\omega)].$$

Also,

$$E[f(X^n(0), (X^n(\cdot), \pi^n(\cdot)))] \to E[f(X(0), (X(\cdot), \pi(\cdot)))].$$

Therefore ψ is the regular conditional law of $(X(\cdot), \pi(\cdot))$ given $X(0)$. Hence $\mathcal{L}(X(\cdot), \pi(\cdot)) \in M_i^0(\nu)$.

We have proved that every limit point of $M_i^\alpha(\mu)$ as $\alpha \to 0$ and $\mu \to \nu$ is in $M_i^0(\nu)$. To prove compactness, let $\mathcal{L}(X^n(\cdot), \pi^n(\cdot)) \in M_i^0(\nu)$ be such that

$$\mathcal{L}(X^n(\cdot), \pi^n(\cdot)) \to \mathcal{L}(X(\cdot), \pi(\cdot)).$$

Pick $\alpha_n \downarrow 0, \nu_n \to \nu$ and $\mathcal{L}(\tilde{X}^n(\cdot), \tilde{\pi}^n(\cdot)) \in M_i^{\alpha_n}(\nu_n)$ such that

$$|\alpha_n| < 2^{-n},$$
$$\rho(\nu_n, \nu) < 2^{-n},$$
$$\rho'(\mathcal{L}(\tilde{X}^n(\cdot), \tilde{\pi}^n(\cdot)), \mathcal{L}(X^n(\cdot), \pi(\cdot))) < 2^{-n},$$

where ρ, ρ' denote the Prohorov metrics on $\mathcal{P}(E), \mathcal{P}(C([0,\infty);E) \times \mathcal{U})$ resp. Then

$$\mathcal{L}(\tilde{X}^n(\cdot), \tilde{\pi}^n(\cdot)) \to \mathcal{L}(X(\cdot), \pi(\cdot)).$$

Argue exactly as above to conclude that $\mathcal{L}(X(\cdot), \pi(\cdot)) \in M_i^0(\nu)$. □

In particular, we have the following result.

LEMMA **4.2** *For each ν, $\{\mathcal{M}_i^0(\nu), i \geq 0\}$ is a nested decreasing family of nonempty compact sets.*

The following results mimic their counterparts in the preceding section. First we note the following straightforward result without proof.

LEMMA **4.3** *Let C_n be a sequence of compact convex subsets of a compact set in $\mathcal{P}(C([0,\infty);E)\times\mathcal{U})$ and C the set of its limit points as $n\to 0$. Let f be a bounded linear functional on $\mathcal{P}(C([0,\infty);E)\times\mathcal{U})$ and D_n, resp. D the set of minimizers of f on C_n, resp. C. Then the set of limit points of D_n is a compact convex subset of D.*

COROLLARY **4.1** *For any two elements $\mathcal{L}((X(\cdot),\pi(\cdot))),\mathcal{L}((X'(\cdot),\pi'(\cdot)))$ of $\mathcal{M}^0_\infty(\nu)$, $X(\cdot),X'(\cdot)$ have the same one dimensional marginals. Furthermore, there exists a $\mathcal{L}((\hat{X}(\cdot),\hat{\pi}(\cdot)))\in\mathcal{M}^0_\infty(\nu)$ such that*

$$\hat{\pi}(t)\ \ (=\hat{\pi}(t,du))=v(t,\hat{X}(t))\ \ (=v(t,\hat{X}(t),du))$$

for some measurable $v:[0,\infty)\to V$ (i.e., $\hat{\pi}(\cdot)$ is a 'Markov control').

PROOF. Lemma 4.3 implies that the minimum of F_i on $M^0_{i-1}(\nu)$ is attained on $M^0_i(\nu)$ for $i>0$. Now mimicking the arguments of the preceding section we get the result. □

LEMMA **4.4** *If $\mathcal{L}((X(\cdot),\pi(\cdot)))\in\mathcal{M}^0_\infty(\nu)$ and $\nu'\overset{def}{=}\mathcal{L}(X(t))$ for some $t>0$, then $\mathcal{L}((X(t+\cdot),\pi(t+\cdot)))\in\mathcal{M}^0_\infty(\nu')$.*

PROOF. It suffices to prove the claim for $\mathcal{M}^0_i(\nu),\mathcal{M}^0_i(\nu')$ in place of $\mathcal{M}^0_\infty(\nu),\mathcal{M}^0_\infty(\nu')$ resp. for all $i\geq 0$. Furthermore, it suffices to consider $\nu=\delta_x$ for some $x\in E$. Fix $i\geq 0$. Let $\mathcal{L}((X(\cdot),\pi(\cdot)))\in\mathcal{M}^0_i([x])$. From the definition of $\mathcal{M}^0_i([x])$, there exist $x_n\to x,\alpha_n\to 0$, and $\mathcal{L}((X^n(\cdot),\pi^n(\cdot)))\in\mathcal{M}^{\alpha_n}_i([x_n])$ such that

$$\mathcal{L}((X^n(\cdot),\pi^n(\cdot)))\to\mathcal{L}((X(\cdot),\pi(\cdot)))$$

in law. Then

$$\mathcal{L}((X^n(t+\cdot),\pi^n(t+\cdot)))\to\mathcal{L}((X(t+\cdot),\pi(t+\cdot)))$$

in law. The claim now follows from Lemma 3.3 above. □

Define $q(x,t,B)\overset{def}{=}P(X(t)\in B)$ for $B\in\mathcal{B}(E)$ and any $\mathcal{L}((X(\cdot),\pi(\cdot)))\in\mathcal{M}^0_\infty([x])$. The exact choice of the latter is immaterial by Corollary 4.1.

LEMMA **4.5** *$\{q(x,t,.),x\in E,t\geq 0\}$ satisfy the Chapman-Kolmogorov equations.*

PROOF. This is immediate from Lemma 4.4. $\square$

Consider a Markov process $X^*(\cdot)$ with the transition kernel $q(\cdot,\cdot,\cdot)$.

LEMMA **4.6** *For $X^*(\cdot)$ constructed as above with $\mathcal{L}(X^*(0)) = \nu$ and $v(\cdot,\cdot)$ as in Corollary 4.1, $(X^*(\cdot), v(\cdot, X^*(\cdot)))$ satisfies the relaxed controlled martingale problem for (A,ν).*

COROLLARY **4.2** *Without loss of generality, we may replace $v(t, X^*(t))$ in Lemma 4.6 by $v^*(X^*(t))$ for a measurable $v^* : E \to V$.*

5 The existence result

We begin by adapting for the present set-up the existence results from Bhatt and Borkar (1996), section 3. For a stationary $\mathcal{L}((X(\cdot),\pi(\cdot)))$, define the associated *ergodic occupation measure* $\varphi \in \mathcal{P}(E \times U)$ by:

$$\int f\varphi(dxdu) \stackrel{def}{=} E[\int_U f(X(t),u)\pi(t,du)].$$

Note that (2.3) then becomes $\int k d\varphi$. Let $\mathcal{G}$ denote the set of all ergodic occupation measures. From Theorem 2.1, pp. 1538-1541, of Bhatt and Borkar (1996), we have:

LEMMA **5.1** *$\mathcal{G}$ is closed convex and is characterized as*

$$\mathcal{G} = \{\mu \in \mathcal{P}(E \times U) : \int Af d\mu = 0 \ \forall f \in \mathcal{D}(A)\}.$$

Define $\eta : [0,\infty) \to \mathcal{P}(E \times U)$, $\eta(t) = \eta(t, dxdu)$ by

$$\int f\eta(t) \stackrel{def}{=} \frac{1}{t}\int_0^t E[\int_U f(X(s),u)\pi(s,du)]ds, \ f \in C_b(E \times U).$$

We make the following 'stability' assumption:

(A1) G is compact and for any compact $B \subset \mathcal{P}(E)$ and $\mathcal{L}((X(\cdot),\pi(\cdot))) \in \bar{\mathcal{M}}(\nu), \nu \in B$ (in particular, for a fixed $\nu \in \mathcal{P}(E)$), $\{\mathcal{L}(X(t)), t \geq 0\}$ is tight.

This implies in particular that $\eta(t), t \geq 0$, is tight and therefore relatively compact in $\mathcal{P}(E \times U)$ by Prohorov's theorem.

LEMMA **5.2** *Any limit point η^* of $\eta(t)$ in $\mathcal{P}(E \times U)$ as $t \to \infty$ is in $\mathcal{G}$.*

PROOF. Let $t \to \infty$ along an appropriate subsequence in the formula

$$\frac{E[f(X(t))]}{t} - \frac{E[f(X(0))]}{t} = \frac{1}{t}\int_0^t E[\bar{A}f(X(s), \pi(s))]ds, \ f \in \mathcal{D}(A),$$

to conclude that $\int Af d\eta^* = 0 \ \forall f \in \mathcal{D}(A)$. The claim now follows from Lemma 5.1 above. □

This enables us to establish the following basic existence result in the spirit of Lemma 3.1, p. 1553, of Bhatt and Borkar (1996).

LEMMA **5.3** *Under (A1), there exists a stationary ergodic $\mathcal{L}((\bar{X}(\cdot), \bar{\pi}(\cdot)))$ that is optimal for the ergodic control problem.*

PROOF. By the above lemma,

$$\liminf_{t\to\infty} \int k d\eta(t) \geq \inf_{\eta \in G} \int k d\eta.$$

By compactness of G, the infimum on the right is a minimum. From Theorem 2.1 of Bhatt and Borkar (1996), we know that the minimum will correspond to a stationary pair $(X(\cdot), \pi(\cdot))$. Considering the ergodic decomposition of the latter, the claim follows. □

Our aim is to refine this result to the existence of an optimal ergodic $\mathcal{L}((X(\cdot), \pi(\cdot)))$ so that $X(\cdot)$ is a *time-homogeneous* Markov process (whence $\pi(\cdot)$ may be taken to be a stationary Markov control as observed earlier). Let $\gamma \stackrel{def}{=} \min_{\mu \in \mathcal{G}} \int k d\mu$ denote the optimal cost and $\nu^* \stackrel{def}{=} \mathcal{L}(\bar{X}(t))$ for $\bar{X}(\cdot)$ as in Lemma 5.3.

Recall the time-homogeneous Markov processes $X^\alpha(\cdot), \alpha > 0$, of section 3, with the associated control processes $\{v^\alpha(X^\alpha(\cdot))\}$ resp. From now on we consider these with $\mathcal{L}(X^\alpha(0)) = \nu^*$. We shall need the following assumption:

(A2) Each $X^\alpha(\cdot)$ is asymptotically stationary.

LEMMA **5.4** *For all $\alpha > 0$,*

$$\alpha E[\int_0^\infty e^{-\alpha t} \int_U k(X^\alpha(t), u) v^\alpha(X^\alpha(t), du) dt] \leq \gamma. \tag{5.1}$$

PROOF. This is immediate from the optimality of $\mathcal{L}((X^\alpha(\cdot), v^\alpha(X^\alpha(\cdot))))$ for the $\alpha-$discounted cost and the fact that the corresponding cost for $\mathcal{L}((\bar{X}(\cdot), \bar{\pi}(\cdot)))$ is γ/α for all $\alpha > 0$. □

LEMMA **5.5** *As $\alpha \to 0$,*

$$\alpha E[\int_0^\infty e^{-\alpha t} \int_U k(X^\alpha(t), u) v^\alpha(X^\alpha(t), du) dt] \to \gamma.$$

PROOF. By the above lemma,

$$\limsup_{\alpha\to 0} \alpha E[\int_0^\infty e^{-\alpha t} \int_U k(X^\alpha(t), u) v^\alpha(X^\alpha(t), du) dt] \leq \gamma.$$

Consider the 'discounted occupation measures' $\mu^\alpha \in \mathcal{P}(E \times U)$ defined by

$$\int f d\mu^\alpha \stackrel{def}{=} \alpha E[\int_0^\infty e^{-\alpha t} \int_U f(X^\alpha(t), u) v^\alpha(X^\alpha(t), du) dt], \ f \in C_b(E \times U).$$

As shown in Bhatt and Borkar (1996), these satisfy

$$\int (Af - \alpha f) d\mu^\alpha + \int f d\nu^* = 0, \ f \in \mathcal{D}(A). \tag{5.2}$$

It follows from (A1) that $\{\mu^\alpha\}$ are tight. Letting $\alpha \to 0$ in (5.2) leads to

$$\int Af d\mu^* = 0 \ \forall f \in \mathcal{D}(A),$$

for any limit point μ^* of μ^α as $\alpha \to 0$, implying $\mu^* \in G$ by Lemma 5.1. Thus

$$\begin{aligned} & \liminf_{\alpha\to 0} \alpha E[\int_0^\infty e^{-\alpha t} \int_U k(X^\alpha(t), u) v^\alpha(X^\alpha(t), du) dt] \\ = & \limsup_{\alpha\to 0} \int k d\mu^\alpha \geq \int k d\mu^* \geq \gamma. \end{aligned}$$

The claim follows. □

Let $\mathcal{L}((\hat{X}^\alpha(\cdot), v^\alpha(\hat{X}^\alpha(\cdot))))$ for $\alpha > 0$ denote the limiting stationary laws of $\mathcal{L}((X^\alpha(\cdot), v^\alpha(X(\cdot))))$, implicit in the statement of assumption (A2) above. By (A1), these are tight. Let $\alpha(n) \downarrow 0$ be a subsequence such that

$$\mathcal{L}((\hat{X}^{\alpha(n)}(\cdot), v^{\alpha(n)}(\hat{X}^{\alpha(n)}(\cdot)))) \to \mathcal{L}((X'(\cdot), \pi'(\cdot)))$$

(say), which will also be stationary. Let

$$\hat{\nu}^n \stackrel{def}{=} \mathcal{L}(\hat{X}^{\alpha(n)}(t)) \to \hat{\nu} \stackrel{def}{=} \mathcal{L}(X'(t)).$$

Since $\mathcal{L}((\hat{X}^{\alpha(n)}(\cdot), v^{\alpha(n)}(\hat{X}^{\alpha(n)}(\cdot)))) \in \mathcal{M}_\infty^{\alpha(n)}(\hat{\nu}^n)$, it follows that

$$\mathcal{L}((X'(\cdot), \pi'(\cdot))) \in \mathcal{M}_\infty^0(\hat{\nu}).$$

Consider $\mathcal{L}((X^*(\cdot), v^*(X^*(\cdot))))$ as in the preceding section with $\mathcal{L}(X^*(0)) = \hat{\nu}$. Then $\mathcal{L}((X^*(\cdot), v^*(X^*(\cdot)))) \in \mathcal{M}_\infty^0(\hat{\nu})$. Thus by Corollary 4.1, it has the same one dimensional marginal as $\mathcal{L}((X'(\cdot), \pi'(\cdot)))$, viz., the constant marginal $\hat{\nu}$. Since it is also time-homogeneous Markov, it follows that it is stationary. Furthermore,

$$\begin{aligned} & E[\int_U k(\hat{X}^{\alpha(n)}(t), u) v^{\alpha(n)}(\hat{X}(t), du))] \\ \to & E[\int_U k(\hat{X}'(t), u) \pi'(t, du))] \\ = & E[\int_U k(X^*(t), u) v^*(X^*(t), du))] \\ = & \lim_{t\to\infty} \frac{1}{t} \int_0^t E[\int_U k(X^*(s), u) v^*(X^*(s), du) ds], \end{aligned}$$

implying that the latter equals γ. That is, $\mathcal{L}((X^*(\cdot), v^*(X^*(\cdot))))$ is an optimal stationary pair. By considering its ergodic decomposition if necessary, we have proved:

THEOREM **5.1** *Under (A1), (A2), there exists an optimal ergodic, time-homogeneous Markov solution to the ergodic control problem.*

6 Concluding remarks

We have established the existence of an optimal time-homogeneous Markov and ergodic solution to the ergodic control problem under a certain set of conditions. A careful look at the arguments of the final section shows that it is of the nature: 'if an optimal ergodic solution exists, then so does one which is also time-homogeneous Markov'. Condition (A1) played a crucial role in establishing the former. In specific cases, however, one may be able to replace it by other more convenient conditions for the purpose, see, e.g., section 3 of Bhatt and Borkar (1996) for one such instance.

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